A SYSTEMATIC COUPLED-CLUSTER CALCULATION OF THE GROUND STATE OF U(1) LATTICE GAUGE MODELS IN (1+1) AND (2+1) DIMENSIONS

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Dedicated to Janusz Dąbrowski in honour of his 65th birthday

We have applied the microscopic coupled-cluster method (CCM) of many-body quantum theory to the U(1) lattice gauge models in (1+1) and (2+1) dimensions. The mode couplings and plaquette correlations are studied in detail by means of a hierarchical truncation scheme. Good numerical results for the ground-state energy have been obtained for a range of coupling constants varying from strong to weak. The systematic and nonperturbative natures of the CCM are emphasized.

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1. Introduction

Lattice gauge models have been a subject of study for almost two decades by various techniques. Such techniques include perturbation theory [1] and various analytical continuations from or resummations of the perturbation series [2], finite lattice calculations [3], Monte Carlo calculations [4, 5], variational methods [6], and renormalization group methods [7]. Recently, two of the most universal techniques of quantum many-body theory, namely the method of correlated basis functions (CBF) [8] and the coupled-cluster method (CCM) [9], have also been applied to the Hamiltonian models of lattice gauge field theory, and have shown some promising results [10, 11].

In our previous paper [11], we have shown that within the one-plaquette approximation (*i.e.*, one in which no inter-plaquette correlations are retained) the U(1) lattice gauge model becomes the well-known Mathieu problem; and within a subtruncation scheme we were able to obtain essentially

the exact numerical solutions of the Mathieu equation. In this paper we extend our analysis to include the two-plaquette correlations. We also employ a similar sub-truncation scheme to study in a systematic manner the mode couplings and the plaquette correlations. We focus on the local correlations in this paper. A computer algebraic technique is employed to derive and solve the coupled CCM equations.

The outline of this paper is as follows. In Section 2 we give the general analysis of the CCM for U(1) models and briefly outline the one-plaquette approximations. Section 3 is devoted entirely to the two-plaquette approximation scheme. We employ a local approximation within this two-plaquette scheme in order to study the effect of high-order mode–mode couplings. We conclude with a summary and discussion in Section 4.

2. Coupled-cluster formalism for U(1) models and its one-plaquette approximation

For most applications a parallel can be drawn between the coupled-cluster method (CCM) and perturbation theory. Both usually start from a non-interacting (unperturbed) state and the correlations are incorporated *via* excitations from this non-interacting state or the model state as it is often called. The key difference between the two theories, however, lies in the fact that in the CCM the correlation operator takes the form of an exponentiated function. This feature not only gives the CCM the property of extensitivity for such quantities as the ground-state energy, but also takes the CCM completely out of the realm of perturbative techniques, as we shall see more clearly later. Both these and other features of the CCM have been well documented. The interested reader is referred to Ref. [9] and the references quoted therein.

Firstly in this section, we briefly outline our CCM analysis for the U(1) lattice gauge theory. We consider in this paper only the cases of (1+1) dimensions (1D chain, *i.e.*, a linear array of plaquettes) and of (2+1) dimensions (2D square lattice). The Hamiltonian for U(1) models is usually written as

$$H = -\frac{1}{2}\sum_{l}\frac{\partial^2}{\partial A_l^2} + \lambda\sum_{p}(1-\cos B_p)\,, \tag{2.1}$$

where the first summation is over all links l of the corresponding lattice , and the second over all plaquettes p; λ is the coupling constant, and $\lambda \to 0$ is referred to as the strong-coupling limit, while $\lambda \to \infty$ is the weak coupling limit; A_l is a vector potential defined on the link l, whereas the magnetic field B_p is a plaquette variable defined by the four values of A_l with $l \in p$, *i.e.*, $B_p \equiv A_1 + A_2 - A_3 - A_4$, where the order for the 4 link indices is counterclockwise around the plaquette. For the U(1) model we can easily

express H in terms of the plaquette variables alone for any dimensionality. Since we consider only the 1D and 2D models in this paper, we quote the corresponding Hamiltonians,

$$H = \sum_p \left[-2\frac{\partial^2}{\partial B_p^2} + \lambda(1 - \cos B_p) \right] + \frac{1}{2} \sum_{p,\rho} \frac{\partial^2}{\partial B_p \partial B_{p+\rho}}\,; \quad -\pi \leq B_p \leq \pi\,, \tag{2.2}$$

where ρ is a lattice vector connecting nearest-neighboring plaquettes of the lattice. In 1D ρ has two values, while in 2D it has four.

The well-known Mathieu equation can be derived from the above equation in the case of a single plaquette,

$$-2\frac{d^2}{dB^2}\psi_n(B) + \lambda(1 - \cos B)\psi_n(B) = \varepsilon_n \psi_n(B)\,; \quad -\pi \leq B \leq \pi\,, \tag{2.3}$$

and in the strong-coupling ($\lambda \to 0$) limit the Hamiltonian reduces to the simple unperturbed form,

$$H_0 = -\frac{2d^2}{dB^2}\,; \quad -\pi \leq B \leq \pi\,. \tag{2.4}$$

This unperturbed Hamiltonian has two sets of eigenstates, namely $\{\cos mB\,;\ m = 0, 1, 2, \ldots\}$ with even parity, and $\{\sin mB\,;\ m = 1, 2, \ldots\}$ with odd parity.

We now take the ground state of H_0, which is simply a constant and chosen to be unity, as our non-interacting model state for the U(1) model. The ground state of the many-body systems described by Eq. (2.2) is then constructed from these simple strong-coupling single-plaquette wave functions as the starting-point of our CCM analysis. Thus, the exact many-body ket ground state of Eq. (2.2) is written in the CCM form as,

$$|\Psi_0(\{B_p\})\rangle = \mathrm{e}^{S(\{B_p\})}|\Phi\rangle\,, \quad S(\{B_p\}) = \sum_{k=1}^{N_p} S_k(\{B_p\})\,, \tag{2.5}$$

where $|\Phi\rangle$ is the constant many-plaquette model state, N_p is the total number of plaquettes in the lattice, and the k-body correlation operators S_k are decomposed as

$$S_1 = \sum_{n=1}^{\infty} \sum_{p=1}^{N_p} \mathcal{S}_p(n) \cos(nB_p)\,, \tag{2.6a}$$

$$S_2 = \frac{1}{2!} \sum_{n_1,n_2=1}^{\infty} \sum_{p_1,p_2=1}^{{}^{\prime}N_p} \Big[\mathcal{S}^{(1)}_{p_1p_2}(n_1, n_2) \cos(n_1 B_{p_1}) \cos(n_2 B_{p_2}) + \mathcal{S}^{(2)}_{p_1p_2}(n_1, n_2) \sin(n_1 B_{p_1}) \sin(n_2 B_{p_2}) \Big]\,, \tag{2.6b}$$

and similarly for the higher-order partitions S_k with $k > 2$. The coefficients $\{\mathcal{S}_{p_1 \ldots (n_1, \ldots)}\}$ are to be determined by the coupled sets of CCM equations discussed below, and the prime on the summation in Eq. (2.6b) excludes the terms with $p_1 = p_2$. We note that the choices of Eqs (2.6a), (2.6b) are dictated by the requirements that $|\Psi_0\rangle$ should be invariant under the transformation $B_p \rightarrow B_p + 2\pi$ and should have even parity under the interchange of the sign of all $\{B_p\}$ variables simultaneously.

In deriving the coupled CCM equations for the correlation coefficients $\{\mathcal{S}_p\}$, $\{\mathcal{S}_{p_1 p_2}\}$ *etc.*, one needs to define an inner product of wave functions $\langle \tilde{g}(\{B_p\})|$ and $|f(\{B_p\})\rangle$ as,

$$\langle \tilde{g}|f\rangle = \int_{-\pi}^{\pi} \prod_{p=1}^{N_p} \left(\frac{dB_p}{2\pi}\right) \tilde{g} \cdot f \,. \tag{2.7}$$

Thus $\langle \Phi|\Phi\rangle = 1$, if the constant value of $|\Phi\rangle$ is chosen to be unity.

From the ground-state (g.s.) Schrödinger equation $H|\Psi_0\rangle = E_g|\Psi_0\rangle$, or equivalently

$$\mathrm{e}^{-S} H \mathrm{e}^{S} |\Phi\rangle = E_g |\Phi\rangle \,, \tag{2.8}$$

one obtains the ground-state energy equation (or zero-plaquette equation) by taking the inner product with the model state,

$$E_g = \langle \Phi | \mathrm{e}^{-S} H \mathrm{e}^{S} | \Phi \rangle \,; \tag{2.9}$$

and the one-plaquette equation for the coefficients $\{\mathcal{S}_p\}$ by taking the inner product with states comprising single-plaquette wave functions,

$$\langle \Phi | \cos(nB_p) \mathrm{e}^{-S} H \mathrm{e}^{S} | \Phi \rangle = 0 \,. \tag{2.10}$$

Similar equations for the two-plaquette coefficients consist of two sets of equations, and will be discussed in the next section.

Before we solve the CCM coupled equations (2.10), it is worth mentioning that within the context of the above CCM parametrization there are clearly two distinct kinds of correlations in play. In the first place one has the mode-coupling terms between different modes $\{\cos(nB_p)\,,\ \sin(nB_p)\}$ specified by the index n on the same plaquette p. Secondly, one has the more physical correlations between different plaquettes specified by the indices $\{p_i\}$. Clearly mode-coupling is included even at the one-plaquette level (*i.e.*, as specified by S_1 alone), whereas one needs to include at least S_2 as well in order to describe plaquette correlations.

While the CCM parametrization of the ground state given by Eqs (2.5)–(2.6) is in principle exact, in practice one clearly needs a truncation scheme. The most common CCM truncation scheme is the so-called SUBn scheme,

in which only those correlations described by the cluster partitions $\{S_k\}$ with $k \leq n$ are included, and those with $k > n$ are set to zero. For example, in the lowest-order SUB1 approximation, *i.e.*, the one-plaquette scheme, we make the replacement $S \to S_{\mathrm{SUB1}} = S_1$, and Eqs (2.9) and (2.10) become

$$E_g = \lambda N_p - \sum_{p=1}^{N_p} \sum_{n=1}^{\infty} n^2 \mathcal{S}_p^2(n) \,. \tag{2.11}$$

and

$$\sum_{p=1}^{N_p} \left[-\frac{1}{2}\lambda \delta_{m,1} + m^2 \mathcal{S}_p(m) + \frac{1}{2} \sum_{n,n'=1}^{\infty} nn' \mathcal{S}_p(n) \mathcal{S}_p(n') (\delta_{m,n+n'} - \delta_{m,|n-n'|}) \right] = 0 \,; \qquad m = 1, 2, \ldots . \tag{2.12}$$

After an extension of the definition of these coefficients to include the negative modes, and taking advantage of the lattice translational invariance to introduce the definition,

$$a_m \equiv m \mathcal{S}_p(m) \,; \quad \text{with } -a_m, \quad \text{and } a_0 = 0 \,, \tag{2.13}$$

we can then readily rewrite Eqs (2.11) and (2.12) as,

$$\left(\frac{E_g}{N_p} - \lambda \right) \delta_{m,0} + \frac{1}{2}\lambda (\delta_{m,1} + \delta_{m,-1}) - m a_m - \frac{1}{2} \sum_{n=-\infty}^{\infty} a_n a_{m-n} = 0 \,, \tag{2.14}$$

which is valid for all integers m.

One can now readily recover the Mathieu equation (2.3) from Eq. (2.14) by a lattice Fourier transform for the coefficients $\{a_m\}$, [11], where the one-body ground-state eigenvalue ε_0 is replaced by the intensive quantity E_g/N_p. This is not surprising since no multi-plaquette correlation effects have been included in our SUB1 approximation. To go beyond the one-plaquette scheme (or the Mathieu problem), one needs to include 2-plaquette (and higher-order) correlations. This is considered in the SUB2 scheme in the following section.

We solve Eq. (2.12) numerically by a hierarchical sub-truncation scheme, the so-called SUB1(n) scheme in which one retains at the n-th level of approximation only those coefficients a_m with $|m| \leq n$, and sets the remainder with $|m| > n$ to zero. Solutions for any finite n and λ to the general SUB1(n) approximation to Eq. (2.12) are easily obtained numerically by a simple interaction method [11]. Here we show in Fig. 1 the ground-state energy per plaquette, E_g/N_p, as a function of λ, for several SUB1(n) schemes, in

order to demonstrate the systematic and rapid improvement as n increases. The full SUB1 values, which are the *exact* solution to the Mathieu equation (2.3), are actually well represented, to the level of accuracy shown, by the SUB1(20) results for the whole range of λ values displayed. The convergence with index n of the SUB1(n) results to the full SUB1 limit is clearly quite rapid, even in the large-λ (weal-coupling) limit. This is contrasted with the corresponding perturbation theory results for the ground state of the Mathieu problem which have a natural radius of convergence at a value $\lambda_0 \leq 1.46877$ [12]. This perturbation series gives, to eight order,

$$\varepsilon_0 \xrightarrow[\lambda\to\infty]{} \lambda - \frac{1}{4}\lambda^2 + \frac{7}{256}\lambda^4 - \frac{29}{4608}\lambda^6 + \frac{68687}{37748736}\lambda^8 . \qquad (2.15)$$

We have explicitly verified that in this limit the SUB1(4) approximation for E_g/N_p exactly reproduces this series to the order shown. In general, the $\lambda \to 0$ limiting form of the g.s. energy in SUB1(n) approximation exactly reproduces the result from $2n$-th order perturbation theory, PT($2n$). We have also included several PT($2n$) results in Fig. 1 for comparison.

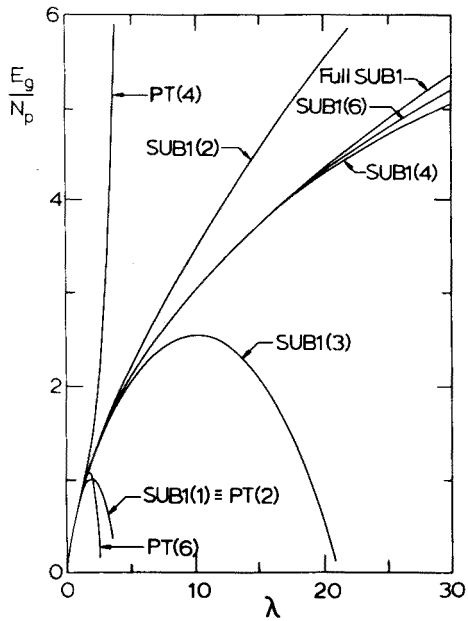

Fig. 1. Ground-state energy per plaquette of the U(1) model as a function of λ for several SUB1(n) approximations, including the full SUB1 scheme which corresponds to the exact solution of the Mathieu problem, and some results from $2n$-th order perturbation theory, PT($2n$), given by Eq. (2.15).

We shall not discuss the one-plaquette approximation any more except to mention that recently we have also obtained results for both the excitation

energy gap (glueball mass) and plaquette energy within this one-plaquette approximation, and have shown they are also exact for the corresponding Mathieu problem. We now switch our attention to the SUB2 approximations which include the plaquette correlations.

3. Two-plaquette approximation

The two-plaquette approximation of our CCM analysis is considered under the so called SUB2 scheme which is defined by the truncation of the correlation operator S in Eq. (2.5) as,

$$S \to S_{\mathrm{SUB2}} = S_1 + S_2 \,, \tag{3.1}$$

where S_1 and S_2 are written as in Eq. (2.6). Similar to Eq. (2.10) of the one-plaquette equation in the SUB1 scheme, in replacing S with S_{SUB2} one can drive the two-plaquette SUB2 equations from the Schrödinger equation (2.8) by taking the inner product with states comprising the corresponding two-plaquette variables, *i.e.*

$$\langle \cos(m_1 B_{p_1}) \cos(m_2 B_{p_2}) | \mathrm{e}^{-S_{\mathrm{SUB2}}} H \mathrm{e}^{S_{\mathrm{SUB2}}} | \Phi \rangle = 0 \,; \tag{3.2}$$

for the coefficients $\mathcal{S}^{(1)}_{p_1 p_2}(n_1, n_2)$, and

$$\langle \sin(m_1 B_{p_1}) \sin(m_2 B_{p_2}) | \mathrm{e}^{-S_{\mathrm{SUB2}}} H \mathrm{e}^{S_{\mathrm{SUB2}}} | \Phi \rangle = 0 \,; \tag{3.3}$$

for the coefficients $\mathcal{S}^{(2)}_{p_1 p_2}(n_1, n_2)$, and where in both equations, $m_1, m_2 = 1, 2, \ldots$, and $p_1 \neq p_2$.

Comparing with the SUB1 equations (2.12), the final forms of the SUB2 equations (3.2) and (3.3) are quite involved. In the following we consider a further sub-approximation within this SUB2 scheme. However, we first need to simplify the notations for the two-plaquette coefficients. Using the lattice symmetry properties, one can write

$$\mathcal{S}^{(1)}_{p_1 p_2}(n_1, n_2) \equiv b_r(n_1, n_2) \,, \quad \mathcal{S}^{(2)}_{p_1 p_2}(n_1, n_2) \equiv c_r(n_1, n_2) \,, \quad r \equiv p_2 - p_1 \,. \tag{3.4}$$

Clearly, r is simply an integer in the 1D array of plaquettes, but a lattice vector of the square lattice for the 2D case.

The approximation we make is a lowest-order local approximation in which one retains those coefficients with $|r| = 1$ only. The coefficients involved are therefore the one-plaquette coefficients $\{a_n\}$, as defined previously by Eq. (2.13), and the nearest-neighbor two-plaquette coefficients $\{b_1(n_1, n_2)\}$ and $\{c_1(n_1, n_2)\}$. These so-called SUB2-1 equations are given as follows.

Firstly, the zero-plaquette equation (or energy equation) now becomes,

$$\frac{E_g}{N_p} = \lambda - \sum_{n=1}^{\infty} a_n^2 - \frac{z}{4} \sum_{n,n'=1}^{\infty} nn' b_1(n,n') c_1(n,n') - \frac{z}{2} \sum_{n,n'=1}^{\infty} n^2 [b_1^2(n,n') + c_1^2(n,n')] . \tag{3.5}$$

Secondly, in this SUB2-1 scheme the one-plaquette equations become

$$\begin{aligned} &-\frac{\lambda}{2}(\delta_{1,m} + \delta_{-1,m}) + m a_m \\ &-\frac{1}{2} \sum_{n=-\infty}^{\infty} \left[a_n a_{n+m} + zn a_n b_1(n,m) + \frac{z}{4} m a_n c_1(n,m) \right] \\ &-\frac{z}{8} \sum_{n,n'=-\infty}^{\infty} \Big[\{n(n+m)+n'^2\} \{b_1(n,n') b_1(n+m,n') + c_1(n,n') c_1(n+m,n')\} \\ &\quad + \frac{1}{2} nn' \{b_1(n,n') c_1(n+m,n') + b_1(n,n'+m) c_1(n,n')\} \Big] = 0 . \end{aligned} \tag{3.6}$$

Finally, the two-plaquette equations, which consist of two sets of equations, became respectively,

$$\begin{aligned} &\frac{1}{4} m_1 m_2 c_1(m_1,m_2) + \frac{1}{2}(m_1^2 + m_2^2) b_1(m_1,m_2) \\ &-\frac{1}{2} \sum_{n=-\infty}^{\infty} \Big[\frac{1}{4} a_n \{m_1 c_1(m_1, n+m_2) + m_2 c_1(m_2, n+m_1)\} \\ &\quad + n a_{n+m_1} b_1(n,m_2) + n a_{n+m_2} b_1(n,m_1) \Big] \\ &-\frac{1}{8} \sum_{n,n'=-\infty}^{\infty} \Big[\frac{1}{2} \{n(n'+m_2) + n'(n+m_1)\} b_1(n,n') c_1(n+m_1, n'+m_2) \\ &\quad + \{n(n+m_1) + n'(m_2+n')\} \{b_1(n,n') b_1(n+m_1, n'+m_2) \\ &\quad + c_1(n,n') c_1(n+m_1, n'+m_2)\} \Big] = 0 ; \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} &\frac{1}{4}[a_{m_1} a_{m_2} + m_1 m_2 b_1(m_1,m_2)] + \frac{1}{2}(m_1^2 + m_2^2) c_1(m_1,m_2) \\ &+\frac{1}{2} \sum_{n=-\infty}^{\infty} \Big[\frac{1}{4} m_2 a_{n+m_1} b_1(m_2,n) + \frac{1}{4} m_1 a_{n+m_2} b_1(m_1,n) \end{aligned}$$

$$+ na_{n+m_1}c_1(n,m_2) + na_{n+m_2}c_1(n,m_1)\Big]$$
$$+\frac{1}{4}\sum_{n,n'=-\infty}^{\infty}\Big[\frac{1}{4}nn'b_1(n,n'+m_2)b_1(n',n+m_1)$$
$$+\frac{1}{4}(n+m_2)(n'+m_1)c_1(n,n'+m_1)c_1(n',n+m_2)$$
$$+ n(n+m_1)b_1(n,n'+m_2)c_1(n',n+m_1)$$
$$+ n(n+m_2)b_1(n,n'+m_1)c_1(n',n+m_2)\Big] = 0\,, \qquad (3.8)$$

where z is the coordination number of the lattice, *i.e.*, $z = 2$ for 1D and $z = 4$ for the 2D square lattice, respectively. In deriving the above equations, we have used the lattice symmetries for the coefficients, namely

$$b_r(n,m) = b_r(m,n), \qquad c_r(n,m) = c_r(m,n)\,; \qquad (3.9)$$

and

$$b_r(-n,m) = b_r(n,-m) = b_r(n,m), \quad c_r(-n,m) = c_r(n,-m) = -c_r(n,m)\,. \qquad (3.10)$$

One can see that there is still the mode-coupling to deal with as in the SUB1 scheme.

It is also possible to make further truncations beyond those made above. Within the local 1-mode approximation, for example, we retain only the single coefficient, a_1, which is the same as the SUB1(1) approximation considered earlier. In the local 2-mode approximation, however, 4 coefficients are retained. They are a_1 and a_2 from the one-plaquette equations, and $b_1(1,1)$ and $c_1(1,1)$ from the 2-plaquette correlations. The equations in this particular local 2-mode approximation, denoted as SUB2-1(2), are given, together with the energy equation, by,

$$\frac{E_g}{N_p} = \lambda - a_1^2 - a_2^2 - \tfrac{z}{4}b_1c_1 - \tfrac{z}{2}(b_1^2 + c_1^2)\,; \qquad (3.11a)$$

$$-\tfrac{\lambda}{2} + a_1 - a_1a_2 - za_1\big(b_1 + \tfrac{1}{4}c_1\big) = 0\,; \qquad (3.11b)$$

$$2a_2 + \tfrac{1}{2}a_1^2 = 0\,; \qquad (3.11c)$$

$$(1 - a_2)\big(b_1 + \tfrac{1}{4}c_1\big) = 0\,. \qquad (3.11d)$$

$$\tfrac{1}{4}a_1^2 + (1 + a_2)\big(\tfrac{1}{4}b_1 + c_1\big) = 0\,; \qquad (3.11e)$$

where we have simplified the notation further in this SUB2-1(2) scheme by writing $b_1 \equiv b_1(1,1)$ and $c_1 \equiv c_1(1,1)$. We note that the SUB2-1(2) approximation reproduces the correct coefficients in the strong-coupling limit of the U(1) models up to the fourth order [5, 2],

$$\frac{E_g}{N_p} \xrightarrow[\lambda \to \infty]{} \begin{cases} \lambda - \frac{1}{4}\lambda^2 + \frac{89}{3840}\lambda^4 + O(\lambda^6), & 1D; \\ \lambda - \frac{1}{4}\lambda^2 + \frac{73}{3840}\lambda^4 + O(\lambda^6), & 2D. \end{cases} \tag{3.12}$$

In a similar fashion the local 3-mode approximation, the SUB2-1(3) scheme, and other local higher-order multi-mode approximations, can also be made within the SUB2 equations. The SUB2-1(3) scheme in particular retains 7 independent coefficients. The 3 additional coefficients are $a_3, b_1(1,2)(= b_1(2,1))$ and $c_1(1,2)(= c_1(2,1))$. Generally, for the SUB2-1(n) scheme, the number N_n of independent variables is given by

$$N_n = \left[n\left(\tfrac{1}{2}n + 1\right)\right], \tag{3.13}$$

where the symbol $[A]$ denotes the integer part of the number A.

It is not difficult to see that as the number n increases in a SUB2-1(n) scheme, both the number of the corresponding coupled equations and the number of terms in each equation increase rapidly. It quickly becomes awkward and time-consuming to derive this equations by hand. Fortunately, this task can be easily assigned to a computer-algebraic technique. We have employed such a technique to derive and solve the SUB2-1(n) equations up to order $n = 10$. Tables I and II show the results for the ground-state energy per plaquette for the 1D and 2D modes, respectively, at various values of λ within several SUB2-1(n) schemes, together with the full SUB1 results (produced by the SUB1(20) scheme to the same accuracy shown) and the results from perturbation theory of both weak- and strong-coupling limits for comparison. In Fig. 2, the same quantity is also shown for the 2D model only. The behaviour for the 1D case is similar. The agreement with other calculations [2, 10] is very good.

Just as the strong-coupling perturbation series of Eq. (2.15) for the ground-state energy of the Mathieu problem has a finite radius of convergence, so it seems likely that its counterparts for the 1D and 2D cases of U(1) lattice gauge theory from Eq. (3.12) also do, as can be seen from the Tables. Much work in modern field theory goes into attempts to analytically continue such series outside their natural boundaries. A typical recent such attempt for the (2+1)-dimensional U(1) model [2] starts from the strong-coupling perturbation series of Eq. (3.12), utilizing the known coefficients up to $O(\lambda^{18})$, as input to generalized Padé approximants. These results are also included in Table II for comparison. As stated in the previous section for the one-plaquette approximation (*i.e.*, the Mathieu problem), we

TABLE I

Ground-state energy per plaquette at various values of λ for the U(1) model in (1+1) dimensions. Shown are the results of the SUB2-1(n) schemes with $n = 2, 3, 4, 5, 6, 8$ and 10, defined in the text, together with the full SUB1 results (given by the SUB1(20) scheme to the accuracy shown) and the results of both strong ($\lambda \to 0$) and weak ($\lambda \to \infty$) coupling limits in perturbation theory, PT4(S) and PT(W), given by Eqs (3.12) and (3.14).

Method	λ								
	0.5	1	2	3	4	5	6	8	10
SUB1	0.4391	0.7724	1.2430	1.5828	1.8597	2.1000	2.3156	2.6966	3.0315
SUB2-1(2)	0.4389	0.7689	1.1980	1.3684	1.1115	−0.3116	−6.3126	—	—
SUB2-1(3)	0.4389	0.7703	1.2319	1.5567	1.8022	1.9833	2.1047	2.1756	2.0250
SUB2-1(4)	0.4389	0.7702	1.2320	1.5615	1.8243	2.0409	2.2184	2.4663	2.6365
SUB2-1(5)	0.4389	0.7702	1.2321	1.5630	1.8308	2.0586	2.2556	2.5708	2.7884
SUB2-1(6)	0.4389	0.7702	1.2322	1.5637	1.8344	2.0694	2.2800	2.6506	2.9723
SUB2-1(8)	0.4389	0.7702	1.2322	1.5638	1.8345	2.0698	2.2811	2.6539	2.9796
SUB2-1(10)	0.4389	0.7702	1.2322	1.5637	1.8343	2.0693	2.2802	2.6523	2.9780
PT4(S)	0.4389	0.7732	1.3708	2.6273	5.9333	13.236	27.038	86.933	—
PT(W)	0.5744	0.8624	1.2697	1.5822	1.8457	2.0778	2.2877	2.6603	2.9886

again emphasize that our own SUB2-1(n) approximants themselves represent a natural extension of perturbation theory. They comprise, in effect, a well-defined analytic continuation or resummation of perturbation theory results within the context of a natural and consistent hierarchy of approximations. They may be contrasted with the rather *ad hoc* approaches based on Padé and other resummation techniques, which usually find it difficult to approach the weak-coupling limit with the correct asymptotic form unless this is built in from the start.

Finally, perturbation theory in the weak-coupling limit gives

$$\frac{E_g}{N_p} \xrightarrow[\lambda\to\infty]{} C_0\sqrt{\lambda} - \tfrac{1}{8}C_0^2 + O(\lambda^{-1/2}), \tag{3.14}$$

where the values of the constant C_0 are 0.9833 in 1D [5], and 0.9581 in 2D [10]. We note that the zero-dimensional analogue (namely, the Mathieu problem or our SUB1 approximation) has a similar weak-coupling limit, but with $C_0 = 1$. As can be seen from the Tables and from Fig. 2, the SUB2-1(2) results are not much better than their counterparts from perturbation theory, as is the case for the SUB1(1) scheme for the Mathieu

TABLE II

Same as Table II but for the (2+1)-dimensional model on a square lattice, and with additional results ot 8-th order strong-coupling perturbation theory, PT8(S), from Ref. [2]. The results from the correlated basis function (CBF) approach of Dabringhaus, Ristig and Clark [10], and from an analytic continuation of the strong-coupling perturbation series by Hamer, Oitmaa, and Zheng [2], denoted as HOZ, are also included.

Method	λ								
	0.5	1	2	3	4	5	6	8	10
SUB1	0.4391	0.7724	1.2430	1.5828	1.8597	2.1000	2.3156	2.6966	3.0315
SUB2-1(2)	0.4386	0.7652	1.1468	1.1280	0.3019	−2.8326	−15.108	—	—
SUB2-1(3)	0.4387	0.7681	1.2216	1.5371	1.7691	1.9282	2.0153	1.9720	1.6422
SUB2-1(4)	0.4387	0.7681	1.2214	1.5428	1.7994	2.0123	2.1921	2.4585	2.6365
SUB2-1(5)	0.4387	0.7681	1.2216	1.5442	1.8043	2.0237	2.2105	2.4977	2.6753
SUB2-1(6)	0.4387	0.7681	1.2217	1.5453	1.8099	2.0404	2.2482	2.6188	2.9512
SUB2-1(8)	0.4387	0.7681	1.2217	1.5454	1.8100	2.0404	2.2477	2.6142	2.9358
SUB2-1(10)	0.4387	0.7681	1.2216	1.5452	1.8095	2.0393	2.2456	2.6096	2.9278
CBF	0.4387	0.7677	1.2167	1.5335	1.7929	2.0201	2.2255	—	—
PT4(S)	0.4387	0.7690	1.3042	2.2898	4.8667	10.632	21.638	—	—
PT8(S)	0.4387	0.7673	1.1358	-0.7375	-20.873	—	—	—	—
HOZ	—	—	1.215	—	1.785	—	2.2	—	—
PT(W)	0.5627	0.8434	1.2402	1.5447	1.8015	2.0276	2.2321	2.5917	2.9150

equation. However, as n increases the SUB2-1(n) results improve rapidly. It demonstrates again that the produce correct numerical results in the weak-coupling (large λ) region, the retention of the higher-order mode-couplings is necessary.

Different sub-approximation schemes can also be made from the SUB2 equations. For example, one can include only low-order mode-mode coupling but retain long-range plaquette-plaquette correlations. Thus, for example, one might retain the coefficients $b_r(1,1)$ and $c_r(1,1)$ with all possible values of r. From the experience of the application of the CCM to spin lattice models [13] this approximation should in practice suffice to predict a possible phase transition. We do not discuss it further here, except to note that our preliminary results do not suggest a critical behavior for any finite value of λ for either the 1D or 2D U(1) models. These findings are in agreement with the general consensus that these systems exhibit no phase transition.

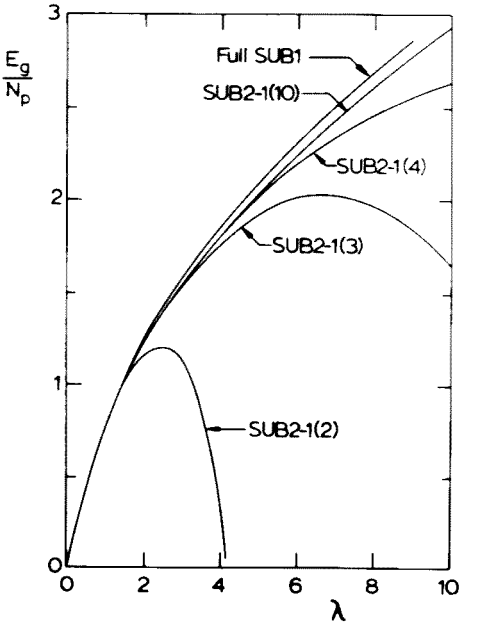

Fig. 2. Ground-state energy per plaquette of the U(1) model in (2+1) dimensions as a function of λ for several SUB2-1(n) approximations ($n = 2, 3, 4, 10$), and the full SUB1 scheme which corresponds to the exact solution of the Mathieu problem.

4. Summary and conclusion

In this paper we have extended our CCM analysis to include the plaquette correlations in the ground state of the U(1) lattice gauge models. We focus on a local approximation scheme and systematically include the higher-order mode couplings which are essential for the weak-coupling region.

From the numerical results presented to date it should be apparent that the key advantage of our CCM analysis for the U(1) lattice gauge models, as for the many systems in quantum many-body theory to which it has already received wide applications, lies in its systematic microscopic approach. The method is manifestly nonperturbative from the outset, although, as we have shown, easy contact can always be regained with perturbation theory. We have demonstrated that the CCM results are valid outside the realm of validity of perturbation theory. In this sense the method provides a very natural and automatic extension of the perturbative results. By contrast, Padé and similar resummation techniques can be extremely unreliable, particularly when no or only limited information is available for the asymptotic behavior in the continuum (weak-coupling) limit. We believe that a similar CCM analysis should also be applicable to such non-Abelian theories as the

SU(2) lattice gauge models, and we hope to report similar results for these models soon.

Furthermore, although the actual coupled equations for the retained cluster configurations coefficients naturally rapidly become awkward and time-consuming to generate by hand as the order of the approximation increases, nevertheless, they are extremely amenable to generation by computer-algebraic techniques. Our first attempt to employ computer-algebraic techniques has proved to be very fruitful. It is apparent to us that more comprehensive applications of such computer-algebraic techniques within the framework of our CCM analysis may produce results which are fully competitive with those obtainable by alternative techniques.

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