Vol. 24(1993) *ACTA PHYSICA POLONICA B* No 12

ENERGY OF W DISTRIBUTION IN TOP QUARK DECAYS*

M. JEŻABEK

Institute of Nuclear Physics, Kawiory 26a, PL-30055 Cracow, Poland
and

Institut f. Theoretische Teilchenphysik, Universität Karlsruhe
D-76128 Karlsruhe, Germany

CH. JÜNGER

Institut f. Theoretische Teilchenphysik, Universität Karlsruhe
D-76128 Karlsruhe, Germany

(Received August 4, 1993)

A relatively simple analytical formula is derived for the energy spectrum of W boson in top quark decays $t \to Wb$ including $\mathcal{O}(\alpha_s)$ radiative corrections. We discuss the accuracy of this formula and compare it to a more general albeit more complicated one derived in (A. Czarnecki, M. Jeżabek, J.H. Kühn, *Acta Phys. Pol.* **B20**, 961 (1989); (E) **B23**, 173 (1992)). A Monte Carlo algorithm for generation of W energy spectrum is briefly described.

PACS numbers: 12.15. Ji

1. Introduction and summary

Radiative QCD corrections to the energy distribution of $\bar{f}f'$ in $t \to b\bar{f}f'$ and $t \to bg\bar{f}f'$ decays have been calculated some time ago [1]. In the meantime the lower limit for top quark mass m_t has been pushed up by CDF and D0 collaborations well above the threshold for $t \to bW$ channel. Although the results of [1] are applicable also in this regime it seems reasonable and useful to derive a new formula assuming dominance of decays into real W.

* Work partly supported by the Polish Committee for Scientific Research (KBN) Grants 203809101 and 223729102, and by EEC Contract ERB-CIPA-CT-92-2077.

Such a formula, albeit less general than that given in [1], can be very useful in studies of top quark physics at future e^+e^- colliders [2].

Our present approach is based on an appropriately modified narrow width ($\Gamma_W = 0$) approximation, where Γ_W is the width of W boson. In contrast to [1], where the rates are manifestly infrared finite, we introduce an explicit infrared cutoff λ on the effective mass squared of the system b quark + gluon. Thus the formula for $\mathcal{O}(\alpha_s)$ contribution from virtual gluons also depends on λ. This apparent failure turns out to be an advantage in Monte Carlo simulations, which are indispensable for a precise determination of the strong coupling constant α_s and m_t from the total $\sigma(e^+e^- \to \bar{t}t)$ and differential cross sections [3–6]. We wrote a Monte Carlo program based on the results of the present article and found a very good agreement with the formulae given in [1].

We have checked also that massless b approximation which is known to be a satisfactory one for the total decay rate and m_t above 120 GeV [7, 8], cannot be used in calculations of W energy distribution. One reason is purely kinematical: Born distribution in the narrow width approximation has a Dirac delta shape, i.e. the energy of W is fixed by two body kinematics. A shift from realistic $m_b = 4.7$ GeV to $m_b = 0$ results in a non-negligible shift in this energy. We attempted to correct the massless formula (see Appendix A) for this trivial effect but the result was still a rather poor approximation. Thus, we conclude that for realistic top and bottom masses one has to use the complete $\mathcal{O}(\alpha_s)$ result rather than its massless approximation.

Our article is organized as follows. In Section 2 we derive our formula for the enegy of W distribution in the narrow width approximation. In Section 3 we include a non-zero W width and describe our Monte Carlo program based on our calculations. Then, we compare the results of this program with those of [1]. In Appendix A our formulae for $m_b \to 0$ are given.

2. Energy of W distribution in narrow width approximation

We use throughout the same notation as in [9]. We stay in the top quark rest frame and some variables are expressed in units of m_t. In particular y — effective mass squared of $\bar{f}f'$ system, z — the mass squared of bg system and $\varepsilon = m_b/m_t$. We define also the energy of W :

$$x_W = \frac{E_W}{m_t} = \frac{1}{2}(1 + y - z)\,, \tag{1}$$

which for twobody Born decay mode $t \to bW$ is replaced by

$$\overline{x}_W = \frac{E_W}{m_t} = \frac{1}{2}(1 + y - \varepsilon^2)\,, \tag{2}$$

and $x_W \le \bar{x}_W$. In general "barred" quantities are evaluated at $z = \varepsilon^2$. Other useful variables are:

$$w_3 = \sqrt{x_W^2 - y} = \tfrac{1}{2}\sqrt{\lambda(1,y,z)}, \tag{3}$$

$$\lambda(u,v,w) = u^2 + v^2 + w^2 - 2(uv + uw + vw) \tag{4}$$

the lenght of W momentum[1], which is equal to

$$p_3 = \sqrt{x_h^2 - z} \tag{5}$$

the momentum of bg system of energy $x_h = 1 - x_W$, the light cone variables

$$\begin{aligned} w_\pm &= x_W \pm w_3\,, \\ p_\pm &= x_h \pm p_3\,, \end{aligned} \tag{6}$$

and rapidities

$$\begin{aligned} Y_W &= \tfrac{1}{2}\ln\frac{w_+}{w_-}\,, \\ Y_p &= \tfrac{1}{2}\ln\frac{p_+}{p_-}\,. \end{aligned} \tag{7}$$

In the narrow width approximation $\gamma = \Gamma_W/m_W \to 0$ one can replace a factor resulting from W propagator by Dirac delta function

$$\frac{1}{\pi}\frac{\gamma y_0}{(y-y_0)^2 + \gamma^2 y_0^2} \longleftrightarrow \delta(y - y_0)\,, \tag{8}$$

where $y_0 = (m_W/m_t)^2$. Thus, when the decay through real W dominates the effective mass is close to $\sqrt{y_0}$. In this section we consider y as fixed and give the formula for the differential rate $\left.\frac{\mathrm{d}\Gamma}{\mathrm{d}x_W}\right|_y$. Then, in Section 3 we relax this assumption using (8). For fixed y the energy distribution of W including $\mathcal{O}(\alpha_s)$ corrections can be written as follows:

$$\left.\frac{\mathrm{d}\Gamma}{\mathrm{d}x_W}\right|_y = \frac{G_F m_t^3}{8\sqrt{2}\pi}\left[\delta(z-\varepsilon^2)\left(\mathrm{F}_0 - \frac{2\alpha_s}{3\pi}\bar{\mathcal{G}}_1(y,\lambda)\right) + \frac{2\alpha_s}{3\pi}\Theta(z - \varepsilon^2 - \lambda)\Theta(x_W - \sqrt{y})\,\bar{g}_1(y,z)\right], \tag{9}$$

---

[1] In unpolarized case discussed here we choose z-axis in the direction of W.

where z is fixed through (1),

$$\mathrm{F}_0(y,\varepsilon) = \tfrac{1}{2}\sqrt{\lambda(1,y,\varepsilon^2)}\,C_0(y,\varepsilon)\,, \tag{10}$$

$$C_0(y,\varepsilon) = 4[(1-\varepsilon^2)^2 + y(1+\varepsilon^2) - 2y^2]\,. \tag{11}$$

$$\begin{aligned}
\tilde{\mathcal{G}}_1 =& g_1 C_0(y,\varepsilon)\bar{x}_h + g_2 C_0(y,\varepsilon)\bar{p}_3 + g_3\bar{x}_h\bar{p}_3 + g_4\bar{p}_3 + g_5\bar{Y}_p + g_6\\
g_1 =& 4\bar{Y}_p^2 - 2\mathrm{Li}_2(\bar{w}_0) + 4\mathrm{Li}_2\left(\bar{w}_-/\bar{w}_+\right)\\
& - 4\mathrm{Li}_2\left(\frac{\bar{p}_-\bar{w}_-}{\bar{p}_+\bar{w}_+}\right) + 2\mathrm{Li}_2(\bar{w}_+) - 8\bar{Y}_p\bar{Y}_w\\
& + 4\bar{Y}_p\left(2\ln\varepsilon - \ln\lambda + 2\ln(1-\bar{p}_-/\bar{p}_+) - \ln y\right)\\
g_2 =& 4 - 6\ln\varepsilon + 4\ln\lambda\\
g_3 =& 24(1-\varepsilon^2)\ln\varepsilon\\
g_4 =& 8\left(-1 + 2\varepsilon^2 - \varepsilon^4 - y - \varepsilon^2 y + 2y^2\right)\\
& - \left[18\,\varepsilon^2\left(2y^2 - y - 1\right) + 4\,\varepsilon^2 C_0(y,\varepsilon)\right]\frac{\lambda}{\varepsilon^2\left(\varepsilon^2 - \lambda\right)}\\
g_5 =& 4\left(-1 + \varepsilon^2 + \varepsilon^4 - \varepsilon^6 - 3y + 2\varepsilon^2 y - 3\varepsilon^4 y + 9y^2 + 9\varepsilon^2 y^2 - 5y^3\right)\\
g_6 =& \left[9\,\varepsilon^2\left(2y^2 - y - 1\right) + 2\,\varepsilon^2 C_0(y,\varepsilon)\right]\frac{1+y}{1-y}\ln\left(1 + \lambda/\varepsilon^2\right) -\\
& \left[7\left(2y^2 - y - 1\right) + 2\,C_0(y,\varepsilon)\right](1-y)\ln\left(1+\lambda/\varepsilon^2\right)\,,
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
\tilde{g}_1 =& 2\,a_1\,p_3(z) + 4\,a_2\,Y_p(z) - \frac{4\,\varepsilon^4 C_0(y,\varepsilon)}{z^2(z-\varepsilon^2)}\,p_3(z) + \frac{4\,\bar{x}_h C_0(y,\varepsilon)}{z-\varepsilon^2}\,Y_p(z)\\
a_1 =& \frac{\varepsilon^2}{z^2}\left[-9 + 15\,\varepsilon^2 - 8\,\varepsilon^4 - y\,(9 + 7\,\varepsilon^2 - 18y)\right]\\
& + \frac{1}{z}\left[-7 + \varepsilon^2\,(20 - 5\varepsilon^2 - 11y) + 7y\,(2y-1)\right]\\
& + 2y - 3(1+\varepsilon^2)\\
a_2 =& 2 + \varepsilon^2(\varepsilon^2 - 5) - 2y\,(2y-1) + \left(1 + \varepsilon^2 + 2y\right)\,z\,.
\end{aligned} \tag{13}$$

In the above formulae λ denotes an infrared cutoff on the effective mass $z \geq \varepsilon^2 + \lambda$. Let us sketch now the derivation of Eq. (9): The contribution resulting from real gluon radiation (Θ-piece) is obtained by direct integration of the fully differential decay rate, whereas the (Born + soft) contribution (δ-piece) is derived from the requirement that integrating (9) over x_W one obtains the narrow width limit of the expression for the total rate given in [7]; see also (16) in the following section. The formula (9) simplifies considerably in the limit $m_b \to 0$, *cf.* Appendix A. However, for realistic b quark masses this is a rather poor approximation.

3. Finite W width and results

We generalize now the results of the previous section and include a nonzero W width. Let us note that y is not fixed for $\gamma \neq 0$. The double differential distribution

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}x_W\,\mathrm{d}y} = \frac{\gamma}{\pi}\frac{y_0}{(y-y_0)^2+\gamma^2 y_0^2}\frac{\mathrm{d}\Gamma}{\mathrm{d}x_W}\bigg|_y \tag{14}$$

is, however, closely related to the narrow width result (9), *cf.* (8). We use the standard model result for Γ_W:

$$\Gamma_W = \frac{G_F {m_W}^3}{6\sqrt{2}\pi}\left(9+6\frac{\alpha_s}{\pi}\right). \tag{15}$$

Integrating (14) over x_W we obtain of course the standard model result for $\mathrm{d}\Gamma/\mathrm{d}y$ [7,8][2]

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}y} = \frac{{G_F}^2 {m_t}^5}{192\pi^3}\left(9+6\frac{\alpha_s}{\pi}\right)\frac{1}{(1-y/y_0)^2+\gamma^2}\left[\mathrm{F}_0(y,\varepsilon)-\frac{2\alpha_s}{3\pi}\mathrm{F}_1(y,\varepsilon)\right], \tag{16}$$

$0 \le y \le (1-\varepsilon)^2$, where F_0 is defined in (10) and

$$\begin{aligned}
\mathrm{F}_1(y,\varepsilon) = & \tfrac{1}{2}C_0(y,\varepsilon)(1+\varepsilon^2-y)\Big[2\pi^2/3+4\mathrm{Li}_2\,(u_w)-4\mathrm{Li}_2\,(u_q) \\
& \qquad -4\mathrm{Li}_2\,(u_q u_w)-4\ln u_q\ln(1-u_q) \\
& \qquad -2\ln u_w \ln u_q+\ln y\ln u_q+2\ln\varepsilon\ln u_w\Big] \\
& -2\mathrm{F}_0(y,\varepsilon)\left[\ln y+3\ln\varepsilon-2\ln\lambda(1,y,\varepsilon^2)\right] \\
& +4(1-\varepsilon^2)\left[(1-\varepsilon^2)^2+y(1+\varepsilon^2)-4y^2\right]\ln u_w \\
& +\big[3-\varepsilon^2+11\varepsilon^4-\varepsilon^6+y(6-12\varepsilon^2+2\varepsilon^4) \\
& -y^2\,(21+5\varepsilon^2)+12y^3\big]\ln u_q \\
& +6\sqrt{\lambda(1,y,\varepsilon^2)}(1-\varepsilon^2)(1+\varepsilon^2-y)\ln\varepsilon \\
& +\sqrt{\lambda(1,y,\varepsilon^2)}\left[-5+22\varepsilon^2-5\varepsilon^4-9y(1+\varepsilon^2)+6y^2\right],
\end{aligned} \tag{17}$$

---

[2] we derived (9) using this condition.

where

$$u_q = \frac{\bar{p}_-}{\bar{p}_+} \qquad u_w = \frac{\bar{w}_-}{\bar{w}_+}. \tag{18}$$

We can also integrate (14) over y. In this way we obtain a new formula for

$$\frac{d\Gamma}{dx_W} = \int\limits_0^{(1-\varepsilon)^2} dy\, \frac{d\Gamma}{dx_W\, dy}. \tag{19}$$

We compared (19) with the result of [1] and found a perfect numerical agreement for small λ. For $\lambda = 10^{-8}$ the relative error is 10^{-6}.

The formulae (9) and (16) can be also used as a starting point for Monte Carlo simulations. A key observation is that (16) gives the distribution of y for $0 \le y \le (1-\varepsilon)^2$ whereas (9) gives relative probabilities for x_W at fixed y. The distribution (14) can be generated as follows: y is generated first according to (16). Then, for given y, x_W is generated according to (9). Both steps can be performed by a standard combination of importance sampling and von Neumann rejection. The only difficulty is to find the value of the infrared cutoff λ such that λ is small enough and the δ-piece in (9) remains positive. This well known difficulty limits relative accuracy of our Monte Carlo to about 1%.

In Fig. 1 we show the normalized distribution $\Gamma^{-1}\, d\Gamma/dx_W$ for $m_t = 120$ GeV obtained from (19) for $\lambda = 10^{-8}$. The curve obtained using the result from [1] is identical up to the resolution of the plot.

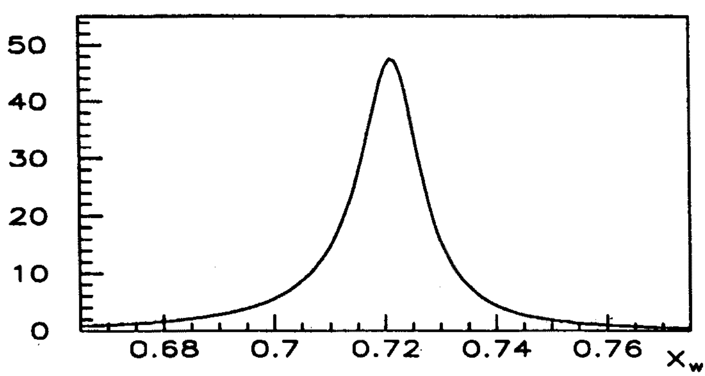

Fig. 1. Normalized energy distribution of W: $\Gamma^{-1}\, d\Gamma/dx_W$ for $m_t = 120$ GeV evaluated from (19).

In Fig. 2 we compare the analytical result (19) ($\lambda = 10^{-8}$) with our

Monte Carlo program for $\lambda = 3 \cdot 10^{-4}$. We plot the ratio (in percent)

$$1 - \frac{\mathrm{d}\Gamma/\mathrm{d}x_W|_n}{\mathrm{d}\Gamma/\mathrm{d}x_W|_a}, \tag{20}$$

where the subscript "n" refers to the Monte Carlo and "a" to the result obtained from (19).

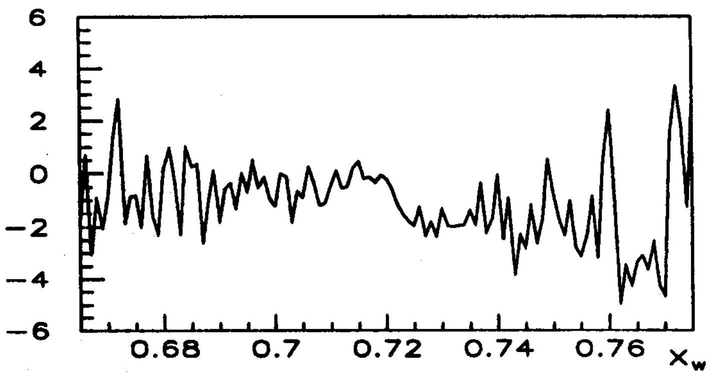

Fig. 2. Comparison of Monte Carlo for infrared cutoff $\lambda = 3 \cdot 10^{-4}$ and analytic result (19) for $\lambda = 10^{-8}$. The ratio (20) is shown as function of x_W.

APPENDIX A

In this Appendix we give the limit of (9) for $m_b \to 0$. These formulae can be used for $t \to s$ or $t \to u$ transitions. It would be, however, a rather poor approximation to use these formulae for the dominant $t \to b$ transition. For $\varepsilon \to 0$ (9) is replaced by

$$\left.\frac{\mathrm{d}\Gamma}{\mathrm{d}x_W}\right|_y = \frac{G_F m_t{}^3}{8\sqrt{2}\pi} \left\{\delta(z)\, F_0 \left[1 - \frac{2\,\alpha_s}{3\,\pi} G_1(y,\lambda)\right] + \frac{2\,\alpha_s}{3\,\pi}\, \Theta(z-\lambda)\, \Theta\left(x_W - \sqrt{y}\right)\, g(z,y)\right\}. \tag{21}$$

$$F_0 = 2(1-y)^2\,(2y+1)\,. \tag{22}$$

$$G_1(y,\lambda) = \frac{2}{3}\pi^2 + \frac{5}{2} + 2\,\mathrm{Li}_2(y) + 4\,\ln^2(1-y) - 7\,\ln(1-y) + \ln^2(\lambda) + \frac{1}{2}\,\ln(\lambda)\,[7 - 8\,\ln(1-y)] + \frac{5+4y}{1+2y}\,\ln(1-y)\,. \tag{23}$$

$$g(z,y) = p_3(z)\left[2\,(2y-3) - 14\,(1-y)(2y+1)\,\frac{1}{z}\right] + 4\,Y_p(z)\left[(2y+1)(2-2y+z) + \frac{F_0}{z}\right]. \qquad (24)$$

Then, integrating the double differential distribution (14) over y one obtains $\mathrm{d}\Gamma/\mathrm{d}x_W$ in the massless limit $m_b = 0$.

REFERENCES

[1] A. Czarnecki, M. Jeżabek, J.H. Kühn, *Acta Phys. Pol.* **B20**, 961 (1989). (E) **B23** (1992) 173

[2] P.M. Zerwas (ed.), e^+e^- Collisions at 500 GeV: The Physics Potential, DESY Orange Report DESY 92–123A, Hamburg 1992.

[3] J.M. Strassler, M.E. Peskin, *Phys. Rev.* **D43**, 1500 (1991).

[4] Y. Sumino, K. Fujii, K. Hagiwara, H. Murayama, C.-K. Ng, *Phys. Rev.* **D47**, 56 (1992).

[5] M. Jeżabek, J.H. Kühn, T. Teubner, *Z. Phys.* **C56**, 653 (1992); M. Jeżabek, T. Teubner, *Z. Phys.* **C** (1993) in print; Karlsruhe Univ. preprint TTP 93-11.

[6] M. Martinez *et al.*, in: Proceedings of the Workshop on Physics and Experiments at Linear e^+e^- Colliders, Waikoloa, Hawaii, April 1993.

[7] M. Jeżabek, J.H. Kühn, *Nucl. Phys.* **B314**, 1 (1989).

[8] M. Jeżabek, J.H. Kühn, *Phys. Rev.* **D**,(Rapid Commun.) (1993) in print; Karlsruhe Univ. preprint TTP 93-4.

[9] M. Jeżabek, J.H. Kühn, *Nucl. Phys.* **B320**, 20 (1989).