ON THE SIGNATURE SYMMETRY IN U(2n)–MODEL*

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The properties of the base functions resulting from the solutions of fermionic Hamiltonian possessing unitary symmetry are analyzed. The attention is focused on the solutions of nuclear cranking model Hamiltonian in a single j-shell space. It appears that, in this case, the group theoretical treatment offers a lot of simplifications as compared to any other method. In particular, we analyze the properties of the base functions under the signature symmetry transformation proving that signature is sharply defined inside each irreducible representation.

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1. Introduction

Recently, a method has been developed [1] for finding an exact solution to the cranking Hamiltonian H^{ω}. The first version of the calculation was applied to axially symmetric nuclear mean field with the two-body nucleon-nucleon pairing force. The method of solution follows from the observation

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that in a case of single j-shell, the monopole pairing force and an external rotation about a fixed axis the total cranking Hamiltonian H^ω can be build out of the generators forming [2] the Lie algebra of the unitary group in $(2n)$-dimensions (where $n = j + \frac{1}{2}$ denotes half the number of single-particle levels). It turns out that [2] the highest weight for each irreducible representation of the group is fully characterized by a single integer number p. This is due to the Pauli principle and to the many-fermion structure of all the basic states forming each irreducible representation. Another unitary group in $(2n)$-dimension labelled by the particle-number as the highest weight can be also employed [3] for finding the solution to the cranking Hamiltonian H^ω. As still another possibility the symmetry of the orthogonal group $SO(4n)$ can be applied [4]. The present paper, however, is based on the method discussed in Ref. [1].

Let us now turn to the signature symmetry of the model. We shall assume that our Hamiltonian H^ω is left invariant with respect to the rotation R_x through an angle 180° about an axis (say, x-axis) perpendicular to the symmetry axis of the single particle potential (say, z-axis). In other words the Hamiltonian H^ω is left invariant with respect to the transformation

$$R_x = \exp(-i\pi j_x)\,, \tag{1}$$

and the eigenstates of H^ω could, in principle, be also labelled by the signature quantum number r_x, the eigenvalue of R_x.

Our paper aims at proving that all the states belonging to a given p-*irrep* (*i.e.*, the irreducible representation characterized by the integer number p, its highest weight) have the same signature r_x, and, furthermore, the signature r_x can be simply expressed as

$$r_x = i^{p-n}\,, \tag{2}$$

where p and n were already defined.

2. Description of the model

The cranking Hamiltonian H^ω defining our model contains three terms, namely the single-particle axially deformed Hamiltonian, $H_{\rm sp}$, in a single-j shell model space, the two-body monopole pairing force $H_{\rm pair}$ and the cranking term equal $-\omega j_x$:

$$\hat{H}^\omega = \hat{H}_{\rm sp} + \hat{H}_{\rm pair} - \omega\hat{j}_x\,. \tag{3}$$

All three parts appearing in the Hamiltonian can be expressed in terms of the creation and annihilation operators $c_k^\dagger$ and c_k (*cf.* Refs [2] and [1]). Alternatively, the set of operators

$$N_{kl} = c_k^\dagger c_l, \quad B_{kl}^\dagger = c_k^\dagger c_l^\dagger \text{ and } B_{kl} = c_l c_k\,, \tag{4}$$

where $k, l = 1, 2, \ldots, 2n$, may be employed. The Lie algebra defined by the above set of operators is actually broader than that of SU($2n$). In fact, it is the algebra of special orthogonal group SO($4n$) $\supset$SU($2n$) [2, 4]. The generators of the SU($2n$) can be expressed as certain linear combinations $A_{kl}(k, l = 1, 2, \ldots, 2n)$ of the N_{kl}, $B^{\dagger}_{kl}$ and B_{kl} operators (see Ref. [2] or [1]):

$$\begin{aligned} A_{k,l} &= \tfrac{1}{2}\left(N_{2k-1,2l-1} + (-)^{k+l}N_{2k,2l}\right) + (-)^{l+1}\tfrac{1}{2} \\ &\quad \times \left(N_{2k-1,2l} + (-)^{k+l}N_{2k,2l-1}\right) \\ A_{l+n,k+n} &= -\tfrac{1}{2}\left(N_{2k-1,2l-1} + (-)^{k+l}N_{2k,2l}\right) + (-)^{l+1}\tfrac{1}{2} \\ &\quad \times \left(N_{2k-1,2l} + (-)^{k+l}N_{2k,2l-1}\right) + \delta_{k,l} \\ A_{l,l+n} &= \tfrac{1}{2}\left(B^{\dagger}_{2k-1,2l-1} + (-)^{k+l+1}B^{\dagger}_{2k,2l}\right) + (-)^{l}\tfrac{1}{2} \\ &\quad \times \left(B^{\dagger}_{2k-1,2l} + (-)^{k+l+1}B_{2k,2l-1}\right) \\ A_{l+n,k} &= (A_{k+n,l})^{\dagger}, \end{aligned} \tag{5}$$

with $k, l = 1, 2, \ldots, n$. Generators $A_{k,l}$ obey the commutation relations

$$[A_{k,l},\ A_{m,n}] = \delta_{l,m}A_{k,n} - \delta_{k,n}A_{m,l};\ \text{and}\ A^{\dagger}_{k,l} = A_{l,k}, \tag{6}$$

which are the standard relations characteristic to the Lie algebra of U($2n$).

The cranking Hamiltonian H^{ω} given by Eq. (3) can be expressed in terms of the generators $A_{k,l}$. The corresponding expressions are given in Ref. [1] as well as the results of the diagonalization. Thus, they will not be repeated here. Instead, we shall concentrate on the *p-irrep* basis and its relation with the signature quantum number r_x.

The basic vectors in the representation space can be conveniently expressed in terms of the Gelfand and Tsetlin (GT) patterns [5], (see also Refs [1] and [2]):

$$\begin{pmatrix} m_{1,2n} & m_{2,2n} & \ldots & m_{2n-2,2n} & m_{2n-1,2n} & m_{2n,2n} \\ m_{1,2n-1} & m_{2,2n-1} & \ldots & m_{2n-2,2n-1} & m_{2n-1,2n-1} & \\ \ldots & \ldots & \ldots & \ldots & & \\ \vdots & \vdots & & & & \\ m_{1,2} & m_{2,2} & & & & \\ m_{1,1} & & & & & \end{pmatrix}. \tag{7}$$

It can be shown [2] that for the many-fermion states the numbers $m_{k,l}$ entering the pattern may be equal to either 0 or 1. Furthermore, numbers $m_{k,l}$ have to obey certain inequalities:

$$m_{i,k} \geq m_{i,k-1} \geq m_{i+1,k}, \text{ where } k = 2, 3, \ldots, 2n;\ i = 1, 2, \ldots, k-1, \tag{8}$$

$(k = 2, 3, \ldots, 2n;\ i = 1, 2, \ldots, (k-1))$. Thus, each row of the pattern contains several unities followed by several zeros. In this situation the whole GT pattern may be generally replaced by a single column vector [1]:

$$\begin{pmatrix} \nu_{2n} \\ \nu_{2n-1} \\ \vdots \\ \nu_2 \\ \nu_1 \end{pmatrix}. \tag{9}$$

The integers ν_k $(k = 1, 2, \ldots, 2n)$ are the sums:

$$\nu_k \sum_{i=1}^{k-1} m_{ik}\,, \tag{10}$$

or in other words each ν_k is equal to the number of unities in the k-th row of the pattern. It also follows from inequalities (8) that for any two neighbouring numbers

$$k \geq \nu_k \quad \text{and} \quad \begin{cases} \nu_k = \nu_{k-1}; & \text{or} \\ \nu_k = \nu_{k-1} + 1 \end{cases}. \tag{11}$$

Quantity ν_{2n} defines the highest weight of the *p-irrep*, thus

$$\nu_{2n} \equiv p\,. \tag{12}$$

The generators $A_{k,l}$ of the algebra act on vectors (9) leading to other vectors within the same representation. The corresponding formulae are given in Ref. [1]. Here, we give only the formula for the "diagonal" operators (*i.e.*, $A_{k,l}$ for $k = l$)

$$A_{l,l} \begin{pmatrix} \nu_{2n} \\ \nu_{2n-1} \\ \vdots \\ \nu_2 \\ \nu_1 \end{pmatrix} = (\nu_l - \nu_{l-1}) \begin{pmatrix} \nu_{2n} \\ \nu_{2n-1} \\ \vdots \\ \nu_2 \\ \nu_1 \end{pmatrix}, \tag{13}$$

$l = 1, 2, \ldots, n$ (for $l = 1$ quantity ν_0 equals 0). In particular the operator

$$\mathcal{O} \equiv \sum_{l=1}^{2n} A_{l,l} \tag{14}$$

has eigenvalue $\nu_{2n} = p$ when acting on the arbitrary state (9). Consequently:

$$\mathcal{O}\begin{pmatrix} \nu_{2n}=p \\ \nu_{2n-1} \\ \vdots \\ \nu_2 \\ \nu_1 \end{pmatrix} = \sum_{k=1}^{n}(A_{k,k}+A_{k+n,k+n})\begin{pmatrix} \nu_{2n}=p \\ \nu_{2n-1} \\ \vdots \\ \nu_2 \\ \nu_1 \end{pmatrix} = p\begin{pmatrix} \nu_{2n}=p \\ \nu_{2n-1} \\ \vdots \\ \nu_2 \\ \nu_1 \end{pmatrix}. \tag{15}$$

3. Relation between signature r_x and highest weight p

The enumeration of the single-particle states used here follows the convention adopted in Ref. [1]. Thus all the $|j,m\rangle$ states are ordered according to the scheme: the states $|1\rangle$ and $|2\rangle$ correspond to $|j,m=1/2\rangle$ and $\overline{|j,m=1/2\rangle}$, the states $|3\rangle$ and $|4\rangle$ correspond to $|j,m=3/2\rangle$ and $\overline{j,m=3/2\rangle}$ and so on. The state $\overline{|j,m\rangle}$ denotes the time reversed state $\hat{T}|j,m\rangle$, where $\hat{T}$ is time-reversal operator. The above pairs correspond to states which are degenerate in the absence of rotation ($\omega = 0$).

It can be shown that

$$R_x|j,m\rangle = -i(-)^{m+1/2}\overline{|j,m\rangle}. \tag{16}$$

Indeed, using the angular momentum algebra (*cf.* Eq. (1) and Ref. [6])

$$R_x|j,m\rangle = R_yR_z|j,m\rangle = -i(-)^{m+1/2}\overline{|j,m\rangle} = -i(-1)^k\overline{|j,m\rangle}, \tag{17}$$

where $m = k - 1/2$.

Now let us concentrate on a pair of states

$$|jm\rangle = |2k-1\rangle \quad \text{and} \quad \overline{|jm\rangle} = (-1)^{j+m}|j-m\rangle = |2k\rangle, \tag{18}$$

($k = 1,2,\ldots,n$). Instead of the above two sets we can take two linear combinations, the symmetric and antisymmetric one:

$$\begin{aligned} |s_k\rangle &= \frac{1}{\sqrt{2}}\big(|2k-1\rangle + |2k\rangle\big) \\ |a_k\rangle &= \frac{1}{\sqrt{2}}\big(|2k-1\rangle - |2k\rangle\big). \end{aligned} \tag{19}$$

Formulas given above imply that:

$$\begin{aligned} R_x|s_k\rangle &= -i(-1)^k|s_k\rangle \\ R_x|a_k\rangle &= +i(-1)^k|a_k\rangle. \end{aligned} \tag{20}$$

If all the $|jm\rangle$, $\overline{|jm\rangle}$ pairs ($m = 1/2,\ 3/2, \ldots, j$) are transformed in this way we obtain representation which diagonalizes the operator R_x.

Now one can consider four possibilities concerning the occupation of the above states $|s_k\rangle$ and $|a_k\rangle$:

(1) Both states $|s_k\rangle$ and $|a_k\rangle$ are empty.
(2) Both states $|s_k\rangle$ and $|a_k\rangle$ are occupied.
(3) State $|s_k\rangle$ is occupied while $|a_k\rangle$ is empty.
(4) State $|s_k\rangle$ is empty while $|a_k\rangle$ is occupied.

This is easy to see that the contribution $r^{(k)}$ to the total signature r_x coming from the pair $|s_k\rangle$ and $|a_k\rangle$ equals:

$$r^{(k)} = 1, 1, -i(-1)^k \quad or \quad +i(-1)^k \tag{21}$$

in the four cases (1), (2), (3) and (4), respectively. Let us now to introduce the operator

$$\mathcal{N}^{(k)} = N_{2k-1,2k} + N_{2k,2k-1}\,. \tag{22}$$

Since this operator leads to the exchange of the state $|s_k\rangle$ into $|a_k\rangle$ and *vice versa* its eigenvalues are 0, 0, 1, or -1 in the four cases (1), (2), (3) and (4), respectively. Using transformation (5) we can see that eigenvalues w^k of the operator

$$A_{k,k} + A_{k+n,k+n} - 1 = (-1)^{k+1}\mathcal{N}^{(k)} \tag{23}$$

are equal 0, 0, $(-1)^{k+1}$ and $(-1)^k$ in case (1), (2), (3) and (4), respectively. Here $k = 1, 2, \ldots, n$. The above numbers determine the eigenvalues of the operator $A_{k,k} + A_{k+n,k+n} - 1$ in the four cases (1)–(4).

Comparing these four values of $w^{(k)}$ with the corresponding expressions (21) for the partial signatures $r^{(k)}$ we obtain one equation

$$i^{w^{(k)}} = r^{(k)} \tag{24}$$

valid for all four cases. Now, the total signature r_x of the arbitrary basic states of the *p-irrep*

$$r_x = \prod_{k=1}^{n} i^{w^{(k)}} = i^{\sum_{k=i}^{n} w^{(k)}} = i^{p-n} \tag{25}$$

since $\sum_{k=1}^{n} w^{(k)}$ is an eigenstate of the operator

$$\sum_{k=1}^{n}(A_{k,k} + A_{k+n,k+n} - 1) = p - n\,. \tag{26}$$

Generally speaking if the cranking Hamiltonian H^{ω} commutes with the symmetry operator R_x then either *(i)* the energy matrix within any *irrep* becomes block diagonal with respect to the eigenvalues of R_x, or *(ii)* all the states within a given *irrep* are eigenstates of the symmetry operator R_x with the same eigenvalue r_x defined by the properties of the *irrep*. Our argument in the above derivation indicates that the alternative *(ii)* takes place.

The $(2n)$-dimensional Hilbert space defined by a single j-shell ($j = n - 1/2$) can be decomposed into the set of *p-irrep*'s with $p = 1, 2, \ldots, 2n$ with signatures equal to $r_x = i^{-n}, i^{-n+1}, \ldots, i^{n}$ according to formula (25) and dimensions: $\binom{2n}{0}, \binom{2n}{1}, \ldots, \binom{2n}{2n}$, respectively. In our case (*cf.* Table 1 of Ref. [1]) the basic states of the arbitrary $p - irrep$ do not correspond to a given particle number N. In fact, the particle number operator $\hat{N}$ is not defined sharply within the *irrep* [1, 2]. However, formula (25) in N-idependent.

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REFERENCES

[1] W. Satuła, Z. Szymański, to be published in Proceedings of International Workshop on Nuclear Structure Models, Oak Ridge, 1992.
[2] J. Krumlinde, Z. Szymański, *Nucl. Phys.* **A221**, 93 (1974).
[3] O. Burglin, J. Dudek, N. El-Aouad, B. Heydon, N. Rowley, W. Satuła, Z. Szymański, to be published.
[4] W. Satuła, Z. Szymański, *Acta Phys. Pol.* **B24**, 61 (1993).
[5] I.M. Gel'fand, M.L. Tsetlin, *Dokl. Akad. Nauk SSSR* **71**, 825 (1950).
[6] A. Bohr, B.R. Mottelson, *Nuclear Structure*, Vol. 1 and 2, Benjamin, New York 1969, 1975.