CLIFFORD ALGEBRAS AND ALGEBRAIC COMPOSITENESS OF FUNDAMENTAL FERMIONS

W. Królikowski

Institute of Theoretical Physics, Warsaw University
Hoża 69, 00-681 Warszawa, Poland

Dedicated to Jan Rzewuski on the happy occasion of his 75th birthday

(*Received September 22, 1992*)

We demonstrate in some detail, how an idea of leptons and quarks composed of algebraic partons (defined by a sequence of Clifford algebras) can justify the existence of three and only three families of these fundamental fermions. In this argument, the theory of relativity, the probability interpretation of quantum mechanics and the Pauli exclusion principle, all extended to the algebraic partons, play a crucial role. The Lorentz group turns out to be realizable intrinsically in two different ways, the algebraic partons corresponding to the new way. We describe also a semiempirical mass spectral formula for charged leptons composed of algebraic partons. With the use of experimental m_e and m_μ it gives $m_\tau = 1783.47$ MeV or $m_\tau = 1776.80$ MeV, the second option in excellent agreement with new measurements of m_τ.

PACS numbers: 12.90. +b, 12.50. Ch, 14.40. Jj

1. Introduction

According to the general feeling, the most puzzling feature of today's particle physics is perhaps the phenomenon of three families of leptons:

$$\begin{array}{llll} \nu_e & \nu_\mu & \nu_\tau(?) & (\text{charge} \quad 0) \\ e^- & \mu^- & \tau^- & (\text{charge} \ -1) \end{array} \tag{1}$$

and quarks:

$$\begin{array}{llll} u & c & t(?) & (\text{charge} \quad 2/3) \\ d & s & b & (\text{charge} \ -1/3) \end{array} \tag{2}$$

differing apparently by nothing but their masses. Among them, the tauonic neutrino ν_τ and top quark t are not yet observed directly, though indirect evidence leaves practically no doubt as to their existence. In particular, the recent CERN measurements of total decay width for Z^0 gauge boson manifesting itself as a resonance at about 91 GeV of CM energy in the process

$$e^+ e^- \longrightarrow Z^0 \longrightarrow \text{anything}\,, \tag{3}$$

have shown that the number of different neutrino versions lighter than $\frac{1}{2}m_Z \simeq 46$ GeV is just three. Moreover, this result has given us a strong argument that the number of all lepton and quark families is equal to three. In fact, *(i)* it is rather natural to imagine that (Dirac) neutrinos are very light (or massless) and *(ii)* the numbers of different versions of neutrinos, charged leptons, up quarks and down quarks are required to be equal in order to ensure the internal consistency of the standard model of electroweak and strong interactions (cancellation of the fatal chiral anomalies).

In this presentation, we are going to demonstrate in some detail that there are three different, physically distinguished versions of the Dirac equation

$$[\Gamma \cdot (p - gA) - M]\,\psi = 0\,, \tag{4}$$

where

$$\{\Gamma^\mu,\, \Gamma^\nu\} = 2g^{\mu\nu}\,. \tag{5}$$

Here, $g\Gamma \cdot A$ symbolizes the standard-model coupling, identical for all three versions, while the mass operator M may depend on the version. So, we shall be tempted to connect these versions with the three experimental families of leptons and quarks.

Our argument will express an idea of algebraic compositness of fundamental fermions that accepts an act of algebraic abstraction from the familiar notion of spatial compositness (so useful, for instance, in the case of pseudoscalar and vector mesons built up of quark-antiquark pairs moving in the physical space).

2. An example of spatial compositness

For an illustration, let us consider the Duffin–Kemmer–Petiau equation describing a particle with spin $0 \oplus 1$. In the free case, it can be written in the form

$$\left[\tfrac{1}{2}(\gamma_1 + \gamma_2) \cdot P - M\right]\psi(X) = 0\,, \tag{6}$$

where γ_1^μ and γ_2^μ are two sets of commuting Dirac matrices,

$$\left\{\gamma_i^\mu, \gamma_i^\nu\right\} = 2g^{\mu\nu}\,, \quad [\gamma_1^\mu, \gamma_2^\nu] = 0\,, \tag{7}$$

so that

$$\gamma_1^\mu = \gamma^\mu \otimes \mathbf{1}\ , \ \gamma_2^\mu = \mathbf{1} \otimes \gamma^\mu \text{ or } \mathbf{1} \otimes \gamma^{\mu C} \tag{8}$$

with γ^μ and $\mathbf{1}$ being the usual Dirac 4×4 matrices and $\gamma^{\mu C} = C^{-1}\gamma^\mu C = -\gamma^{\mu T}$. Here, $\psi = (\psi_{\alpha_1\alpha_2})$ displays an algebraic structure expressed in terms of two Dirac bispinor indices α_1 and α_2.

In the case of pseudoscalar and vector mesons (then $\gamma_2^\mu = \mathbf{1}\otimes\gamma^{\mu C}$), Eq. (6) may be considered as a wave equation for the motion of meson centre of mass, in the approximation where the meson internal structure is neglected. In fact, Eq.(6) can be readily derived as a point-like limiting form of the following two-body wave equation for a quark-antiquark pair [1]:

$$\left[\gamma_1 \cdot (\tfrac{1}{2}P + p) + \gamma_2 \cdot (\tfrac{1}{2}P - p) - m_1 - m_2 - S(x)\right] \psi(X,x) = 0\,, \tag{9}$$

where (for simplicity) masses are assumed equal: $m_1 = m_2$ (what, for instance, is the case for a pair of a quark and an antiquark of the same sort). Here, the internal interaction $S(x)$ can be related to the more familiar internal interaction $I(x)$ appearing in the Bethe–Salpeter equation [2]

$$\left\{\left[\gamma_1 \cdot (\tfrac{1}{2}P + p) - m_1\right] \left[\gamma_2 \cdot (\tfrac{1}{2}P - p) - m_2\right] - I(x)\right\} \psi(X,x) = 0\,. \tag{10}$$

The relation is

$$S(x)\psi(X,x) \equiv \left[\frac{1}{\gamma_1 \cdot \left(\frac{1}{2}P + p\right) - m_1} + \frac{1}{\gamma_2 \cdot \left(\frac{1}{2}P - p\right) - m_2}\right] I(x)\, \psi(X,x) \tag{11}$$

with $I(x)$ being an integral operator acting in the internal four-dimensional space,

$$I(x)\psi(X,x) \equiv \int d^4x'\, I(x,x')\, \psi(X,x')\,. \tag{12}$$

As it was shown many years ago by Jan Rzewuski and myself [3], the Bethe–Salpeter equation (10) is equivalent in the case of equal times ($x^0 = 0$, $X^0 = t$) to the one-time two-body wave equation having the conventional form of the state equation,

$$\left\{i\tfrac{\partial}{\partial t} - \gamma_1^0\left[\vec{\gamma_1}\cdot\left(\tfrac{1}{2}\vec{P}+\vec{p}\right) + m_1\right] - \gamma_2^0\left[\vec{\gamma_2}\cdot\left(\tfrac{1}{2}\vec{P}-\vec{p}\right) + m_2\right] - V(\vec{x})\right\}\psi(\vec{X},\vec{x},t) = 0. \tag{13}$$

Here, $V(\vec{x})$ is an internal interaction deducible perturbatively step by step from the Bethe-Salpeter interaction $I(x)$, and is given by an integral operator acting in the internal three-dimensional space,

$$V(\vec{x})\, \psi(\vec{X},\vec{x},t) \equiv \int d^3\vec{x'}\, V(\vec{x},\vec{x'})\psi(\vec{X},\vec{x'},t)\,. \tag{14}$$

In the case of an instantaneous interaction Eq. (13) becomes the familiar Salpeter equation [4].

Thus, we can see that the one-body Duffin–Kemmer–Petiau equation (6), where $[\gamma_1^\mu, \gamma_2^\nu] = 0$ is assumed, may be derived via the two-body Bethe–Salpeter equation (as its point-like limiting form) from the conventional quantum field theory. So, in this case, the algebraic structure $\psi = (\psi_{\alpha_1\alpha_2})$ coexists with the spatial structure and, therefore, one can speak of the *spatial compositness*, a notion so familiar from the history of matter research.

3. An example of algebraic compositness

From the above argument it is readily seen that, if $\{\gamma_1^\mu, \gamma_2^\nu\} = 0$ were assumed instead of $[\gamma_1^\mu, \gamma_2^\nu] = 0$, Eq. (6) would not be derivable from the conventional quantum field theory via a two-body wave equation (as its point-like limiting form). The proof of this statement goes as follows. In the new case, the counterpart of Eq. (6) (with the convenient coefficient $1/\sqrt{2}$ in place of $1/2$) would read [5]

$$\left[\frac{1}{\sqrt{2}}(\gamma_1^\mu + \gamma_2^\mu) \cdot P - M\right]\psi(X) = 0\,, \tag{15}$$

where

$$\{\gamma_i^\mu, \gamma_j^\nu\} = 2\delta_{ij} g^{\mu\nu}\,, \tag{16}$$

so that

$$\gamma_1^\mu = \gamma^\mu \otimes 1\,,\ \gamma_2^\mu = \gamma^5 \otimes i\gamma^5\gamma^\mu \tag{17}$$

with $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Such an equation might be considered as a point-like limiting form of the two-body wave equation

$$\left[\sqrt{2}\gamma_1 \cdot (\tfrac{1}{2}P + p) + \sqrt{2}\gamma_2 \cdot (\tfrac{1}{2}P - p) - m_1 - m_2 - S(x)\right]\psi(X, x) = 0\,, \tag{18}$$

but the latter, in contrast to Eq. (9), could not be derived from the conventional quantum field theory. This is a consequence of the fact that the particle kinetic-energy operators in the Fock space

$$\gamma_i^0\,(\vec{\gamma}_i \cdot \vec{p}_i + m) \tag{19}$$

all commute, if they are derived from the field kinetic-energy operator

$$\int d^3\vec{x}\, \psi^+(x)\gamma^0(\vec{\gamma}\cdot\vec{p} + m)\psi(x)\,, \tag{20}$$

so, in such a case, all γ_i^μ must commute for different i (at least, when massive particles are considered; if a relativistic interaction with an external scalar field is introduced, also massless particles cannot escape from this conclusion).

Thus, though Eq. (15) (with Eq. (16)) may be investigated for some hypothetical particles, it cannot be considered as a point-like limiting form of a two-body wave equation following from the conventional field theory. So, $\psi = (\psi_{\alpha_1\alpha_2})$ displays an algebraic structure that, now, does not coexist with any spatial structure (at any rate, in the framework of the conventional quantum field theory [6]). This illustrates, therefore, the notion of *algebraic compositness*. In Eq. (15) the Dirac bispinor indices α_1 and α_2 describe *"algebraic partons"*, agents of the idea of this compositness.

Let us observe that the logical relationship between the notions of spatial compositness and algebraic compositness reminds (to some extent) the logical relationship between the notions of orbital angular momentum and spin. In fact, in these cases we have to do with similar acts of algebraic abstraction from some notions of spatial character.

It is interesting to note that due to the Clifford algebra (16) the matrices

$$\Gamma^\mu = \frac{1}{\sqrt{2}}\left(\gamma_1^\mu + \gamma_2^\mu\right) \tag{21}$$

appearing in Eq. (15) satisfy the Dirac algebra (5). This implies that Eq. (15) has the form of the Dirac equation (4) (in the free case). Thus, the hypothetical particles described by Eq. (15), when coupled to the magnetic field, should display (magnetically "visible") spin 1/2 though any of them is a composite of two algebraic partons of spin 1/2. There exists, therefore, another (magnetically "hidden") spin 1/2. It is related to the matrices $\left(\frac{1}{\sqrt{2}}\right)\left(\gamma_1^\mu - \gamma_2^\mu\right)$ also fulfilling the Dirac algebra (5) and anticommuting with the matrices Γ^μ.

Note further that the matrices (21) may be represented in the convenient form

$$\Gamma^\mu = \gamma^\mu \otimes \mathbf{1}\,, \tag{22}$$

if the representation (17) is changed into

$$\gamma_{1,2}^\mu = \frac{1}{\sqrt{2}}\left(\gamma^\mu \otimes \mathbf{1} \pm \gamma^5 \otimes i\gamma^5\gamma^\mu\right)\,. \tag{23}$$

So, Eq. (15) can be rewritten as

$$\left(\gamma_{\alpha_1\beta_1} \cdot P - \delta_{\alpha_1\beta_1} M\right)\ \psi_{\beta_1\alpha_2}(X) = 0\,, \tag{24}$$

where the second Dirac bispinor index α_2 is free. Such an equation is known as the Dirac form [7] of the Kähler equation [8].

4. A sequence of Dirac-type equations

As is not difficult to see, the Dirac algebra (5) admits the remarkable sequence $N = 1, 2, 3, \ldots$ of representations

$$\Gamma^{\mu} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \gamma_i^{\mu} , \tag{25}$$

where the matrices $\gamma_i^{\mu}, i = 1, 2, 3, \ldots, N$, satisfy the sequence $N = 1, 2, 3, \ldots$ of Clifford algebras

$$\{\gamma_i^{\mu}, \gamma_j^{\nu}\} = 2\delta_{ij} g^{\mu\nu} . \tag{26}$$

With the matrices (25), Eq. (4) gives us a sequence $N = 1, 2, 3, \ldots$ of Dirac-type equations [9]. Of course, for $N = 1$ Eq. (4) (with the matrices (25) inserted) is the usual Dirac equation, while for $N = 2$ it is equivalent to the Dirac form of the Kähler equation already discussed in Section 3 (in the free case). For $N = 3, 4, 5, \ldots$ it provides us with new Dirac-type equations.

Except for $N = 1$ the representations (25) are reducible since they may be realized in the convenient form

$$\Gamma^{\mu} = \gamma^{\mu} \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(N-1)\text{times}} \tag{27}$$

with γ^{μ} and $\mathbf{1}$ standing for the usual Dirac 4×4 matrices. It is so, because for any $N > 1$ one can introduce, beside $\Gamma_1^{\mu} \equiv \Gamma^{\mu}$ given in Eq. (25), $N - 1$ other Jacobi-type independent combinations $\Gamma_2^{\mu}, \ldots, \Gamma_N^{\mu}$,

$$\Gamma_2^{\mu} = \frac{1}{\sqrt{2}} \left(\gamma_1^{\mu} - \gamma_2^{\mu}\right) , \Gamma_3^{\mu} = \frac{1}{\sqrt{6}} \left(\gamma_1^{\mu} + \gamma_2^{\mu} - 2\gamma_3^{\mu}\right) , \ldots, \tag{28}$$

such that

$$\{\Gamma_i^{\mu}, \Gamma_j^{\nu}\} = 2\delta_{ij} g^{\mu\nu} \tag{29}$$

(in consequence of Eq. (26)). In particular, for $N = 3$ one may use the representation

$$\Gamma_1^{\mu} = \gamma^{\mu} \otimes \mathbf{1} \otimes \mathbf{1} , \; \Gamma_2^{\mu} = \gamma^5 \otimes i\gamma^5 \gamma^{\mu} \otimes \mathbf{1} , \; \Gamma_3^{\mu} = \gamma^5 \otimes \gamma^5 \otimes \gamma^{\mu} . \tag{30}$$

In the representation (27), the Dirac type equation (4) for any N can be rewritten as

$$[\gamma \cdot (p - g A) - M]_{\alpha_1 \beta_1} \psi_{\beta_1 \alpha_2 \cdots \alpha_N} = 0 , \tag{31}$$

where $M_{\alpha_1\beta_1} = \delta_{\alpha_1\beta_1}M$. Here, $\psi = \left(\psi_{\alpha_1\alpha_2\ldots\alpha_N}\right)$ carries N Dirac bispinor indices α_i, $i = 1, 2, \ldots, N$, of which only the first one is acted on by the Dirac matrices γ^μ and so is coupled to the particle's momentum and to the standard model gauge fields (among others, to the electromagnetic field). The rest of them are free. Thus, only α_1 is "visible", say, in the magnetic field, while $\alpha_2, \ldots, \alpha_N$ are "hidden". In consequence, a particle described by Eq. (4) or (31) can display, say, in the magnetic field only a "visible" spin 1/2, though it possesses also $N-1$ hidden spins 1/2.

Our *first crucial assumption* we are going to make is that the physical Lorentz group of the theory of relativity, if applied to the particle described by Eq. (4) or (31) for any N, is generated both by the particle's visible and hidden degrees of freedom. Then, the Lorentz group generators for any N have the form

$$J^{\mu\nu} = L^{\mu\nu} + \frac{1}{2}\sum_{j=1}^{N} \Sigma_j^{\mu\nu}\,, \tag{32}$$

where $L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$ and

$$\Sigma_j^{\mu\nu} = \frac{i}{2}\left[\Gamma_j^\mu\,, \Gamma_j^\nu\right] = \begin{cases} i\Gamma_j^0\Gamma_j^l & \text{for } \mu = 0\,, \nu = l \\ \varepsilon^{klm}\Sigma_j^m & \text{for } \mu = k\,, \nu = l \end{cases} \tag{33}$$

with $(1/2)\vec{\Sigma}_j = (1/2)\left(\Sigma_j^m\right)$ being the spin 1/2 related to the Dirac bispinor index α_j. Note that Σ_j^m commute for different j (though Γ_j^μ do not). The same is true for the chiralities $\Gamma_j^5 = i\Gamma_j^0\Gamma_j^1\Gamma_j^2\Gamma_j^3$, and Γ_j^5 commute also with all $\vec{\Sigma}_j$. In the chiral representation (where γ^5 is diagonal), the Dirac bispinor indices $\alpha_i = 1, 2, 3, 4$, $i = 1, 2, \ldots, N$, of ψ are defined by four pairs of eigenvalues ± 1 of Σ_i^3 and Γ_i^5

It is not difficult to see that the Lorentz group generators (32) can be rewritten for any N as

$$J^{\mu\nu} = L^{\mu\nu} + \frac{1}{2}\sum_{j=1}^{N} \sigma_j^{\mu\nu}\,, \tag{32'}$$

where

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma_j^\mu, \gamma_j^\nu\right] = \begin{cases} i\,\gamma_j^0\gamma_j^l & \text{for } \mu = 0, \nu = l \\ \varepsilon^{klm}\sigma_j^m & \text{for } \mu = k, \nu = l \end{cases} \tag{33'}$$

with γ_j^μ satisfying the Clifford algebra (26) and so anticommuting for different j. Note that for γ_j^μ commuting for different j (as they appear in the conventional problem of N Dirac particles) the Lorentz-group intrinsic generators have the identical form (33'). Thus, the Lorentz group, though it turns out to be realizable intrinsically in two different ways, does not distinguish by itself between these ways. In contrast, the Poincaré group does (for $N > 1$). In fact, the intrinsic generators for translations, having in the commuting case the form $\frac{1}{2}\sum_{j=1}^{N}\gamma_j^{\mu}(1-\gamma_j^5)$ with $\gamma_j^5 = i\gamma_j^0\gamma_j^1\gamma_j^2\gamma_j^3$, get for $N > 1$ no counterparts in the anticommuting case, because then their formal counterparts, instead of commuting, anticommute for different $\mu = 0, 1, 2, 3$.

Due to our crucial assumption, the form $\psi^+\Gamma_1^0\Gamma_1^{\mu}\psi$ is no relativistic covariant for $N > 1$, though Eq. (4) with $\Gamma^{\mu} \equiv \Gamma_1^{\mu}$ implies that always

$$\partial_{\mu}\psi^+\Gamma_1^0\Gamma_1^{\mu}\psi = 0 \tag{34}$$

(in fact, it is a component of a more complicated relativistic covariant, namely, the component $\mu_2 = \ldots = \mu_N = 0$ of the tensor $(-1)^{N(N-1)/2}\psi^+\Gamma_1^0\Gamma_2^0 \ldots\Gamma_N^0\Gamma_1^{\mu_1}\Gamma_2^{\mu_2}\ldots\Gamma_N^{\mu_N}\psi)$. In contrast, the form $\psi^+\Gamma_1^0\Gamma_2^0\ldots\Gamma_N^0\Gamma_1^{\mu}\psi$ is a relativistic vector for any N, but Eq. (4) with $\Gamma^{\mu} \equiv \Gamma_1^{\mu}$ shows that

$$\partial_{\mu}\psi^+\Gamma_1^0\Gamma_2^0\ldots\Gamma_N^0\Gamma_1^{\mu}\psi = 0 \tag{35}$$

only for N odd. Thus, the interplay of the theory of relativity and the probability interpretation of quantum mechanics requires that *(i)* only the odd terms

$$N = 1, 3, 5, \ldots \tag{36}$$

should be present in the sequence of the Dirac-type equation (4) (if these are considered as wave equations), and *(ii)* the probability current should have the form

$$j^{\mu} = \eta_N\psi^+\Gamma_1^0\Gamma_2^0\ldots\Gamma_N^0\Gamma_1^{\mu}\psi. \tag{37}$$

Here, η_N is a phase factor making the matrix of hidden internal parity

$$P_{\rm hidden} = \eta_N\Gamma_2^0\ldots\Gamma_N^0 \tag{38}$$

Hermitian. Since due to Eq. (35) $P_{\rm hidden}$ is a constant of motion, one can consistently impose on the wave function ψ in the wave equation (4) the constraint

$$P_{\rm hidden}\psi = \psi \tag{39}$$

in order to guarantee the probability density to be positive :

$$j^0 = \eta_N\psi^+\Gamma_2^0\ldots\Gamma_N^0\psi > 0. \tag{40}$$

5. Hidden exclusion principle

The form of the Dirac-type equation (4) with $\Gamma^\mu \equiv \Gamma_1^\mu$ distinguishes the visible bispinor index α_1 from $N-1$ hidden bispinor indices $\alpha_2, \ldots, \alpha_N$. As to the latter indices, appearing in this scheme on the equal footing, we will make our *second crucial assumption* that they represent physically nondistinguishable degrees of freedom obeying the Fermi statistics along with the Pauli exclusion principle. Then, the wave functions $\psi = \left(\psi_{\alpha_1 \alpha_2 \ldots \alpha_N}\right)$ in the sequence (36) of the Dirac-type wave equations (4) or (31) should be completely antisymmetric with respect to the hidden indices $\alpha_2 \ldots \alpha_N$. This implies that the sequence (36) must terminate at $N = 5$,

$$N = 1, 3, 5, \tag{41}$$

leaving us with three and only three terms (41) in the sequence of the Dirac-type wave equations (4) or (31).

In the case of $N = 5$ our exclusion principle requires that

$$\psi_{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5} \equiv \varepsilon_{\alpha_2 \alpha_3 \alpha_4 \alpha_5} \psi^{(5)}_{\alpha_1} . \tag{42}$$

Thus, in this case there are $4! = 24$ equivalent nonzero components (carrying the index α_1), all equal (up to the sign) to one Dirac function $\psi^{(5)}_{\alpha_1}$. This reduces the Dirac-type equation (4) or (31) to the usual Dirac equation. Here, of course, spin is 1/2 and it is provided by the visible spin, while four hidden spins sum up to zero.

The case of $N = 3$ is more complicated since then one should consider five candidates for relativistic covariants, *viz.*

$$p_{\alpha_1} = \left(C^{-1}\right)_{\alpha_2 \alpha_3} \psi_{\alpha_1 \alpha_2 \alpha_3} , \qquad s_{\alpha_1} = \left(C^{-1}\gamma^5\right)_{\alpha_2 \alpha_3} \psi_{\alpha_1 \alpha_2 \alpha_3} , \tag{43}$$

$$a^\mu_{\alpha_1} = \left(C^{-1}\gamma^\mu\right)_{\alpha_2 \alpha_3} \psi_{\alpha_1 \alpha_2 \alpha_3} , \quad v^\mu_{\alpha_1} = \left(C^{-1}\gamma^5\gamma^\mu\right)_{\alpha_2 \alpha_3} \psi_{\alpha_1 \alpha_2 \alpha_3} , \tag{44}$$

$$t^{\mu\nu}_{\alpha_1} = \left(C^{-1}\gamma^5 \tfrac{i}{2} \left[\gamma^\mu, \gamma^\nu\right]\right)_{\alpha_2 \alpha_3} \psi_{\alpha_1 \alpha_2 \alpha_3} . \tag{45}$$

Here, C denotes the usual charge conjugation matrix that in the chiral representation (where $\gamma^5 = \mathrm{diag}(1, 1, -1, -1)$) may be written as

$$C = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ i & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} = C^{-1} . \tag{46}$$

Making use of Eq. (30), one can write the hidden internal parity (38) in the form

$$P_{\text{hidden}} = i\Gamma_2^0 \Gamma_3^0 = 1 \otimes \gamma^0 \otimes \gamma^0 , \tag{47}$$

where in the chiral representation

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} . \tag{48}$$

Then, the constraint (39) implies that

$$\begin{aligned} &\psi_{\alpha_1 11} = \psi_{\alpha_1 33}\,, \quad \psi_{\alpha_1 22} = \psi_{\alpha_1 44}\,, \quad \psi_{\alpha_1 12} = \psi_{\alpha_1 34}\,, \quad \psi_{\alpha_1 21} = \psi_{\alpha_1 43}\,, \\ &\psi_{\alpha_1 13} = \psi_{\alpha_1 31}\,, \quad \psi_{\alpha_1 24} = \psi_{\alpha_1 42}\,, \quad \psi_{\alpha_1 14} = \psi_{\alpha_1 32}\,, \quad \psi_{\alpha_1 41} = \psi_{\alpha_1 23}\,. \end{aligned} \tag{49}$$

Thus, the constraint (39) and our exclusion principle (requiring that $\psi_{\alpha_1\alpha_2\alpha_3} = -\psi_{\alpha_1\alpha_3\alpha_2}$) leads to the conclusion that from all components $\psi_{\alpha_1\alpha_2\alpha_3}$ only

$$\psi_{\alpha_1 12} = -\psi_{\alpha_1 21} = \psi_{\alpha_1 34} = -\psi_{\alpha_1 43} \equiv \psi^{(3)}_{\alpha_1} \tag{50}$$

and

$$\psi_{\alpha_1 14} = -\psi_{\alpha_1 41} = \psi_{\alpha_1 32} = -\psi_{\alpha_1 23} \tag{51}$$

may be nonzero. Then, after a simple calculation,

$$p_{\alpha_1} = 0\,, \quad s_{\alpha_1} = -4i\psi_{\alpha_1 12}\,, \tag{52}$$

$$a^\mu_{\alpha_1} = 0\,, \quad v^\mu_{\alpha_1} = \begin{cases} -4i\psi_{\alpha_1 14} & \text{for } \mu = 0 \\ 0 & \text{for } \mu = 1,2,3 \end{cases}, \tag{53}$$

$$t^{\mu\nu}_{\alpha_1} = 0\,. \tag{54}$$

But, the theory of relativity applied to the vector $v^\mu_{\alpha_1}$ given in Eq. (53) requires that $v^0_{\alpha_1} = 0$ since $v^\mu_{\alpha_1} = 0$ for $\mu = 1,2,3$. Hence, $\psi_{\alpha_1 14} = 0$. In this way, we can see that all components $\psi_{\alpha_1\alpha_2\alpha_3}$ must vanish except those in Eq. (50). So, in this case there are 4 equivalent nozero components (carrying the index α_1), all equal (up to the sign) to the Dirac function $\psi^{(3)}_{\alpha_1}$. This reduces the Dirac-type equation (4) or (31) to the usual Dirac equation. Here, spin is evidently 1/2 and it is given by the visible spin, two hidden spins being summed up to zero.

Concluding Sections 4 and 5, we can say that in each of the three allowed cases $N = 1,3,5$ there exists one and only one Dirac particle (for any given color and up/down weak flavor described by the standard model). So, it is very natural to connect these three versions of the Dirac particle with the three experimental families of leptons and quarks.

This happy existence of three and only three versions of the Dirac particle is a consequence of an interplay of the theory of relativity, the probability interpretation of quantum mechanics and the Pauli exclusion principle, all extended to the particle's hidden degrees of freedom. These appear necessarily in the Dirac-type equation (4) for any $N > 1$, when the representation (25) implied by the Clifford algebra (26) is inserted.

Since for the wave functions with $N = 1,3,5$ the number of equivalent nonzero components (carrying the visible bispinor index) is $1, 4, 24$, respectively, the following overall wave function comprising three sectors $N = 1,3,5$ (or three fundamental fermion families) may be constructed:

$$\Psi = \frac{1}{\sqrt{29}} \begin{pmatrix} \psi^{(1)}_{\alpha_1} \\ \sqrt{4}\psi^{(3)}_{\alpha_1} \\ \sqrt{24}\psi^{(5)}_{\alpha_1} \end{pmatrix} = \hat{\rho} \begin{pmatrix} \psi^{(1)}_{\alpha_1} \\ \psi^{(3)}_{\alpha_1} \\ \psi^{(5)}_{\alpha_1} \end{pmatrix} . \tag{55}$$

Here, the sector-weighting (or family-weighting) matrix

$$\hat{\rho} = \frac{1}{\sqrt{29}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & \sqrt{24} \end{pmatrix} \tag{56}$$

is introduced.

6. A form of mass matrix and masses for charged leptons

The three-family wave function (55) leads to the following form of the mass matrix for any triple of fundamental fermions listed in one line in Eqs. (1) and (2):

$$\hat{M} = \hat{\rho}\, \hat{h}\, \hat{\rho} . \tag{57}$$

Here, $\hat{h}$ denotes a Higgs coupling strength matrix, while $\hat{\rho}$ is given as in Eq. (56). So, *a priori*, there are four different matrices (57) corresponding to triples of neutrinos, charged leptons, up quarks and down quarks, respectively.

At present, among all 12 fundamental-fermion masses, the masses m_e, m_μ, m_τ of charged leptons e^-, μ^-, τ^- are the best known. On the base of some numerical experience, we can propose the following phenomenological ansatz (in two options) for the matrix $\hat{h}$ in the case of charged leptons :

$$\hat{h} = \begin{pmatrix} h^{(1)} & 0 & 0 \\ 0 & h^{(3)} & 0 \\ 0 & 0 & h^{(5)} \end{pmatrix} \tag{58}$$

with

$$h^{(N)} = M_0 \left(N^2 - \frac{1 \pm \varepsilon^2}{N^2} \right) , \tag{59}$$

where $N = 1, 3, 5$. Here, $M_0 > 0$ and ε^2 denote two real constants independent of N. Then, the eigenvalues of the mass matrix (57) take the form

$$\begin{aligned} \mp m_e &\equiv M^{(1)} = \mp \frac{M_0}{29} \varepsilon^2 , \\ m_\mu &\equiv M^{(3)} = \frac{4}{9} \frac{M_0}{29} \left(80 \mp \varepsilon^2 \right) , \\ m_\tau &\equiv M^{(5)} = \frac{24}{25} \frac{M_0}{29} \left(624 \mp \varepsilon^2 \right) , \end{aligned} \tag{60}$$

since the Dirac masses are defined as nonnegative (*a priori*, the second option seems to be more attractive). In this way, from the system of three equations (60) we obtain in terms of experimental m_e and m_μ the predictions (in two options) for the mass m_τ,

$$m_\tau = \frac{6}{125} \left(351 m_\mu \pm 136\, m_e \right) = \begin{cases} 1783.47 & \text{MeV} \\ 1776.80 & \text{MeV} \end{cases} , \tag{61}$$

and for the parameters M_0 and ε^2,

$$M_0 = \frac{29}{320} \left(9 m_\mu \pm 4 m_e \right) = \begin{cases} 86.3629 & \text{MeV} \\ 85.9924 & \text{MeV} \end{cases} \tag{62}$$

and

$$\varepsilon^2 = \frac{320 m_e}{9 m_\mu \pm 4 m_e} = \begin{cases} 0.171590 \\ 0.172329 \end{cases} . \tag{63}$$

Let us note an excellent agreement between the predictions (61) for m_τ and its experimental value

$$m_\tau = 1784.1^{+2.7}_{-3.6}\ \text{MeV} \tag{64}$$

cited for several years by Particle Data Group [10] or

$$m_\tau = (1776.9 \pm 0.4 \pm 0.3)\text{MeV} , \quad m_\tau = (1776.3 \pm 2.4 \pm 1.4)\text{MeV} \tag{65}$$

reported recently by Beijing Electron-Positron Collider Group [11] and ARGUS Collaboration [12], respectively.

This supports the phenomenological ansatz (59) operating with the number N of *"algebraic partons"* involved in the families $N = 1, 3, 5$ and described by the Dirac bispinor indices as these appear in the Clifford algebras (26) or, more conveniently, (29). The algebraic partons are agents of the idea of *algebraic compositness.* In the picture which emerges from our argument, they are building blocks of fundamental fermions in such a way that any fundamental fermion with $N = 1, 3, 5$ is composed of one "visible" algebraic parton of spin 1/2 and $N-1=0,2,4$ "hidden" algebraic partons of spins 1/2, the latter forming relativistic scalars.

Since the hidden bispinor indices correspond to the "relative" Jacobi-type combinations (28) of γ_i^μ, they represent "relative" algebraic degrees of freedom "inside" fundamental fermions. So, the hidden algebraic partons of spins 1/2 are excitations of these relative algebraic degrees of freedom. Similarly, the visible algebraic parton of spin 1/2 is related to the "centre-of-mass" algebraic degree of freedom represented by the visible bispinor index corresponding to the "centre-of-mass" Jacobi-type combination (25) of γ_i^μ. Of course, the centre-of-mass algebraic degree of freedom coexists with the particle's spatial degrees of freedom.

REFERENCES

[1] W. Królikowski, *Acta Phys. Pol.* **B18**, 111 (1987); *Nuovo Cimento* **100A**, 701 (1988).
[2] *Cf. e.g.* H.A. Bethe, E.E. Salpeter, in *Encyclopedia of Physics*, Vol. **35**, Springer, Berlin-Göttingen-Heidelberg 1957.
[3] W. Królikowski, J. Rzewuski, *Nuovo Cimento* **2**, 203 (1955); **4**, 974 (1956).
[4] E.E. Salpeter, *Phys. Rev.* **87**, 328 (1952).
[5] Królikowski, *Acta Phys. Pol.* **B17**, 813 (1986).
[6] For a formal quantization scheme beyond the conventional quantum field theory *cf.* W. Królikowski, *Acta Phys. Pol.* **B22**, 613 (1991); [E., *Acta Phys. Pol.* **B23**, 83 (1992)].
[7] T. Banks, Y. Dothan, D. Horn, *Phys. Lett.* **B117**, 413 (1982).
[8] E. Kähler, *Rendiconti di Matematica* **21**, 425 (1962); *cf.* also D. Ivanenko, L. Landau, *Z. Phys.* **48**, 341 (1928), published soon after the discovery of Dirac equation (P.A.M. Dirac, *Proc. Roy. Soc.* **117**, 610 (1928)). I am indebted to Andrzej Trautman for his calling my attention to this early paper.
[9] W. Królikowski, *Acta Phys. Pol.* **B21**, 871 (1990); *Phys. Rev.* **D45**, 3222 (1992).
[10] Particle Data Group, Review of Particle Properties, *Phys.Rev.* **D45**, Part 2 (June 1992).
[11] Nading Qi, *Bull. Am. Phys. Soc.* **37**, 957 (April 1992), caption only; information in: *CERN Courier* **32**, 13 (July 1992).
[12] H. Albrecht *et al.* (ARGUS Collaboration), DESY report 92-086 (June 1992).