AN INFINITE LIE ALGEBRA ASSOCIATED WITH THE QUANTUM COULOMB FIELD*

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The theory of the quantum Coulomb field associates with each Lorentz frame, *i.e.*, with each unit, future oriented time-like vector u, the operator of the number of transveral infrared photons $N(u)$ and the phase $S(u)$ which is the coordinate canonically conjugated with the total charge Q: $[Q, S(u)] = ie$, e being the elementary charge. It is shown that the operators $N(u)$, $(Q/e)S(u)$ and Q^2 form an infinite Lie algebra. One can conclude from this algebra that $\Delta N(u) = (4/\pi)Q^2$, where Δ is the Laplace operator in the Lobachevsky space of four-velocities u, thus relating the total charge Q with the number of infrared photons.

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A charged particle, when scattered, produces an infrared electromagnetic field which behaves at the spatial infinity like the inverse of distance. Gervais and Zwanziger [1] gave a clean way to separate the infrared field from the rest : performing the rescaling $\lim_{\lambda\to\infty} \lambda A_\mu(\lambda x)$ one obtains an electromagnetic potential which is homogeneous of degree -1 and represents thus a pure infrared field free from high frequency contaminations. Now, from the physical point of view, the infrared electromagnetic field is a free dynamical system: scattered charges produce infrared fields but infrared fields do not scatter charges; there is no back reaction. This allows to treat the infrared field as a free field; such a treatment is physically rigorous. This

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means also that the infrared field can be quantized in the usual way [2]. The resulting field theory is remarkable as it contains the quantum analog of the classical Coulomb field and allows to make meaningful and meaningfully arrived at statements on the magnitude of the fine structure constant $e^2/\hbar c$. In particular the value $e^2/\hbar c = \pi$ is seen to be critical as it separates two kinematically distinct regimes of the quantum Coulomb field [3].

In the quantum theory of infrared fields the classical Coulomb field

$$A_0 = \frac{Q}{r}, \quad \boldsymbol{A} = 0, \tag{1}$$

is a global solution of the classical equation of motion, to be used, for the sake of completeness, in the procedure of quantization. The theory is summarized in [2] on page 364 in the form of all relevant canonical commutation relations, written in a fixed Lorentz reference frame which we imagine as being at rest. We repeat this summary here for the reader's convenience:

$$[Q, S_0] = ie, \quad [Q, c_{lm}] = 0, \quad [S_0, c_{lm}] = 0, \tag{2}$$

$$[c_{lm}, c^{\dagger}_{l'm'}] = 4\pi e^2 \delta_{ll'} \delta_{mm'} . \tag{3}$$

Here Q is the total charge, e is the elementary charge, S_0 is the phase, a coordinate canonically conjugated with the operator Q/e; $c^{\dagger}_{lm}$ and c_{lm} are, respectively, creation and annihilation operators for transversal infrared photons; they are numbered by the numbers $l = 1, 2, \ldots$ and $m = -l, -l+1, \ldots, l$ known from the theory of angular momentum.

The vacuum state $|0\rangle$ is defined by the relations (Ref. [2], p. 364)

$$c_{lm}|0\rangle = 0, \quad \langle 0|c^{\dagger}_{lm} = 0, \quad Q|0\rangle = 0, \quad \langle 0|Q = 0, \tag{4}$$

which are standard for transversal photons but nonstandard for the total charge Q. The reason for this is that the total charge Q is a Lorentz invariant quantity while the phase S_0 is not; hence any definition of the vacuum state involving the phase S_0 would necessarily violate the Lorentz invariance.

The operator

$$N = \frac{1}{4\pi e^2} \sum_{lm} c^{\dagger}_{lm} c_{lm} \tag{5}$$

gives the number of transversal infrared photons; its spectrum consists of nonnegative integers, $N = 0, 1, 2, \ldots$.

Now, the whole theory is Lorentz invariant as it may be represented as a quantum field theory of a massless scalar field "living" in the three-dimensional hyperboloid of unit space-like four-vectors. Therefore, all the previous relations may be written in exactly the same form in another Lorentz reference frame :

$$[Q, S_0'] = ie, \quad N' = \frac{1}{4\pi e^2} \sum_{lm} c_{lm}'^{\dagger} c_{lm}', \tag{6}$$

etc.

Prime over a quantity denotes the same quantity in the moving reference frame which, without loss of generality, will be assumed to move in the z-direction. There is no prime over Q since $Q = Q'$ is a Lorentz scalar.

We can formulate now the main goal of this paper: to relate primed quantities N', S_0' with the nonprimed ones N, S_0. It will be seen that the appropriate relation has the form of an infinite Lie algebra.

In what follows it will be convenient to simplify notation and to replace the pair of indices (lm) by a single index α. Thus the quantum mechanics of charge is simplified to the following form

$$[Q, S_0] = ie, \quad [Q, c_\alpha] = 0, \quad [S_0, c_\alpha] = 0, \tag{7}$$

$$[c_\alpha, c_\beta^\dagger] = 4\pi e^2 \delta_{\alpha\beta}, \tag{8}$$

and

$$c_\alpha|0\rangle = 0, \quad \langle 0|c_\alpha^\dagger = 0, \quad Q|0\rangle = 0, \quad \langle 0|Q = 0. \tag{9}$$

We shall prove two lemmas which together give the Lorentz transformation of the amplitudes c_α and S_0.

Lemma 1.

The amplitude c_α' is a linear combination of the amplitudes c_α and Q *but does not contain the phase* S_0

$$c_\alpha' = \sum_\beta A_{\alpha\beta} c_\beta + B_\alpha Q. \tag{10}$$

Moreover, the matrix $A_{\alpha\beta}$ is unitary.

Indeed, since $c_\alpha'|0\rangle = 0$, c_α' must be a linear combination of those unprimed amplitudes which annihilate the vacuum state, *i.e.*, c_α and Q. Unitarity of the matrix $A_{\alpha\beta}$ follows from the fact that the Lorentz transformation (10) must preserve the canonical commutation relations

$$[c_\alpha', c_\beta'^\dagger] = 4\pi e^2 \delta_{\alpha\beta}. \tag{11}$$

Lemma 2.

The amplitude S_0' is a linear combination of the amplitudes S_0, c_α and $c_\alpha^\dagger$ but *does not contain the charge* Q

$$S_0' = S_0 - \frac{1}{4\pi ie}\sum_{\alpha\beta}\left(\bar{B}_\alpha A_{\alpha\beta}c_\beta - B_\alpha \bar{A}_{\alpha\beta}c_\beta^\dagger\right). \tag{12}$$

It is easy to check that this expression is indeed consistent with the Lorentz invariance of the commutation relations which involve the phase S_0: calculating $[Q, S_0']$, $[c_\alpha', S_0']$ and $[c_\alpha'^\dagger, S_0']$ and using (10) one finds identity in each case. This does not show that the term proportional to the charge Q on the right hand side is absent. Using, however, the explicit expression of S_0' through the solutions of the wave equation on the three-dimensional hyperboloid of space-like unit four-vectors given in Ref. [2], one finds indeed that the phase S_0' does not contain the total charge Q.

To summarize, we have the following expressions for the Lorentz transformation of all the amplitudes :

$$c_\alpha' = \sum_\beta A_{\alpha\beta}c_\beta + B_\alpha Q\,, \tag{13}$$

$$c_\alpha'^\dagger = \sum_\beta \bar{A}_{\alpha\beta}c_\beta^\dagger + \bar{B}_\alpha Q\,, \tag{14}$$

$$S_0' = S_0 - \frac{1}{4\pi ie}\sum_{\alpha\beta}\left(\bar{B}_\alpha A_{\alpha\beta}c_\beta - B_\alpha \bar{A}_{\alpha\beta}c_\beta^\dagger\right), \tag{15}$$

where the matrix $A_{\alpha\beta}$ is unitary. We have

$$N' = \frac{1}{4\pi e^2}\sum_\alpha c_\alpha'^\dagger c_\alpha'\,. \tag{16}$$

Putting into (16) c_α' and using unitarity of the matrix $A_{\alpha\beta}$ one obtains

$$N' = N + \frac{Q}{4\pi e^2}\sum_{\alpha\beta}\left(\bar{B}_\alpha A_{\alpha\beta}c_\beta + B_\alpha \bar{A}_{\alpha\beta}c_\beta^\dagger\right) + \frac{Q^2}{4\pi e^2}\sum_\alpha |B_\alpha|^2\,. \tag{17}$$

It is shown in Ref. [2] that

$$\sum_\alpha |B_\alpha|^2 = 8e^2(\lambda\coth\lambda - 1)\,, \tag{18}$$

where λ is the hyperbolic angle between the time axis of the moving frame and that of the rest frame.

Let us take the commutator $[N, S_0']$. Using (12) and the obvious relations $[N, c_\alpha^\dagger] = c_\alpha^\dagger$, $[N, c_\alpha] = -c_\alpha$, one finds

$$[N, S_0'] = \frac{1}{4\pi i e} \sum_{\alpha\beta} \left(\bar{B}_\alpha A_{\alpha\beta} c_\beta + B_\alpha \bar{A}_{\alpha\beta} c_\beta^\dagger\right). \tag{19}$$

Hence

$$N' = N + i\frac{Q}{e}[N, S_0'] + 2\frac{Q^2}{\pi}(\lambda \coth \lambda - 1). \tag{20}$$

Since the total charge Q commutes with the operator N, the last equation (20) can be written in the form

$$[N, \frac{Q}{e} S_0'] = i\big(N - N' + 2\frac{Q^2}{\pi}(\lambda \coth \lambda - 1)\big). \tag{21}$$

This equation has already the form characteristic for the Lie algebras. To close the algebra we need further $[N, N']$ and $[(Q/e)S_0, (Q/e)S_0']$.

One has immediately from the previous formula (21) and canonical commutation relations

$$[N, N'] = i\frac{Q}{e}(S_0' - S_0), \quad [\frac{Q}{e} S_0, \frac{Q}{e} S_0'] = i\frac{Q}{e}(S_0 - S_0'). \tag{22}$$

An objection may be raised that the operator $(Q/e)S_0$ is "ill defined" as it can be represented in the rest frame as the differential operator $(i\frac{\partial}{\partial S_0} S_0)$, S_0 being an angular variable with the period 2π. However, one sees that the right hand sides of all the commutators contain only *differences* of two phases which are perfectly well defined. Such differences may be also introduced in the left hand sides; for example, the commutator $[N, (Q/e)S_0']$ can be written as $[N, (Q/e)(S_0' - S_0)]$ since $[N, (Q/e)S_0] = 0$. The same can be done for all other commutators, which shows that all the commutators written above have perfectly well defined meaning.

It will be useful to change notation. We introduce the unit, time-like, future oriented vector u which indicates the time axis of the rest frame and a similar vector v which indicates the time axis of the moving frame. All such vectors together form the Lobachevsky space of four-velocities. Using this notation we have

$$\big[N(u), N(v)\big] = i\frac{Q}{e}\big(- S(u) + S(v)\big), \tag{23}$$

$$\left[N(u), \frac{Q}{e}S(v)\right] = i\left(N(u) - N(v) + 2\frac{Q^2}{\pi}(uv)\right), \tag{24}$$

$$\left[\frac{Q}{e}S(u), \frac{Q}{e}S(v)\right] = i\frac{Q}{e}\left(S(u) - S(v)\right), \tag{25}$$

where $(uv) = \lambda \coth \lambda - 1$, λ being the hyperbolic angle between u and v: $g_{\mu\nu}u^{\mu}v^{\nu} = \cosh \lambda$. Commutators involving Q^2 are omitted because they are completely obvious.

The Lobachevsky space of four-velocities is a three-dimensional locally Euclidean space of constant negative curvature with the metric induced by the metric of the four-dimensional space-time in which it is immersed. One can execute in this space all invariant operations of tensor analysis. Let us apply the Laplace operator Δ_v (v indicates that it is to be applied at the point v) to both sides of the commutator

$$\left[N(u), \frac{Q}{e}S(v)\right] = i\left(N(u) - N(v) + 2\frac{Q^2}{\pi}(uv)\right). \tag{26}$$

One finds $\Delta_v(uv) = 2$ by direct calculation. Thus

$$\left[N(u), \frac{Q}{e}\Delta_v S(v)\right] = i\left(- \Delta_v N(v) + 4\frac{Q^2}{\pi}\right). \tag{27}$$

But $\Delta_v S(v) = 0$, as can be shown again from the explicit expression for $S(v)$ given in Ref. [2]. Hence, dropping the index v which is not necessary anymore we have

$$\Delta N = 4\frac{Q^2}{\pi}. \tag{28}$$

This equation is remarkable as it connects the total charge Q with the operator N of the number of transversal infrared photons which has an absolute scale, its spectrum consisting of nonnegative integers.

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