Some Recent Results in Classical and Quantal Chaos

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1 Introduction

I will discuss certain results concerning one-body dissipation in dynamic nuclear processes. First, the problem of one-body dissipation will be reviewed within the context of the Independent Particle Model of nuclear dynamics; this will include a brief discussion of the Wall Formula for one-body dissipation, derived in the 1970's by Jan Błocki, Władek Świątecki, and others.[1] Next, an alternate approach to this problem, within the same, purely classical model, will be discussed, along with some applications. Finally, I will briefly present some preliminary results on the quantal version of one-body dissipation.

2 One-Body Dissipation, the Independent Particle Model, and the Wall Formula

The results referred to in the title of this talk have been obtained over the past year within the framework of the Independent Particle Model of nuclear dynamics. We therefore begin with a very brief review of this model. The Independent Particle Model is based on an explicit separation between the collective degrees of freedom of the nucleus and the individual nucleons which compose it: one treats the nucleus as a gas of non-interacting particles bouncing around inside of a box whose shape is allowed to change with time. (Fig. 1) The independent particles represent the nucleons, and the shape of the box represents the shape of the nucleus, i.e. its collective degrees of freedom. Since this shape may change with time — in heavy-ion collisions, for instance, or during the process of fission — the walls of the box are not stationary; we will use the variable $\dot{n}$ to denote the normal outward velocity of the wall, at a given point on the surface of the box. Thus, by specifying $\dot{n}$ over the entire surface of the box, one specifies exactly how the shape of the box is changing at a given instant in time. In all of the results which I will present, it is assumed that while the shape of the box changes with time, its volume stays constant.

The Independent Particle Model provides a framework within which we may approach the problem of one-body dissipation in nuclear dynamics: as the particles bounce off of the moving walls, they gain or lose energy; as discussed below, there is a net increase in the kinetic energy of the gas of particles, which represents a flow of energy from the collective to the individual degrees of freedom, i.e. dissipation. Since the mechanism for this tranfer of energy is the interaction between an individual nucleon and the collective degrees of freedom — rather than, say, two-body interactions between nucleons — we call it *one-body* dissipation. Thus, one approaches the problem of one-body dissipation by posing the following question: What is the effect of the time-dependence of the shape of the box, on the total kinetic energy of the particles bouncing around inside? In the case when the walls of the box move slowly in comparison with the particles, the answer to this question is given by the Wall Formula for one-body dissipation, which provides an expression for the rate of change of E_T, the total energy of the particles:

$$\frac{dE_T}{dt} = \rho\bar{v} \oint \dot{n}^2 \, d\sigma \qquad (1)$$

Here, ρ is the total mass density of the particles inside of the box, $\bar{v}$ is the average speed of these particles, and $\dot{n}$ — the normal outward velocity of the walls of the box — is squared and integrated over the entire surface of the box. This simple formula was originally derived [1] by treating each infinitesimal surface element of the box as a tiny piston, moving either into or away from the gas of particles, according to the value of $\dot{n}$; by summing the contributions from the individual pistons to the total rate at which the energy of the gas changes, one obtains the Wall Formula.

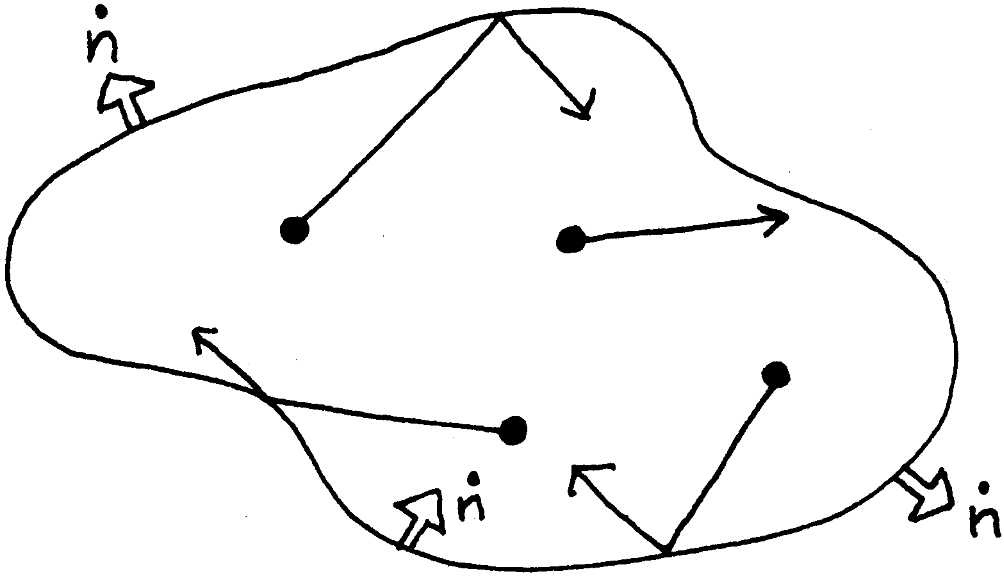

Fig. 1

In this derivation of the Wall Formula, one makes the crucial assumption that the distribution of particles inside of the box is always uniform, and the distribution of velocities isotropic. It turns out that this is equivalent to assuming that the motion of the particles inside of the box is *chaotic*. (This is determined by the shape of the box: some shapes give rise to predominantly chaotic motion, others to regular motion.) I will not dwell upon this point other than to mention that this is precisely where the concept of chaos enters into this approach to one-body dissipation.

3 An Alternate Approach: a Diffusion Equation for the Distribution of Energies

The Wall Formula has been studied and applied for a number of years. What I would like to discuss now is an alternate approach to one-body dissipation. In this approach, one uses the same model of nuclear dynamics as before, namely, a gas of non-interacting particles bouncing around inside of a box whose shape changes with time. However, instead of working with the *total* energy of the particles, one works with the *distribution* of energies, $\eta(E,t)$. This is defined in the usual way: $\eta(E,t)\,dE$ gives the total number of particles whose energy falls between E and $E+dE$, at time t. The centerpiece of this approach is an equation of motion for the distribution of energies, η. This equation of motion, presented below (eq. 2), is a diffusion equation. Without bothering with the details of its derivation, I would like to motivate for you its form, that is, to present a heuristic explanation of why the distribution of energies evolves by a process of diffusion.

As the point of departure, consider a single particle inside of this box. Since the volume of the box is kept constant as the shape changes, some portions of the wall will be moving toward the interior of the box, others away from the interior. Now, the energy of this particle will change only when it collides with the walls of the box: if the wall is moving toward the particle at the point of collision, then the particle gains energy during the collision; if the motion of the wall is away from the particle, then the particle loses energy. Thus, the energy of the particle changes in small, discrete amounts, at discrete times, corresponding to the collisions between the particle and the walls of the box. If we picture this process in terms of the motion of this particle along the energy axis, then this motion will consist of small, discrete steps, sometimes in the positive direction, sometimes in the negative. The idea that I am trying to convey here is that, if we ignore all degrees of freedom of the particle other than its energy, i.e. if we imagine the motion of the particle along the energy axis, then we can treat this motion as a sort of Random Walk.

Of course, to formally treat this motion as a Random Walk requires more justification than the simple picture presented above. Such justification, it turns out, follows from the assumption that the motion of the particle inside of the box is chaotic. In other words, if we assume that the particle moves around chaotically inside of the box, then — it can be shown — we may formally treat the motion of the particle along the energy axis as a variety of Random Walk. I will not go into the details of this matter, but mention it here to point out exactly where it is that the concept of chaos plays a role in this approach.

Imagine now that we fill our box with a large number of particles which bounce around independently of one another. Projecting out all degrees of freedom other than the energy of each particle, we get a distribution of particles along the energy axis (described by the distribution of energies, η), each independently performing a kind of Random Walk along that axis. Now, from any textbook on stochastic motion you will learn that the distribution of an ensemble of systems, each independently performing random motion, evolves by a process of diffusion. This, in a nutshell, is why the distribution of energies, η, is governed by a diffusion equation. We have a picture in our minds of an ensemble of particles "Randomly Walking" along the energy axis, and this picture suggests to us that the distribution of this ensemble along that axis evolves by a process of diffusion.

Let me now skip over a good deal of algebraic detail and present to you the central result of this approach, which is the following equation for the evolution of η:

$$\frac{\partial \eta}{\partial t} = \frac{\sqrt{2m}}{2V} \oint \dot{n}^2 \, d\sigma \, \frac{\partial}{\partial E} \left[E^2 \frac{\partial}{\partial E} \left(E^{-1/2} \eta \right) \right] \tag{2}$$

Here, m is the mass of an individual particle; V is the volume of the box (assumed constant). The integral

$$\oint \dot{n}^2 \, d\sigma \tag{3}$$

we have seen before: it is the normal outward wall velocity, squared and integrated over the surface of the box. The term to the right of this integral includes the second derivative of η with respect to E, which makes the entire equation a diffusion equation.

Now that we have this equation, we can use apply it. An obvious first question is a comparison with Wall Formula. The total energy of the gas of particles, at time t, is given by:

$$E_T(t) = \int dE\ \eta(E,t)\ E \quad . \tag{4}$$

Differentiating both sides with respect to time, then using the expression for $\partial\eta/\partial t$ given by eq. 2, then performing two integrations by parts, one obtains the following expression for the rate of change of the total energy:

$$\frac{dE_T}{dt} = \rho\bar{v} \oint \dot{n}^2\, d\sigma \quad . \tag{5}$$

This, of course, is exactly the Wall Formula. Thus, with a very simple calculation one obtains, from the diffusion equation governing the distribution of energies (eq. 2), the Wall Formula for the rate of change of the total energy of the gas, which was originally derived using a quite different approach (the "Piston Approach" described above).

I would like now to point out a feature of the Wall Formula in the form here presented. Namely, notice that it includes the factor $\bar{v}$, the average speed of the particles in the container. Now, if the total energy of the particles is changing, then so is their average speed; thus, if we want to integrate the Wall Formula over a finite amount of time, in order to obtain the total amount of energy dissipated into the gas, we need to have $\bar{v}$ as a function of time. The Wall Formula, as presented above, does not reveal how $\bar{v}$ evolves with time, and therefore does not form a closed set of equations. However, in much the same way that we obtained the Wall Formula (eq. 5) from our diffusion equation (eq. 2), we may obtain the following equation for the evolution of $\bar{v}$:

$$\frac{d\bar{v}}{dt} = \frac{3}{4V} \oint \dot{n}^2\, d\sigma \quad . \tag{6}$$

Again, the calculation required here is simple: we write an explicit expression for $\bar{v}$, differentiate with respect to time, apply eq. 2, and perform two integrations by parts to obtain eq. 6. We may call this the "Second Wall

Formula". When combined, the two Wall Formulae form a closed set of equations: one integrates eq. 6 to obtain $\bar{v}$ as a function of time, then plugs the result back into eq. 5 and integrates to obtain the total energy dissipated over a finite period of time.

Let me now discuss a final prediction that follows from the equation of motion for η.[2] The prediction is the following: given such a system of independent particles bouncing around chaotically inside of a box whose shape changes slowly with time, the distribution of particle velocities will, asymptotically with time, tend toward an exponential form. Thus, after a long time, we will have

$$f(\mathbf{v}) \sim e^{-v/c} \quad , \qquad (7)$$

where $c(t)$ is a time-dependent scaling factor, and $f(\mathbf{v})$ is the distribution of velocities (i.e. $f(\mathbf{v})\, d^3v$ is the number of particles found in a volume d^3v around the point $\mathbf{v}$ in velocity space). In a moment, I will show you the results of some numerical investigations of this prediction. However, I would like to mention here that this prediction, along with eq. 6, both of which follow from the diffusion equation for the evolution of η, are both equally well derived using the Piston Approach, by which the Wall Formula was originally derived. Thus, the two approaches represent parallel methods of addressing the same physical situation.

Let me now show you the results of some recent numerical simulations performed by J. Błocki.[3] In these, he simulated the motion of a large number of particles inside of a slowly time-dependent cavity. After a sufficiently long time, the distribution of velocities of these particles was plotted on a logarithmic scale. The six different figures shown in Fig. 2 represent six different types of shape deformations. We see that for four of these, the distribution of velocities, plotted on a logarithmic scale, makes a straight line (expect in the low-energy region). This is in agreement with the prediction mentioned above: the distribution of velocities is exponential. However, note that in the first two figures we do *not* get a straight line, i.e. the prediction fails. The reason for this is the following. For figures (c)-(f) the shape deformations of the cavity were chosen to produce chaotic motion, whereas for figures (a) and (b) the motion of the particles is regular. This again emphasizes the importance of the assumption that the motion of the particles inside of the box is chaotic: when this condition does not hold, the predictions mentioned above fail.

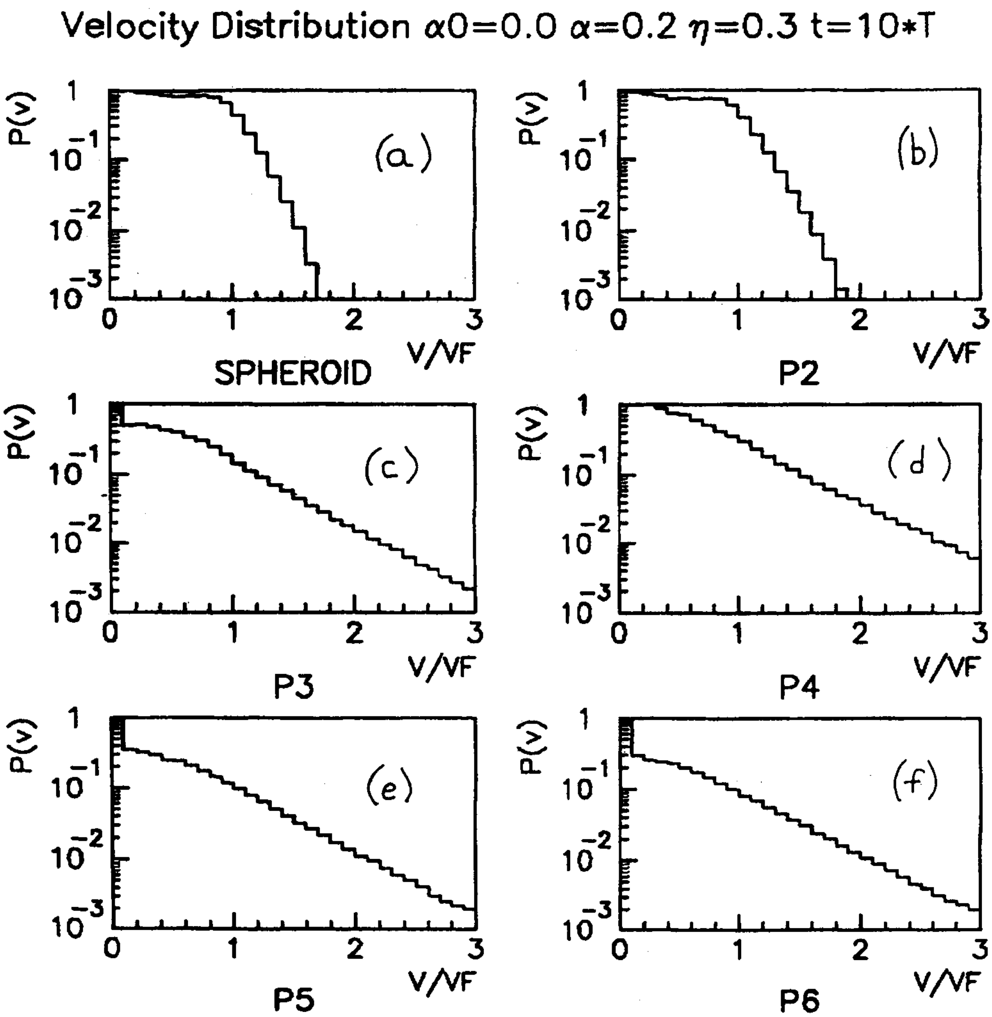
Velocity Distribution α0=0.0 α=0.2 η=0.3 t=10*T
P(v)
V/VF
(a)
SPHEROID
(b)
P2
(c)
P3
(d)
P4
(e)
P5
(f)
P6

Fig. 2

4 Preliminary Quantal Results

Finally, let me present some preliminary results concerning the *quantal* version of this problem. The model here is the same as that used above — a gas of non-interacting particles inside of a time-dependent box — only now the particles are quantized. Thus, we view each particle as a superposition of standing waves (instantaneous energy eigenstates) inside of the cavity; the time-dependent Schrodinger equation and the changing shape of the cavity determine the evolution of this superposition. By numerically solving the Schrodinger equation, we have simulated such a quantal gas. In our simulation, we used a two-dimensional box in the shape of a stadium (a shape which, classically, is known to give rise to chaotic motion), and allowed the aspect parameter — the ratio of the diameter of either endcap to the length of the flat portions (see Fig. 3) — to change sinusoidally with time, with the area of the stadium kept constant. Thus, the stadium is "pumped" periodically. As initial conditions, we filled the lowest twenty energy levels of the cavity. Then, allowing this system to evolve in time, we observed the evolution of the total energy (by which is meant the *expectation value* of the total energy) of the twenty-particle system. In Figure 4 we present some of the results: the increase in the total energy, in units of the total initial energy, is plotted as a function of time, in units of the period of pumping of the cavity. This is shown for six diferent values of the pumping speed, ranging from 50 to 300, in arbitrarily chosen units. In each plot, the thick line represents this quantal increase in energy, while the thin line represents the predictions of the classical Wall Formula. We see that at the lowest speed of 50 (figure a), there is considerable disagreement between the classical prediction and the quantal result. For speeds of 80, 150, and 200 (figures b-d), there seems to be reasonable agreement, while for the highest speeds, 250 and 300 (figures e and f), a systematic discrepancy appears.

While these results should be treated as preliminary, a plausible interpretation is the following. If the stadium were to be pumped infinitely slowly, then no excitation to higher states would occur. Thus, the energy would always be exactly equal to the sum of the lowest twenty eigenenergies, and hence would depend on the quantal details of how these eigenenergies change with the aspect parameter. As we slowly "turn on" the pumping speed, there will appear excitations to higher eigenstates. Nevertheless, for speeds which are slow enough, these excitations will be small, and the individual quantal

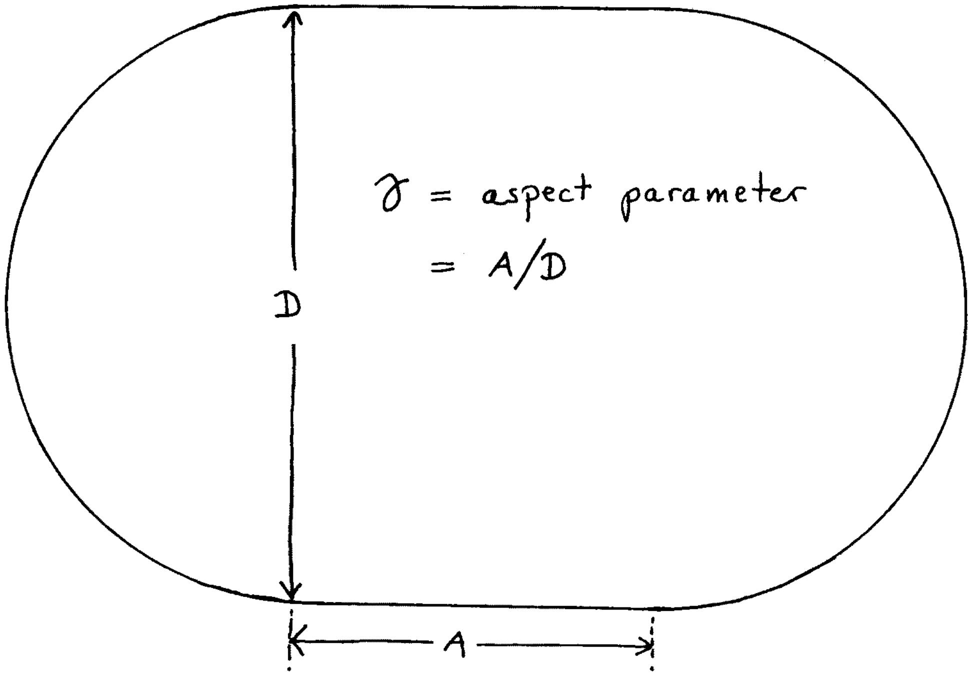

Fig. 3

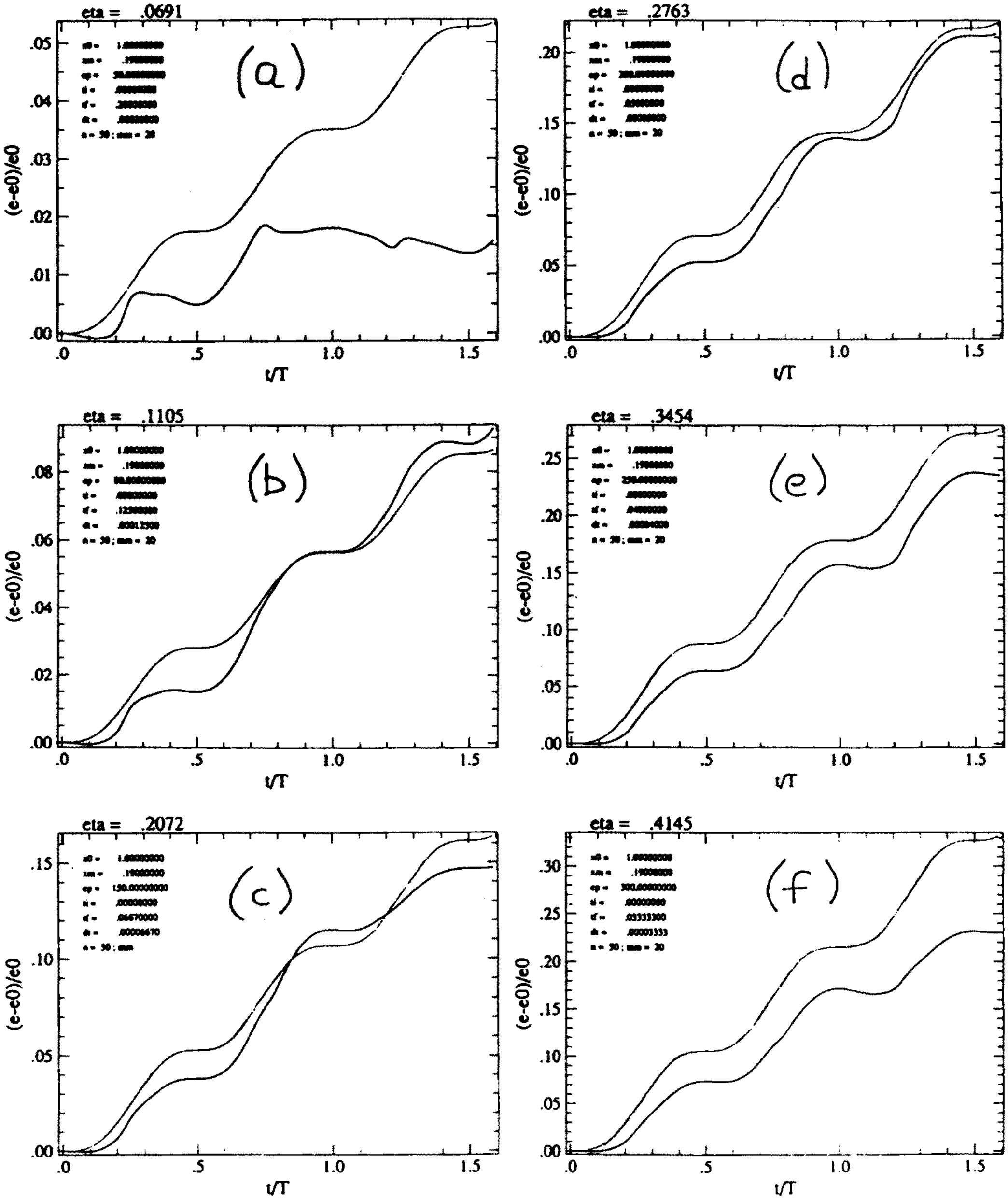

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Fig. 4

energies of the lowest states may still dominate the picture, in which case the purely classical Wall Formula cannot be expected to hold; this may explain the discrepancy in figure (a). As the pumping speed is increased, a greater number of states will participate — due to excitations — and we may expect the classical approximation to be valid. Figures (b) through (d) represent this regime. Finally, for high enough pumping speeds, the wall velocity may become comparable with the average particle velocity, violating one of the assumptions on which the Wall Formula was derived; hence the discrepancy appearing in figures (e) and (f). I would like to stress again that both the numerical results and these interpretations should be treated as preliminary: a considerable amount of work remains to be done before we can claim to have a good, systematic understanding of the quantal version of this problem.

5 References

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