TWO-COMPONENT SPINOR FORMULATION OF THE MAXWELL THEORY

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A two-component spinor form of the conventional electromagnetic Lagrangian density on real Minkowski space is used to obtain the second "half" of Maxwell's equations from a variational principle. It is shown that a somewhat modified form of the free part of this Lagrangian density can also be employed to derive the explicit equations of motion that give rise to the first "half" of the complete theory. The Penrose expression for the electromagnetic energy-momentum tensor $T^{ab}(x)$ is explicitly derived by working out a suitable source-free defining relation. The result is that the spinor components associated with $T^{00}(x)$ and $T^{0k}(x)$, $k = 1,2,3$, are equal to the ordinary energy and linear-momentum densities of the electromagnetic fields, respectively. A set of explicit kinematical integral expressions for the theory is then exhibited.

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1. Introduction

One of the most important features of the relativistic two-spinor approach to Maxwell's theory [1–3] is the fact that all the electromagnetic degrees of freedom are locally represented by a pair of symmetric two-index spinor fields. Since this spinor formulation constitutes essentially the ultimate spacetime description of the theory, it can actually be said that, even at a classical level, the field equations describe the propagation of massless fields carrying spin ± 1. All the information about the degrees of freedom of the fields is thus carried at every spacetime point by the electromagnetic bivector. Another noteworthy result arising in this framework is that the usual Lorentz gauge condition can be explicitly stated as the vanishing of skew symmetric parts of suitably contracted first derivatives of the electromagnetic potential. These statements entail a simplification of certain identities coming directly from the defining expression for the bivector, and can indeed be established in terms of the symmetry property of the fields. Likewise, the complete theory appears as a system of four complex (eight real) first-order linear partial differential equations for each of the fields. In respect of the real-spacetime approach [1, 2] the entire system of field equations can be regarded as being formed by two (Hermitian) SL(2, $\mathbb{C}$)--covariant "halves". The first "half" normally arises from the combination of the above referred identities with four field equations which actually constitute the Bianchi identities of the theory. Roughly speaking, the second "half" consists of statements involving appropriate derivatives of the fields along with the sources.

In this paper, we are basically concerned with deriving the entire electromagnetic theory in real Minkowski space $\mathbb{RM}$ from two variational principles. Indeed the key idea upon which our basic prescription rests, is to look upon the first "half" as a "complementary theory" which can be effectively combined with the other part. Whereas the equations of motion yielding the second "half" involve the two-spinor version of the standard full Maxwell Lagrangian density, those giving rise to first "half" actually carry a slightly modified form of the conventional source-free piece. In either case, the relevant action is defined on a bounded submanifold of $\mathbb{RM}$. Upon working out the dynamical statements, we assume that arbitrary variations of the potential vanish on the boundary of this submanifold. Both fields will be taken to be continuous on the closure of the subset. It will be seen that the wave equations for the fields and potential emerge as a consequence of the two-spinor structure of the field equations. We shall see also that the charge-conservation law arises naturally from the symmetry property of the fields. Starting with a suitable two-spinor definition of the electromagnetic energy-momentum tensor $T^{ab}(x)$, we carry out the actual derivation of Penrose's expression for the tensor [1] in a straightforward way. At this stage, the main procedure consists in working out our defining expression instead of translating the conventional (world) one into spinor terms. It will be shown that those components of Penrose's tensor which are associated with $T^{00}(x)$ and $T^{0k}(x)$, $k = 1, 2, 3$, are equal to the elementary expressions for the energy and linear-momentum densities carried by the fields, respectively. We will particularly arrive at explicit covariant integral expressions for the components of the electromagnetic energy-momentum four-vector which are directly identified with the energy and linear momentum of the fields.

Our paper is organized as follows. Section 2 deals with Maxwell's equations, and is divided into four subsections. In subsection 2.1, we review briefly some well known basic definitions. The first "half" appears at this stage as a set of identities, not as a system of equations of motion. The variational principles are formulated in subsection 2.2. We present the equations of motion for the complete theory together with the charge integrals for the sources in subsection 2.3. In subsection 2.4, the electromagnetic wave equations are derived. Section 3 is concerned with the energy-momentum tensor. We have split it into three subsections. In subsection 3.1, we introduce the explicit two-spinor defining expression for the tensor. In subsection 3.2, we build up the energy expressions. The covariant kinematical expressions for the theory are exhibited in subsection 3.3. Section 2 will also provide us with a framework for elaborating Section 3. In Section 4, we make some final remarks on the theory.

Both the unprimed and primed fields shall be considered herein as classical but will sometimes be referred to as left- and right-handed fields, respectively. The two-spinor conventions and rules as given by Penrose [1] will be adopted throughout this work. We shall use also the natural system of units wherein $c = \hbar = 1$. Of course, the symmetry property of the fields makes the ordering of the relevant upper and lower indices immaterial. Thus we will not order these indices when performing the calculations.

2. Maxwell's equations

2.1. The first "half" as identities

Let τ be a connected open subset of $\mathbb{RM}$. It is convenient to require τ to be also a bounded four-dimensional submanifold of $\mathbb{RM}$ whose closure $\bar{\tau}$ is compact. The reason for this requirement will become manifest later. For the (covariant) electromagnetic bivector on τ, we have the defining expression

$$F_{AA'BB'}(x) = 2\nabla_{[AA'}\Phi_{BB']}(x)\,, \tag{2.1}$$

where $\Phi_{BB'}(x)$ is the (real) electromagnetic potential, and $\nabla_{AA'}$ denotes the ordinary partial derivative operator $\partial/\partial x^{AA'}$. Using the local one-to--one correspondence between the set of all bivectors on τ and the pairs of conjugate symmetric spinors $\{\phi_{AB}(x),\ \bar{\phi}_{A'B'}(x)\}$ we can write (see Ref. [4])

$$F_{AA'BB'}(x) = \varepsilon_{A'B'}\phi_{AB}(x) + \varepsilon_{AB}\bar{\phi}_{A'B'}(x)\,. \tag{2.2}$$

The conjugate spinor fields entering into (2.2) are the so-called electromagnetic spinors. They are here looked upon as dependent insofar as the (six) degrees of freedom carried by the theory are described by either of them at every $x^{AA'} \in \tau$. Their symmetry yields

$$\phi_A^A(x) = 0 = \bar{\phi}_{A'}^{A'}(x)\,. \tag{2.3}$$

We can at once establish the relationships between the fields and the potential by transvecting both (2.1) and (2.2) with $\varepsilon^{A'B'}$ and ε^{AB}. This procedure leads to

$$\phi_{AB}(x) = \nabla_{A'(A}\Phi^{A'}_{B)}(x), \quad \bar{\phi}_{A'B'}(x) = \nabla_{A(A'}\Phi^{A}_{B')}(x). \qquad (2.4)$$

The above identities take a simpler form in case we adopt the following Lorentz–gauge relations

$$\nabla_{A'[A}\Phi^{A'}_{B]}(x) = 0, \quad \nabla_{A[A'}\Phi^{A}_{B']}(x) = 0. \qquad (2.5)$$

To see how this situation actually arises, we make use of the trivial relation

$$\nabla_{A'A}\Phi^{A'}_{B}(x) = \nabla_{A'(A}\Phi^{A'}_{B)}(x) + \tfrac{1}{2}\varepsilon_{AB}\Lambda(x), \qquad (2.6)$$

along with its complex conjugate, with $\Lambda(x) = \nabla_c\Phi^c(x)$. Thus, the identities (2.4) turn out to be re-expressed as

$$\phi_{AB}(x) = \nabla_{A'A}\Phi^{A'}_{B}(x), \quad \bar{\phi}_{A'B'}(x) = \nabla_{AA'}\Phi^{A}_{B'}(x), \qquad (2.7)$$

whence the condition $\Lambda(x) = 0$ can also be re-expressed as the statements (2.3).

The first "half" of the entire theory consists of (2.7) together with the field equations that arise from the simple straightforward computation

$$\nabla_{[a}F_{bc]}(x) = \nabla_{[a}\nabla_b\Phi_{c]}(x) = \nabla_{[[a}\nabla_{b]}\Phi_{c]}(x) = 0, \qquad (2.8)$$

where we have used the (local) commutativity of the ∇'s. These relations evidently amount to the same thing as stating

$$\nabla^a {}^*F_{ab}(x) = 0, \qquad (2.9)$$

with ${}^*F_{ab}(x)$ being the dual electromagnetic bivector which is defined by [1]

$${}^*F_{AA'BB'}(x) = i\left[\varepsilon_{AB}\bar{\phi}_{A'B'}(x) - \varepsilon_{A'B'}\phi_{AB}(x)\right]. \qquad (2.10)$$

We are thus led to

$$\nabla^{A'B}\phi^A_B(x) = \nabla^{AB'}\bar{\phi}^{A'}_{B'}(x), \qquad (2.11)$$

which are the field equations referred to above. These are the Bianchi identities of the theory. Obviously, they form a system of four complex first-order linear partial differential equations on τ. It is of some importance to observe that these statements can be viewed as a consequence of the relations (2.4). We will consider this point explicitly in Section 4.

2.2. Variational principles

We shall now build up the variational principles which yield the equations of motion for the entire theory. It will be convenient to carry out initially the construction of the action for the second "half". For simplicity, we will omit here, as elsewhere, all the arguments of our Lagrangian densities.

The two-spinor form of the conventional full electromagnetic Lagrangian density is written out explicitly as

$$\mathcal{L}_M^{\mathrm{II}} = \frac{1}{8\pi}\left[\phi_{AB}(x)\phi^{AB}(x) + \bar{\phi}_{A'B'}(x)\bar{\phi}^{A'B'}(x)\right] + j^{AA'}(x)\Phi_{AA'}(x)\,. \quad (2.12)$$

In this expression, the four-vector $j^{AA'}(x)$ is the (real) electromagnetic current density which actually plays the role of a source for the fields and potential. This fact will be brought about later when we carry out the explicit derivation of the wave equations (see subsection 2.4). It is obvious that we can re-express (2.12) as

$$\mathcal{L}_M^{\mathrm{II}} = \mathcal{L}_{\mathrm{free}} + \mathcal{L}_{\mathrm{int}}\,, \quad (2.13)$$

where $\mathcal{L}_{\mathrm{free}}$ is the free part of L_M^{II} which carries the conjugate bilinear terms involving the fields, and $\mathcal{L}_{\mathrm{int}}$ stands for the interacting part which appears simply as the inner product of $\Phi_a(x)$ with $j^a(x)$. Now, using (2.4), yields the useful expression

$$\begin{aligned}\mathcal{L}_{\mathrm{free}} = &\frac{1}{8\pi}\Big[\nabla_{A'(A}\Phi_{B)}^{A'}(x)\Big(\nabla_{B'}{}^{(A}\Phi^{B)B'}(x)\Big) \\ &+ \nabla_{A(A'}\Phi_{B')}^{A}(x)\Big(\nabla_{B}{}^{(A'}\Phi^{B')B}(x)\Big)\Big]\,. \end{aligned} \quad (2.14)$$

The formal action for the second "half" then reads

$$S[\mathcal{L}_M^{\mathrm{II}}] = \int_{\mathcal{T}} \mathcal{L}_M^{\mathrm{II}} d^4x\,, \quad (2.15)$$

while the relevant dynamical statement is

$$\delta S[\mathcal{L}_M^{\mathrm{II}}] = \delta\int_{\mathcal{T}} \mathcal{L}_M^{\mathrm{II}} d^4x = 0\,, \quad (2.16)$$

where $d^4x = \frac{1}{4!}e_{abcd}dx^a \wedge dx^b \wedge dx^c \wedge dx^d$ is a four-volume element of τ with e_{abcd} being the usual alternating tensor for the (covariant) standard basis of $\mathbb{RM}$.

To introduce our explicit Lagrangian density for the first "half", it is worthwhile to define

$$\mathcal{L}_{\text{free}} = \mathcal{L}_{\text{lh}} + \mathcal{L}_{\text{rh}}, \tag{2.17}$$

where $\mathcal{L}_{\text{lh}}$ and $\mathcal{L}_{\text{rh}}$ denote, respectively, the left- and right-handed parts of (2.14). These quantities can be clearly rewritten as

$$\mathcal{L}_{\text{lh}} = \frac{1}{8\pi}\varepsilon^{AC}\varepsilon^{BD}\phi_{AB}(x)\phi_{CD}(x), \tag{2.18a}$$

and

$$\mathcal{L}_{\text{rh}} = \frac{1}{8\pi}\varepsilon^{A'C'}\varepsilon^{B'D'}\bar{\phi}_{A'B'}(x)\bar{\phi}_{C'D'}(x). \tag{2.18b}$$

Our modified Lagrangian density is chosen to be

$$\mathcal{L}^{\text{I}}_M = i(\mathcal{L}_{\text{lh}} - \mathcal{L}_{\text{rh}}), \tag{2.19}$$

the corresponding variational statement thus being

$$\delta S[\mathcal{L}^{\text{I}}_M] = \delta \int_\tau \mathcal{L}^{\text{I}}_M d^4x = 0. \tag{2.20}$$

We assume that all the quantities occurring in the above Lagrangian densities are continuous on $\bar{\tau}$ whence both (2.13) and (2.19) are supposedly smooth real SL(2,$\mathbb{C}$)-scalar functions on τ. The δ-variation involved in (2.16) and (2.20) is here regarded as ordinary in the sense that it does not entail any deformation of τ at all. Moreover, when the variational statements are effectively worked out, the current density is held fixed while arbitrary variations of the potential are taken to vanish on the boundary $\partial\tau$ of τ. When working out these statements, we will also assume that the (local) δ-variation commutes with the ∇-operators.

2.3. Equations of motion and charge integrals

In what follows, we shall first see that the basic variational prescription given before enables us to express the entire theory as equations of motion carrying the Lagrangian densities (2.13) and (2.19). It will be particularly shown how our prescription can become more feasible by carrying through the procedure for deriving the second "half" of the field equations. Actually the procedure involving the first "half" appears to be essentially the same as the one referred to above. For this reason, we will write down the corresponding equations of motion without working out explicitly the statement (2.20). Thereafter, the charge integrals of the theory will be exhibited. We will see that the two-spinor integral expressions arising here are identical with those given by Penrose [6] when only electric charges are allowed for. The relevant explicit calculations will not be carried out at this stage. This is because we will consider similar situations later wherein our calculational procedures can be carried through in a more transparent manner (see subsection 3.2).

Expressions (2.12) and (2.13) together with (2.4) suggest writing out (2.16) as

$$\int_{\tau}\left[\frac{\partial\mathcal{L}_{\mathrm{lh}}}{\partial\phi_{AB}(x)}\nabla_{A'(A}\delta\Phi_{B)}^{A'}(x)+\frac{\partial\mathcal{L}_{\mathrm{rh}}}{\partial\bar{\phi}_{A'B'}(x)}\nabla_{A(A'}\delta\Phi_{B')}^{A}(x)\right.$$
$$\left.+\frac{\partial\mathcal{L}_{\mathrm{int}}}{\partial\Phi_{AA'}(x)}\delta\Phi_{AA'}(x)\right]d^4x=0\,. \quad (2.21)$$

Of course, on account of the symmetry of the fields, we can drop the symmetrization round brackets from (2.21) without any loss in generality. Hence, integrating by parts the terms involving the (symmetrized) derivatives of the variations of the potential, after some manipulations we obtain

$$0=\int_{\partial\tau}\left[\frac{\partial\mathcal{L}_{\mathrm{lh}}}{\partial\phi_{AB}(x)}\delta\Phi_{B}^{A'}(x)+\frac{\partial\mathcal{L}_{\mathrm{rh}}}{\partial\bar{\phi}_{A'B'}(x)}\delta\Phi_{B'}^{A}(x)\right]d^3x_{AA'}$$
$$+\int_{\tau}\left[\nabla_{B}^{A'}\frac{\partial\mathcal{L}_{\mathrm{lh}}}{\partial\phi_{AB}(x)}+\nabla_{B'}^{A}\frac{\partial\mathcal{L}_{\mathrm{rh}}}{\partial\bar{\phi}_{A'B'}(x)}+\frac{\partial\mathcal{L}_{\mathrm{int}}}{\partial\Phi_{AA'}(x)}\right]\delta\Phi_{AA'}(x)d^4x\,,$$
$$(2.22)$$

where $d^3x_{AA'}$ is the spinor element of three-hypersurface area of $\partial\tau$ associated with $d^3x_a=\frac{1}{3!}e_{abcd}dx^b\wedge dx^c\wedge dx^d$. Consequently taking $\delta\Phi^{CC'}(x)=0$ on $\partial\tau$, yields the equations of motion

$$\nabla_{B}^{A'}\frac{\partial\mathcal{L}_{\mathrm{lh}}}{\partial\phi_{AB}(x)}+\nabla_{B'}^{A}\frac{\partial\mathcal{L}_{\mathrm{rh}}}{\partial\bar{\phi}_{A'B'}(x)}+\frac{\partial\mathcal{L}_{\mathrm{int}}}{\partial\Phi_{AA'}(x)}=0\,. \quad (2.23)$$

The explicit gauge-invariant field equations for the second "half" then read

$$\nabla^{BA'}\phi_{B}^{A}(x)+\nabla^{AB'}\bar{\phi}_{B'}^{A'}(x)=4\pi j^{AA'}(x)\,. \quad (2.24)$$

More explicitly, we can express the statement (2.22) as

$$\int_{\tau}\left[4\pi j^{AA'}(x)-\nabla^{BA'}\phi_{B}^{A}(x)-\nabla^{AB'}\bar{\phi}_{B'}^{A'}(x)\right]\delta\Phi_{AA'}(x)d^4x$$
$$+\int_{\partial\tau}\left[\phi^{AB}(x)\delta\Phi_{B}^{A'}(x)+\bar{\phi}^{A'B'}(x)\delta\Phi_{B'}^{A}(x)\right]d^3x_{AA'}=0\,, \quad (2.25)$$

which evidently yields (2.24). By virtue of (2.4), Eqs (2.23) take the form

$$\nabla^{A'}_{B}\frac{\partial\mathcal{L}_{\mathrm{lh}}}{\partial\left[\nabla_{B'(A}\Phi^{B'}_{B)}(x)\right]}+\nabla^{A}_{B'}\frac{\partial\mathcal{L}_{\mathrm{rh}}}{\partial\left[\nabla_{B(A'}\Phi^{B}_{B')}(x)\right]}+\frac{\partial\mathcal{L}_{\mathrm{int}}}{\partial\Phi_{AA'}(x)}=0\,. \quad (2.26)$$

Now, a procedure similar to that giving rise to (2.23) yields the invariant equations of motion for the first "half"

$$\nabla^{A'}_{B}\frac{\partial\mathcal{L}_{\mathrm{lh}}}{\partial\phi_{AB}(x)}-\nabla^{A}_{B'}\frac{\partial\mathcal{L}_{\mathrm{rh}}}{\partial\bar{\phi}_{A'B'}(x)}=0\,, \quad (2.27)$$

which effectively reinstates the Bianchi identities (2.11). In a formal way, we can rewrite (2.27) as

$$\nabla^{A'}_{B}\frac{\partial\mathcal{L}_{\mathrm{lh}}}{\partial\left[\nabla_{B'(A}\Phi^{B'}_{B)}(x)\right]}-\nabla^{A}_{B'}\frac{\partial\mathcal{L}_{\mathrm{rh}}}{\partial\left[\nabla_{B(A'}\Phi^{B}_{B')}(x)\right]}=0\,. \quad (2.28)$$

We should emphasize that, as far as the use of (2.26) and (2.28) is concerned, it seems to be appropriate to recall the source-free pieces given in (2.14).

The complete field theory on τ consists of the statements that are obtained by combining (2.11) with (2.24). We thus have the invariant field equations

$$\nabla^{BA'}\phi^{A}_{B}(x)=2\pi j^{AA'}(x)=\nabla^{AB'}\bar{\phi}^{A'}_{B'}(x)\,. \quad (2.29)$$

In the absence of sources ($j_a(x)=0$) the propagation of the fields appears, therefore, to be governed by the massless free-field equations for spin ± 1

$$\nabla^{BA'}\phi^{A}_{B}(x)=0=\nabla^{AB'}\bar{\phi}^{A'}_{B'}(x)\,. \quad (2.30)$$

We shall next derive some particularly simple two-spinor integral expressions for the total charge carried by $j^a(x)$. In this connection, we consider a space-like hypersurface Σ in $\mathbb{RM}$, given by

$$\Sigma=\left\{x^a\in\bar{\tau}\,|\,f(x)=0 \quad \text{with} \quad (\nabla^a f(x))\nabla_a f(x)>0\right\}. \quad (2.31)$$

In fact, it is not strictly necessary to assume here that Σ is the boundary of $\bar{\tau}$. Instead, the crucial point as regards our immediate purposes is to suppose further that the scalar function $f(x)$ and its first derivatives are continuous on Σ, such that (2.31) can be thought of as defining a non-singular three-dimensional submanifold of $\mathbb{RM}$ with a smooth (two-dimensional) boundary $\partial\Sigma\subset\tau$. The total charge of the sources can be expressed in terms of the (covariant) derivative of the electromagnetic bivector by

$$Q[\Sigma]=\frac{1}{24\pi}\int_{\Sigma}\nabla_a F^{ab}(x)e_{bcdh}\,dx^c\wedge dx^d\wedge dx^h\,. \quad (2.32)$$

It is useful to observe that (2.24) can indeed be considered as the two--spinor version of the conventional field equations $\nabla_a F^{ab}(x) = 4\pi j^b(x)$. Now performing some integrations, and using the well-known world-tensor duality relation

$$^*F_{ab}(x) = \tfrac{1}{2}e_{abcd}F^{cd}(x)\,, \tag{2.33}$$

we arrive at the spacetime expression

$$Q[\Sigma] = \frac{1}{4\pi}\int\limits_{\partial\Sigma} {}^*F_{ab}(x)dx^a \wedge dx^b\,, \tag{2.34}$$

which, when combined with (2.10), yields

$$Q[\Sigma] = \frac{1}{2\pi}\text{Im}\int\limits_{\partial\Sigma} \phi_A^B(x)dx_{B'}^A \wedge dx_B^{B'}\,. \tag{2.35a}$$

By using (2.2) we can also re-express (2.32) as

$$\begin{aligned} Q[\Sigma] &= \frac{1}{6\pi}\text{Im}\int\limits_{\Sigma} \nabla^{BA'}\phi_B^A(x)dx_D^{C'} \wedge dx_{AC'} \wedge dx_{A'}^D \\ &= \frac{1}{2\pi}\text{Re}\int\limits_{\Sigma} \nabla^{BA'}\phi_B^A(x)d^3x_{AA'}\,, \end{aligned} \tag{2.35b}$$

where the second equality is due to the Bianchi identities (2.11). It is of some interest to mention that, in the case where also "magnetic" charges are present, an explicit integral for the total "electromagnetic" charge can be obtained by dropping the "Im" from (2.35a). In fact, this expression was particularly used by Penrose [5] towards building up a twistorial charge integral for the theory.

2.4 Wave equations

The electromagnetic theory, as formulated before, gives rise to a system of ten (complex) wave equations on τ. Four of these equations involve $\Phi_{AA'}(x)$ along with $j_{AA'}(x)$, and are deemed to be equivalent to the second "half" of the theory. Actually, they constitute nothing else but the conventional wave equation for the potential which can, in particular, take up the Lorentz gauge condition. The other equations involve the fields together with certain (suitably) contracted first derivatives of the current density. It will be seen that the explicit charge-conservation statement turns out to be an immediate consequence of the latter set of equations whenever the symmetry property of the fields is effectively taken into account.

Let us consider the first of Eqs (2.4). Operating on both sides with $\nabla^A_{C'}$, recalling the first of Eqs (2.29) and using the splitting

$$\nabla^A_{C'}\nabla_{AA'} = \nabla^A_{(A'}\nabla_{C')A} + \nabla^A_{[C'}\nabla_{A']A} = \tfrac{1}{2}\varepsilon_{A'C'}\Box\,, \tag{2.36}$$

we get

$$2\nabla^A_{C'}\phi_{AB}(x) = \Box\,\Phi_{BC'}(x) - \nabla_{BC'}\Lambda(x)\,, \tag{2.37}$$

where $\Box = \nabla_a\nabla^a$, and $\Lambda(x)$ is the same scalar function as that carried by (2.6). Here we have used once again the commutativity of the ∇'s which really implies that the symmetric part $\nabla^A_{(A'}\nabla_{C')A}$ vanishes identically. Therefore, invoking (2.29), we are immediately led to

$$\Box\,\Phi_{AA'}(x) = 4\pi j_{AA'}(x) + \nabla_{AA'}\Lambda(x)\,, \tag{2.38}$$

whence, in the absence of sources, $\Phi_{AA'}(x)$ satisfies the (inhomogeneous) wave equation $\Box\,\Phi_{AA'}(x) = \nabla_{AA'}\Lambda(x)$.

To derive the wave equations involving the fields, we adopt a procedure similar to that yielding (2.38), but now making use of either of Eqs (2.29). In the case of the left-handed field, say, we perform the calculation

$$\begin{aligned}\nabla_{A'C}\nabla^{A'}_B\phi^{AB}(x) &= \left(\tfrac{1}{2}\varepsilon_{CD}\Box + \nabla_{A'(C}\nabla^{A'}_{B)}\right)\phi^{AB}(x)\\ &= \tfrac{1}{2}\varepsilon_{CB}\Box\,\phi^{AB}(x)\\ &= 2\pi\nabla^{A'}_C j^A_{A'}(x)\,,\end{aligned} \tag{2.39}$$

which yields the gauge-invariant equations

$$\Box\,\phi_{AB}(x) = 4\pi\nabla_{A'A}j^{A'}_B(x)\,. \tag{2.40}$$

Similarly, we obtain for the right-handed field

$$\Box\,\bar{\phi}_{A'B'}(x) = 4\pi\nabla_{AA'}j^A_{B'}(x)\,. \tag{2.41}$$

One important consequence of the theory is the fact that it gives rise to the conservation of charge. This result can be immediately seen by invoking (2.3). We have, in effect,

$$\Box\,\phi^A_A(x) = \Box\,\bar{\phi}^{A'}_{A'}(x) = 0 = 4\pi\nabla_{AA'}j^{AA'}(x)\,, \tag{2.42}$$

which clearly involves the statement to which we have referred above. Obviously the divergencelessness of $j^a(x)$ is equivalent to either of the relations

$$\nabla_{A'[A}j^{A'}_{B]}(x) = 0, \qquad \nabla_{A[A'}j^A_{B']}(x) = 0\,, \tag{2.43}$$

whence Eqs (2.40) and (2.41) can be effectively rewritten as

$$\square\,\phi_{AB}(x) = 4\pi\nabla_{A'(A}j^{A'}_{B)}(x), \quad \square\,\bar{\phi}_{A'B'}(x) = 4\pi\nabla_{A(A'}j^{A}_{B')}(x)\,. \tag{2.44}$$

The charge-conservation statement involved in (2.42) can also be established as trivial identities carrying the following conjugate skew-symmetric differential operators

$$\blacksquare_{AB} = \nabla_{A'[A}\nabla^{A'}_{B]} = \tfrac{1}{2}\varepsilon_{AB}\,\square\,, \tag{2.45a}$$

$$\blacksquare_{A'B'} = \nabla_{A[A'}\nabla^{A}_{B']} = \tfrac{1}{2}\varepsilon_{A'B'}\,\square\,. \tag{2.45b}$$

In effect, we have

$$\blacksquare_{AB}\phi^{AB}(x) = \blacksquare_{A'B'}\bar{\phi}^{A'B'}(x) = 2\pi\nabla_a j^a(x) = 0\,. \tag{2.46}$$

The operators (2.45) were actually introduced by Penrose [6] in connection with the problem of establishing the conformal invariance of spinning massless free systems on a general curved spacetime background. In the absence of sources, Eqs (2.44) are thus reduced to

$$\square\,\phi_{AB}(x) = 0, \qquad \square\,\bar{\phi}_{A'B'}(x) = 0\,. \tag{2.47}$$

Under this source-free circumstance each component of the fields appears to satisfy the ordinary homogeneous wave equation. It is evident that this result is equivalent to stating that the free fields satisfy Eqs (2.30).

3. The electromagnetic energy-momentum tensor

We are now in a position to carry out the explicit derivation of Penrose's expression for the electromagnetic energy-momentum tensor. As was said before, the basic idea here is to start with a suitable two-spinor defining expression for the tensor instead of translating its conventional world definition into two-spinor terms. Its usual symmetry and reality properties are indeed taken up at the outset. Its trace-freeness will appear also as a consequence of the symmetry of the fields. Additionally, we will see that its divergence-freeness actually arises as an identity involving the massless free-field equations (2.30). In order to write down the integral expressions for the energy carried by the fields, it will be necessary to impose one further requirement on the region τ. The reason for this is that one of the energy integrals referred to above has to be taken over an appropriate subset of $\mathbb{R}^3$. In accordance with the elementary expression [7], the energy density appears to be positive–definite throughout τ. The energy-momentum four-vector is given as an explicit covariant integral taken over Σ (see (2.31) above). It is observed that the two-spinor components associated with those of the linear momentum of the fields lead to the electromagnetic Poynting vector. The construction of the integrals for the angular momentum of the fields will be carried out in a straightforward way by using the two-spinor expression for $T_{ab}(x)$.

3.1 Penrose's expression

Our explicit starting relation at some $x^{AA'} \in \tau$ is written as

$$\begin{aligned} T_{AA'BB'}(x) =& \varepsilon_{AB}\varepsilon_{A'B'}\mathcal{L}_{\text{free}} \\ &- 2\nabla_{[AA'}\Phi_{CC']}(x)\frac{\partial \mathcal{L}_{\text{free}}}{\partial\left[\nabla^{BB'}\Phi_{CC'}(x)\right]}\,. \end{aligned} \tag{3.1}$$

On the basis of (2.1), we can evidently re-express the above relation as follows

$$\begin{aligned} T_{AA'BB'}(x) =& \varepsilon_{AB}\varepsilon_{A'B'}\mathcal{L}_{\text{free}} \\ &- \left[\varepsilon_{A'C'}\phi_{AC}(x) + \varepsilon_{AC}\bar{\phi}_{A'C'}(x)\right]\frac{\partial \mathcal{L}_{\text{free}}}{\partial\left[\nabla^{BB'}\Phi_{CC'}(x)\right]}\,. \end{aligned} \tag{3.2}$$

The derivative piece can be worked out by using (2.14). We have, in effect

$$\frac{\partial \mathcal{L}_{\text{free}}}{\partial\left[\nabla^{BB'}\Phi_{CC'}(x)\right]} = \frac{1}{4\pi}\left[\varepsilon_{B'}{}^{C'}\phi_B^C(x) + \varepsilon_B{}^C\bar{\phi}_{B'}^{C'}(x)\right]. \tag{3.3}$$

Now, inserting (3.3) into (3.2) and invoking (2.17) along with (2.18), we obtain

$$\begin{aligned} T_{AA'BB'}(x) =& \frac{1}{4\pi}\Big[2\phi_{AB}(x)\bar{\phi}_{A'B'}(x) \\ &- \varepsilon_{A'B'}\phi_{AC}(x)\phi_B^C(x) - \varepsilon_{AB}\bar{\phi}_{A'C'}(x)\bar{\phi}_{B'}^{C'}\Big] \\ &+ \frac{1}{8\pi}\varepsilon_{AB}\varepsilon_{A'B'}\left[\phi_{CD}(x)\phi^{CD}(x) + \bar{\phi}_{C'D'}(x)\bar{\phi}^{C'D'}(x)\right]. \end{aligned} \tag{3.4}$$

Consequently, the relation

$$\varepsilon_{AB}\phi_{CD}(x)\phi^{CD}(x) = 2\phi_{C[A}(x)\phi_{B]}^C(x) = 2\phi_{AC}(x)\phi_B^C(x)\,, \tag{3.5}$$

together with its complex conjugate, yields after a short calculation

$$T_{AA'BB'}(x) = \frac{1}{2\pi}\phi_{AB}(x)\bar{\phi}_{A'B'}(x)\,. \tag{3.6}$$

This constitutes the (gauge invariant) Penrose version of $T_{ab}(x)$. It should be noticed that the relation (3.5) is equivalent to the statement $\phi_{C(A}(x)\phi^{C}_{B)}(x) = 0$.

One feature of the expression (3.6) is indeed that $T_{ab}(x)$ is both symmetric and trace-free. This fact actually arises as an immediate consequence of (2.3). It is worth remarking explicitly that these properties can also be seen as trivial identities coming directly from (3.4). Moreover, the field equations (2.24) yield

$$\nabla^{AA'}T_{AA'BB'}(x) = j^{A}_{B'}(x)\phi_{AB}(x) + j^{A'}_{B}(x)\bar{\phi}_{A'B'}(x)\,, \tag{3.7}$$

whence, for an arbitrary (Hermitian) $\xi^{BB'} \in \tau$, we have

$$\xi^{BB'}\nabla^{AA'}T_{AA'BB'}(x) = 2\,\mathrm{Re}\left[\xi^{BB'}j^{A}_{B'}(x)\phi_{AB}(x)\right]. \tag{3.8}$$

It becomes evident that, taking $j^{a}(x) = 0$ on τ, Eq. (3.7) yields the divergence statement

$$\nabla^{AA'}T_{AA'BB'}(x) = 0\,, \tag{3.9}$$

which seems to agree with the fact that our defining expression for $T_{ab}(x)$ does not involve the sources.

There is a positive–definiteness statement [1] which appears to be essentially due to the reality of the structure (3.6). The statement is that, for any two feature oriented time-like vectors $U^{a}(x)$ and $V^{b}(x)$ on τ, the inequality

$$T_{ab}(x)U^{a}(x)V^{b}(x) \geq 0 \tag{3.10}$$

holds throughout τ. To see this, we use the canonical two-spinor decompositions $\mu^{A}(x)\bar{\mu}^{A'}(x)$, $\nu^{A}(x)\bar{\nu}^{A'}(x)$ of two arbitrary future oriented null vectors, recalling that each of $U^{a}(x)$ and $V^{a}(x)$ can be uniquely expressed in terms of a linear combination of the type

$$Y^{AA'}(x) = a\mu^{A}(x)\bar{\mu}^{A'}(x) + b\nu^{A}(x)\bar{\nu}^{A'}(x)\,, \tag{3.11}$$

where a and b belong both to the set $\mathbb{R}^{+}$ of positive real numbers. After short computation, we thus obtain the positive-definite invariant

$$\begin{aligned} &T_{ab}(x)U^{a}(x)V^{b}(x) \\ &= \frac{1}{2\pi}\Big[\alpha|\phi_{AB}(x)\lambda^{A}(x)\eta^{B}(x)|^{2} + \beta|\phi_{AB}(x)\lambda^{A}(x)\zeta^{B}(x)|^{2} \\ &+ \gamma|\phi_{AB}(x)\tau^{A}(x)\eta^{B}(x)|^{2} + \delta|\phi_{AB}(x)\tau^{A}(x)\zeta^{B}(x)|^{2}\Big] \geq 0\,, \end{aligned} \tag{3.12}$$

provided that α, β, γ and δ belong all to $\mathbb{R}^+$. Of course, the validity of the relation involved in (3.12) can be extended to the whole of $\mathbb{RM}$ if a suitable analyticity condition is imposed on all the quantities involved.

3.2 Electromagnetic energy expressions

Before introducing the explicit integrals for the energy of the fields, we shall carry out the computation of the two-spinor expression associated with $T^{00}(x)$. The latter quantity appears to be identical with the density of energy carried by the fields. We will make explicit use of the spin matrices that establish the correspondence between the standard Minkowski tetrads and the canonical spin bases (see Ref. [1]). The $\sigma_a^{AA'}$-pattern of these particular matrices is given by

$$\left(\sigma_0^{AA'}\right) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \left(\sigma_1^{AA'}\right) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$
$$\left(\sigma_2^{AA'}\right) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \qquad \left(\sigma_3^{AA'}\right) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3.13}$$

The energy density of the fields is defined on τ by

$$\begin{aligned} T^{00}(x) = T_{00}(x) &= \sigma_0^{AA'}\sigma_0^{BB'}T_{AA'BB'}(x) \\ &= \frac{1}{2\pi}\sigma_0^{AA'}\sigma_0^{BB'}\phi_{AB}(x)\bar{\phi}_{A'B'}(x). \end{aligned} \tag{3.14}$$

Using $\left(\sigma_0^{AA'}\right)$ as given in (3.13), we readily obtain the expression

$$T_{00}(x) = \frac{1}{4\pi}\left[|\phi_{00}(x)|^2 + |\phi_{11}(x)|^2 + 2|\phi_{01}(x)|^2\right], \tag{3.15}$$

which clearly is positive-definite on τ. To see that (3.15) is identical with the elementary expression for the electromagnetic energy density, it is convenient to assume that $\tau \cong \mathbb{T} \times \mathbb{V}$, with $\mathbb{T}$ and $\mathbb{V}$ being, respectively, a (one-dimensional) subspace of $\mathbb{R}$ and a (bounded) three-dimensional submanifold of $\mathbb{R}^3$. At this stage, the closure of $\mathbb{V}$ is still taken to be compact so as to make it correspond to a suitable volume in $\mathbb{R}^3$ with a smooth (two-surface) boundary $\partial\mathbb{V}$. We can then achieve at once the relevant relation by using the Penrose complexified three-vector (with $x^0 = t$)

$$\boldsymbol{C}(t, \boldsymbol{x}) = \boldsymbol{E}(t, \boldsymbol{x}) - i\boldsymbol{B}(t, \boldsymbol{x}), \quad t \in \mathbb{T}, \quad \boldsymbol{x} \in \mathbb{V}, \tag{3.16}$$

where $\boldsymbol{E}(t, \boldsymbol{x})$ and $\boldsymbol{B}(t, \boldsymbol{x})$ are the electric and magnetic fields, respectively [1]. It should be observed that, as regards the use of (3.16), the components entering into $\mathbb{RM} \cong \mathbb{R} \times \mathbb{R}^3$ have to be disconnected in such a way that the $\mathbb{R}^3$-component now carries the usual metric signature $(+++)$ instead of $(---)$. The explicit relationships between the (three) independent components of $\phi_{AB}(x)$ and those of $C(t, x)$, are thus provided by

$$\left.\begin{aligned} \phi_{00}(x) &= \tfrac{1}{2}[C_1(t,x) - iC_2(t,x)] \\ \phi_{01}(x) &= -\tfrac{1}{2}C_3(t,x) = \phi_{10}(x) \\ \phi_{11}(x) &= -\tfrac{1}{2}[C_1(t,x) + iC_2(t,x)] \end{aligned}\right\} . \tag{3.17}$$

Now, combining (3.16) with (3.17) and recalling (3.15), we arrive at the elementary energy-density expression

$$T_{00}(t,x) = \frac{1}{8\pi}\left[E^2(t,x) + B^2(t,x)\right] . \tag{3.18}$$

Hence the electromagnetic energy on $\mathbb{T}$, stored in $\mathbb{V}$, is given by

$$\mathcal{E}(t) = \frac{1}{4\pi}\int_{\mathbb{V}} \left[|\phi_{00}(x)|^2 + |\phi_{11}(x)|^2 + 2|\phi_{01}(x)|^2\right] d^3x\,, \tag{3.19}$$

which involves the three-volume form on $\mathbb{V}$ (see also Eq. (3.27) below)

$$d^3x = dx^1 \wedge dx^2 \wedge dx^3 \,. \tag{3.20}$$

We must mention that the complex conjugate of the three-vector (3.16) was also used by Barut [8] to show in a particular way that the Maxwell fields can be viewed as being "composed" of neutrino fields.

Clearly, the energy integral (3.19) is not covariant. To write down a manifestly covariant expression, we have to suppose that $\Sigma \subset \bar{\tau}$ (see (2.31)). The relevant integral for the electromagnetic energy is thus given by the functional

$$\mathcal{E}[\Sigma] = \frac{1}{2\pi}\int_{\Sigma} \sigma_0^{BB'}\phi_B^A(x)\bar{\phi}_{B'}^{A'}(x)d^3x_{AA'}\,, \tag{3.21}$$

where $d^3x_{AA'}$ is a spinor element of three-hypersurface area of Σ (see also (2.22)). The components occurring in the integrand of (3.21) can be written out explicitly, to yield the expression

$$\begin{aligned}\mathcal{E}[\Sigma] = \frac{\sqrt{2}}{4\pi}\int_{\Sigma} \Big\{ &\left[|\phi_{01}(x)|^2 + |\phi_{11}(x)|^2\right] d^3x_{00'} \\ &+ \left[|\phi_{00}(x)|^2 + |\phi_{01}(x)|^2\right] d^3x_{11'} \\ &- 2\,\mathrm{Re}\left[\left(\phi_{00}(x)\bar{\phi}_{0'1'}(x) + \phi_{01}(x)\bar{\phi}_{1'1'}(x)\right) d^3x_{10'}\right]\Big\}\,, \end{aligned} \tag{3.22}$$

which accordingly is reduced to (3.19) when we take $\Sigma \cong \mathbb{V}$ as $x^0 \doteq$ const. To establish this, we carry out the explicit computation

$$\begin{aligned}\mathcal{E}[\Sigma] =& \frac{1}{12\pi}\int_\Sigma \sigma_0^{FF'}\phi_F^E(x)\bar{\phi}_{F'}^{E'}(x)e_{abcd}\sigma^a_{EE'}dx^b \wedge dx^c \wedge dx^d \\ =& \frac{1}{2\pi}\int_\Sigma \sigma_0^{BB'}\phi_B^A(x)\bar{\phi}_{B'}^{A'}(x)\big(\sigma^0_{AA'}dx^1 \wedge dx^2 \wedge dx^3 \\ & - \sigma^1_{AA'}dx^0 \wedge dx^2 \wedge dx^3 + \sigma^2_{AA'}dx^0 \wedge dx^1 \wedge dx^3 \\ & - \sigma^3_{AA'}dx^0 \wedge dx^1 \wedge dx^2\big)\,, \end{aligned} \tag{3.23}$$

whence, on x^0 = constant, Eq. (3.21) turns out to be reduced to

$$\mathcal{E}(t) = \int_{\mathbb{V}} T_{00}(x)dx^1 \wedge dx^2 \wedge dx^3\,. \tag{3.24}$$

In particular, we will see in subsection 3.3 that the integral involved in the expression (3.21) can be immediately identified with the time-component of the energy-momentum four-vector of the theory.

3.3 Covariant kinematical integrals

The manifestly covariant expression for the electromagnetic energy-momentum four-vector is written as

$$p_{AA'}[\Sigma] = \frac{1}{2\pi}\int_\Sigma \phi_A^B(x)\bar{\phi}_{A'}^{B'}(x)d^3x_{BB'}\,. \tag{3.25}$$

Whereas the time-component $p_0[\Sigma] = \sigma_0^{AA'}p_{AA'}[\Sigma] = p^0[\Sigma]$ is immediately seen to be identical with (3.21), the spatial components $p^k[\Sigma] = -p_k[\Sigma]$, are expressed as

$$p^k[\Sigma] = \frac{1}{2\pi}\int_\Sigma \sigma^k_{AA'}\phi^{AB}(x)\bar{\phi}^{A'B'}(x)d^3x_{BB'}\,. \tag{3.26}$$

Whenever Σ is "identified" with $\mathbb{V}$ (see (3.24) above), the covariance of the integrand of (3.26) is lost. Under this circumstance, $p^k[\Sigma]$ appear to be given by

$$p^k(t) = \int_{\mathbb{V}} T_0^k(x)d^3x\,, \tag{3.27}$$

which is the conventional (non-covariant) defining expression for the linear momentum carried by the fields (see, for instance, Ref. [8]).

We shall now proceed to the derivation of the two-spinor expressions for the densities $T^{0k}(x)$. We have, in effect,

$$\begin{aligned} T_0^k(x) &= \frac{1}{2\pi}\sigma_0^{AA'}\sigma^k_{BB'}\phi^B_A(x)\bar{\phi}^{B'}_{A'}(x) \\ &= \frac{\sqrt{2}}{4\pi}\sigma^k_{BB'}\left[\phi^B_0(x)\bar{\phi}^{B'}_{0'}(x) + \phi^B_1(x)\bar{\phi}^{B'}_{1'}(x)\right], \end{aligned} \tag{3.28}$$

such that, rearranging indices and making use of (3.13), we obtain

$$T_{01}(x) = -T_0^1(x) = \frac{1}{2\pi}\,\mathrm{Re}\left\{\phi_{01}(x)\left[\bar{\phi}_{0'0'}(x) + \bar{\phi}_{1'1'}(x)\right]\right\}, \tag{3.29a}$$

$$T_{02}(x) = -T_0^2(x) = \frac{1}{2\pi}\,\mathrm{Im}\left\{\bar{\phi}_{0'1'}(x)\left[\phi_{11}(x) - \phi_{00}(x)\right]\right\}, \tag{3.29b}$$

and

$$T_{03}(x) = -T_0^3(x) = \frac{1}{4\pi}\left[|\phi_{00}(x)|^2 - |\phi_{11}(x)|^2\right]. \tag{3.29c}$$

The expressions for $T_0^k(x)$ as given explicitly above can indeed be looked upon as the standard $\mathbb{R}^3$-components $\{P^k(t,x)\}$ of the elementary Poynting three-vector

$$\boldsymbol{P}(t,\boldsymbol{x}) = \frac{1}{4\pi}\boldsymbol{E}(t,\boldsymbol{x}) \times \boldsymbol{B}(t,\boldsymbol{x}). \tag{3.30}$$

This statement can be readily established by replacing (3.17) into (3.29). Accordingly, we have the following integrals for the electromagnetic power crossing $\partial\mathbb{V}$ at some $t \in \mathbb{T}$

$$\Pi(t) = \int_{\partial\mathbb{V}} P^k(t,x)dS_k = \int_{\mathbb{V}} \nabla_k P^k(t,x)d^3\boldsymbol{x}, \tag{3.31}$$

where dS_k is an element of two-surface area of $\partial\mathbb{V}$ at $\boldsymbol{x}$.

To build up the covariant angular-momentum structures, we have to make use of the explicit expression (3.25) together with the standard defining relation

$$M_{AA'BB'}[\Sigma] = 2\int_{\Sigma} x_{[AA'}T_{BB']CC'}(x)d^3x^{CC'}, \tag{3.32}$$

where the square brackets denote skew-symmetrization over the index-pairs. We thus have the formal integral

$$M_{AA'BB'}[\Sigma] = \int_{\Sigma}\left[\varepsilon_{A'B'}\mu_{ABCC'}(x) + \varepsilon_{AB}\bar{\mu}_{A'B'CC'}(x)\right]d^3x^{CC'}, \tag{3.33}$$

with the conjugate angular-momentum densities being explicitly given by

$$\mu_{ABCC'}(x) = \frac{1}{2\pi} x_{M'(A}\phi_{B)C}(x)\bar{\phi}^{M'}_{C'}(x) = \mu_{(AB)CC'}(x), \quad (3.34a)$$

and

$$\bar{\mu}_{A'B'CC'}(x) = \frac{1}{2\pi} x_{M(A'}\bar{\phi}_{B')C'}(x)\phi^{M}_{C}(x) = \bar{\mu}_{(A'B')CC'}(x). \quad (3.34b)$$

Evidently, the symmetry property of these μ-densities can be stated as the ε-spinor trace relations

$$\mu^{A}_{ACC'}(x) = 0 = \bar{\mu}^{A'}_{A'CC'}(x), \quad (3.35)$$

which actually appear to be equivalent to the bivector property

$$M_{AA'}{}^{AA'}[\Sigma] = 0. \quad (3.36)$$

4. Concluding remarks

In respect of our derivation of the explicit expression (3.6), it should be emphasized that the procedure adopted here was to work out an adequate defining two-spinor expression for the electromagnetic energy-momentum tensor rather than translating the standard classical definition into spinorial terms. Thus, the conventional world-tensor expression

$$T_{ab}(x) = \frac{1}{4\pi}\left[\frac{1}{4}g_{ab}F_{cd}(x)F^{cd}(x) - F_{ac}(x)F^{dc}(x)g_{bd}\right], \quad (4.1)$$

was not taken for granted from the beginning. Instead, this expression actually arose at an intermediate stage of the calculations yielding the corresponding two-spinor formula. One remarkable feature of Penrose's expression is that it does not depend upon whether we take effectively the Lorentz gauge condition into account upon expressing the relevant Lagrangian density. If we had dropped the symmetrization round brackets from (2.14) we would have, in effect, arrived at the same two-spinor expression for $T_{ab}(x)$. Evidently, the same observation holds also for (2.23) and (2.27). Actually, as (3.6) stands, the tensor appears to be particularly invariant under the duality rotations

$$\phi_{AB}(x) \longmapsto e^{-i\theta}\phi_{AB}(x), \quad \bar{\phi}_{A'B'}(x) \longmapsto e^{i\theta}\bar{\phi}_{A'B'}(x), \quad (4.2)$$

with $\theta \in \mathbb{R}$. Hence, as regards the relevant computations, either of the expressions (3.1) and (3.2) seems to be unambiguous, independently of whether the phase factor involved in the above transformations is held fixed. The positiveness of the numerical factor entering explicitly into (3.6) ensures the validity of the statement (3.12), which accordingly tells us that the velocity of the energy flow carried by the Poynting vector cannot exceed the velocity of light. When $j^a(x) = 0$, the divergence-freeness property (3.9) implies the conservation of the energy-momentum four-vector as given covariantly in (3.25). In this case, the relevant defining expression does not depend upon the choice of (space-like) hypersurfaces, the corresponding functional statement being [9]

$$\frac{\delta}{\delta \Sigma} p_{AA'}[\Sigma] = 0\,, \tag{4.3a}$$

which is indeed equivalent to

$$\nabla^{AA'}\left[\phi_{AB}(x)\bar{\phi}_{A'B'}(x)\right] = 0\,. \tag{4.3b}$$

In a similar way, the divergencelessness of $j_a(x)$ amounts to stating

$$\frac{\delta}{\delta \Sigma} Q[\Sigma] = 0\,. \tag{4.3c}$$

That the relations (2.4) are equivalent to the Bianchi identities (2.11) can be easily established by computing

$$\begin{aligned} 2\nabla^{AB'}\phi_A^B(x) &= 2\varepsilon^{BC}\nabla^{AB'}\nabla_{A'(A}\Phi_{C)}^{A'}(x) \\ &= \nabla^{AB'}\phi_A^B(x) + \nabla^{A'B}\bar{\phi}_{A'}^{B'}(x)\,. \end{aligned} \tag{4.4}$$

The two-spinor electromagnetic Lagrangian densities used here afford us one way of setting up the simplest form of Maxwell's theory. In particular, it is clearly seen that the sources for the fields appear to interact with the potential, not with the fields. This feature might indeed be required from the outset as a necessary condition for the theory to remain linear. According to this fact, the linearity of the theory is particularly exhibited by the wave equation involving $\Phi_{AA'}(x)$. In relation to $\mathcal{L}_{\text{free}}$ itself, we should notice that it can be expressed as

$$\begin{aligned} \mathcal{L}_{\text{free}} &= \frac{1}{8\pi}\nabla_a\left[\Phi^b(x)\nabla^a\Phi_b(x) - \Phi^b(x)\nabla_b\Phi^a(x)\right] \\ &= \frac{1}{4\pi}\,\mathrm{Re}\left\{\nabla_{AA'}\left[\Phi_B^{A'}(x)\phi^{AB}(x)\right]\right\}. \end{aligned} \tag{4.5}$$

In fact, the crucial point here is that the (real) vector

$$W^{AA'}(x) = \Phi_B^{A'}(x)\phi^{AB}(x) + \Phi_{B'}^{A}(x)\bar{\phi}^{A'B'}(x) \tag{4.6}$$

is not gauge-invariant, but its divergence is. In effect, whenever $\Phi_a(x)$ is transformed into $\Phi_a(x) + \nabla_a \Upsilon(x)$, with $\Upsilon(x)$ being a well-behaved scalar function on τ, vector $W_a(x)$ undergoes the transformation

$$W_a(x) \longmapsto W_a(x) + F_{ab}(x)\nabla^b \Upsilon(x)\,, \tag{4.7}$$

whence $\nabla_a W^a(x)$ remains invariant.

It should be observed that the assumption upon which the calculations yielding (3.24) are based, destroys the covariance of the expression (3.21). This seems to be due to the splitting of the region $\tau \subset \mathbb{RM}$ into disconnected parts lying in $\mathbb{R}$ and $\mathbb{R}^3$, which also yields the elementary charge integral $\int_{\mathbb{V}} j_0(t, x) d^3x$. Finally, we must stress that the reason for the compactness requirement imposed on $\bar{\tau}$ is the fact that our basic dynamical statements have to be set upon a finite four-volume contained in $\mathbb{RM}$.

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