# DEUTERON RADIUS AND THE TWO-NUCLEON EFFECTIVE RANGE PARAMETERS

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*Dedicated to Janusz Dąbrowski in honour of his 65th birthday*

The square of the deuteron matter radius divided by the square of the scattering length can be expanded in powers of a parameter proportional to the effective range at the deuteron pole. The coefficients of this expansion beyond second order depend on the shape of the potential. We consider the third and fourth order terms for several simple potentials, both local and separable.

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## 1. Introduction

The low energy behavior of the two-nucleon system is an important testing ground for proposed realistic nucleon-nucleon potential models. The deuteron properties and the low-energy scattering parameters are known very precisely from experiment, and thus any interaction suggested to describe the nuclear force ought to predict these quantities correctly. Since nuclear force models are fitted to the elastic scattering data as well as the deuteron properties, they are approximately equivalent on the energy shell. However, some properties, such as the deuteron matter radius, also depend on the nuclear wavefunction in the potential region. Thus, in order to pin down the potential, it is necessary to fit not only nucleon-nucleon phase shifts, but also the deuteron matter radius.

A particular relationship between a pair of low-energy quantities, first studied by Klarsfeld [1], *et al.*, is that of the triplet scattering length, $a_t$, and the deuteron matter radius $r_D$. When these quantities obtained from the various realistic potentials available at the time of their study, are plotted on a scattering length versus matter radius graph, one obtains practically a straight line which passes below and to the right of the experimental region. In other words, the predicted matter radius of the deuteron is too large. The underlying reason for this relationship, which also holds for simple S-state models is not yet known. Explanations for the discrepancy in terms of effects due to meson-exchange [1], relativity [2], or quarks [3] have not been successful. There are indications that non-locality might help [4], or possibly a change in the local attraction along lines suggested by the Moscow group [5]. Thus, recently, Mustafa [6] has found it possible to fit the smaller deuteron matter radius while keeping the fit to scattering observables, but his potential differs by about 15% from OPEP already at internucleon spacing of 2 fm, and by more at smaller distances.

## 2. Relation between deuteron radius and effective range parameters

It has been previously shown [4, 7] that the dimensionless ratio $\sqrt{8}r_D/a_t$ is relatively model independent. The small model dependence can be parameterized in terms of low energy parameters. Various examples of S-state potentials were discussed by the present authors in a previous paper [8], including both local and separable potentials. It was found that simple nonlocal potentials can lead to a reduction in the value of $r_D/a_t$, compared to local potentials, in the direction indicated by experiment. In the present paper, we extend these results. We have found it more convenient to consider not the ratio mentioned above, but its square. To get a brief orientation, consider the case of a sticky core potential. As in Ref. [8], we choose units so that $\hbar^2/2m = 1$, where $m$ is the reduced two-nucleon mass.

## *2.1 Sticky core potential*

A sticky core potential has an infinite repulsion inside a core of radius $c$ and an infinite deep and narrow attraction just outside, at a distance $d = c + \eta$, and there is no potential at larger distances.

Assuming $\eta$ to approach zero, we can write the narrow attraction as

$$V = \frac{\lambda}{d-c}\delta(r-d),$$

At any energy, the S-state radial wavefunction $u(r)$ vanishes inside the core, but the logarithmic derivative of the wavefunction at the outside of the potential:

$$F = \left(\frac{d\ln u}{dr}\right)_{d+} = \left(\frac{u'}{u}\right)_{d+}$$

is independent of energy. In terms of the potential parameters, we have:

$$F = \frac{1+\lambda}{d-c}.$$

The potential can be characterized by its scattering length, $a_t$. Let the zero-energy wave function outside the core be

$$u = 1 - \frac{r}{a_t},$$

then

$$F = -\frac{1}{c - a_t}.$$

Suppose the potential is strong enough to give a bound state at energy $-\alpha^2$. Then the wavefunction outside the potential is proportional to $e^{-\alpha r}$, so that $F = -\alpha$, and $\alpha a_t = 1 + \alpha c$.

For the sticky core potential, we can calculate the deuteron matter radius exactly, and obtain:

$$r_D^2 = \frac{1}{4}\frac{\int_c^\infty e^{-2\alpha r} r^2 dr}{\int_c^\infty e^{-2\alpha r} dr}$$

$$= \frac{1 + 2\alpha c + 2\alpha^2 c^2}{8\alpha^2}.$$

Now we will find it convenient to work with the ratio $8r_{\rm D}^2/a_t^2$.

For the sticky core potential, we obtain:

$$\frac{8r_{\rm D}^2}{a_t^2} = \frac{1+2\alpha c+2\alpha^2c^2}{(1+\alpha c)^2}$$
$$= 1+ \ (\alpha c)^2 - 2(\alpha c)^3 + 3(\alpha c)^4 + \cdots .$$

Finally, let us use the effective range $r_m$ to define

$$z = \frac{\alpha r_m}{2} = 1 - \frac{1}{\alpha a_t},$$

instead of the core radius, as the independent variable.

For the sticky core, we have:

$$z = \frac{\alpha c}{1+\alpha c},$$

and

$$\frac{8r_{\rm D}^2}{a_t^2} = 1+ \ z^2 .$$

*2.2 Series expansion for general potential*

For a more general potential, there are also higher order terms in $z$, and our expansion can be written as:

$$\frac{8r_{\rm D}^2}{a_t^2} = 1+ \ z^2 + \ d_3 z^3 + \ d_4 z^4 + \cdots .$$

However, there is no term linear in $z$, the quadratic term always has coefficient 1. Since, due to the small deuteron binding energy, $z$ is small,

$$z = \frac{0.2316 * 1.755}{2} = 1 - \frac{1}{0.2316 * 5.419} = 0.2032$$

we obtain, in the model independent approximation $(d_3 = d_4 = 0)$ :

$$r_{\rm D} = \frac{a_t}{\sqrt{8}}\sqrt{(1+z^2)} = 1.955 \text{ fm},$$

which is close to the experimental result!

We must, however, include the effect of the higher order terms in $z$ as well. Now it was shown in our earlier work that the $d$ coefficients can be expressed in terms of two parameters $P$ and $J$. $P$ is the well known shape parameter in the expansion of $k$ ctn $\delta_0$ in powers of $k^2$, while $J$ is an integral involving the deuteron wavefunction. Details are given in Ref. [8]. It was shown that

$$d_3 = 2 + 16P_0 - 4J_0 .$$

(In Eq. (2.24) of Ref. [8], the expansion parameter is $\alpha r_0/2$ rather than $z$, but the two expansion parameters are the same to order $\alpha^2$ and hence yield the same expressions for $d_3$.) Here the subscript 0 of $P$ and $J$ indicates that these quantities can be obtained from a potential which gives a zero energy bound state, *i.e.* infinite scattering length. These results can be extended, however, to potentials with finite scattering length, *i.e.* non-vanishing $z$. Thus we can define:

$$P = P_0 + zP_1 + \cdots ,$$

$$J = J_0 + zJ_1 + \cdots .$$

Then it can be shown that

$$d_4 = 4 + 32P_0 + 16P_1 - 4J_1 .$$

### 3. Results

We shall be particularly interested in the value of $z$ that "corresponds" to the deuteron binding energy and the associated $n-p$ triplet scattering length. This is $z_d = 0.2032$. In Table I, we give the expressions and the values of $P_0$ and $P_1$ for 4 local potentials and 3 rank one separable potentials (Yamaguchi, the separable Yukawa and the separable square well). The expressions were obtained by using the computer algebra package, MAPLE. The numerical values of $P_0$ and $J_0$ are, of course, those given for $P$ and $J$ in our previous paper. The expressions for the Hulthen potential contain the value of the Riemann zeta function $\zeta(w)$ [10] evaluated at $w = 3$. In Table II, we give the values of $J_0$ and $J_1$. The values of $d_3$ and $d_4$ in Table III were obtained by the direct expansion of the quantity $8r_{\rm D}^2/a_t^2$ for each model. These values were checked by using the formulae for $d_3$ and $d_4$ in terms of the $P_i$ and $J_i$, $(i = 0, 1)$. We note that, except for the Hulthen potential, $|d_4| < |d_3|$. We have calculated the higher order terms $(d_5, d_6, ...)$ and have found that each series converges rapidly, except for the Hulthen case. The Hulthen potential behaves differently because the radius of convergence of the series is much smaller. Introducing $d = V_0/\mu^2$, where $V_0$ is the depth and $\mu$ is the range [9], we find that as $d$ increases from unity, $z$ increases from zero to a maximum of about 0.24, decreases to zero at $d = 2.43$ and has a pole at about $d = 2.86$. This means that there exists a particular value of $V_0$ ($\mu$ fixed) for which there is a low lying bound state ($\alpha \approx \mu \approx 0.7$) but the corresponding potential gives a scattering length of zero. (The series is not unlike

$$1 - 2z + 4z^2 - 8z^3 + 16z^4 - \cdots .$$

which converges for $|z| < 1/2$.) In Table IV, we give (for the Yamaguchi case) the values of $8r_D^2/a_t^2$ from the series truncated at the term in $z^4$ and the corresponding exact model values. For the "deuteron" value of $z$=0.2032 ($\nu$=0.1658, with $\nu$ being defined as the ratio $\alpha/\beta$, $\beta$ being the range of the Yamaguchi potential), the agreement is very good. Calculating the radius from the series for this value of $z$, we obtain $r_D$=1.9343 fm which agrees to within 0.01% with the exact value of 1.9345 fm. A similar accuracy was obtained for the delta shell potential.

TABLE I

The shape parameter coefficients $P_0$ and $P_1$ for some simple local and separable potentials.

| Potential | $P_0$ | $P_1$ | $P_0$ | $P_1$ |
|---|---|---|---|---|
| Sticky core | $-1/24$ | $-1/24$ | −0.0417 | −0.0417 |
| Square well | $-(\pi^2-6)/12\pi^2$ | $(\pi^2-12)/6\pi^2$ | −0.0327 | −0.0360 |
| Delta shell | $-3/80$ | $-27/640$ | −0.0375 | −0.0422 |
| Hulthen | $-11/216+(2/27)\zeta(3)$ | $-4/81+(4/27)\zeta(3)$ | 0.0381 | 0.1287 |
| Yamaguchi | $-1/54$ | $-2/81$ | −0.0185 | −0.0247 |
| Yamaguchi 2* | $-1/32$ | $-1/32$ | −0.0312 | −0.0312 |
| sep sq well | $(\pi^2-12)/6\pi^2$ | $2(5\pi^4-84\pi^2+336)/3\pi^4$ | −0.0360 | −0.0411 |
| sep Yukawa | 0 | 0 | 0.0000 | 0.0000 |

TABLE II

The coefficients $J_0$ and $J_1$ for the potentials of Table I.

| Potential | $J_0$ | $J_1$ | $J_0$ | $J_1$ |
|---|---|---|---|---|
| Sticky core | $1/3$ | $1/2$ | 0.3333 | 0.5000 |
| Square well | $4(\pi^2-6)/3\pi^2$ | $-8(\pi^2-12)/3\pi^2$ | 0.5228 | 0.5756 |
| Delta shell | $9/20$ | $81/160$ | 0.4500 | 0.5063 |
| Hulthen | $10/9$ | $16/27$ | 1.1111 | 0.5926 |
| Yamaguchi | $10/9$ | $16/27$ | 1.1111 | 0.5926 |
| Yamaguchi 2* | $7/8$ | $3/4$ | 0.8750 | 0.7500 |
| sep sq well | $4(\pi^2-6)/3\pi^2$ | $8(\pi^4+20\pi^2-288)/\pi^4$ | 0.5227 | 0.5586 |
| sep Yukawa | $13/8$ | $3/8$ | 1.6250 | 0.3750 |

TABLE III

The expansion coefficients $d_3$ and $d_4$ for the potentials of Table I ($d_0 = d_2 = 1$, $d_1 = 0$).

| Potential | $d_3$ | $d_4$ | $d_3$ | $d_4$ |
|---|---|---|---|---|
| Sticky core | 0 | 0 | 0.0000 | 0.0000 |
| Square well | $(120 - 14\pi^2)/3\pi^2$ | $4(11\pi^2 - 108)/3\pi^2$ | −0.6138 | 0.0764 |
| Delta shell | $-2/5$ | $1/10$ | −0.4000 | 0.1000 |
| Hulthen | $-88/27 + (32/27)\zeta(3)$ | $-64/81 + (128/27)\zeta(3)$ | −1.8346 | 4.9085 |
| Yamaguchi | $-74/27$ | $52/81$ | −2.7407 | 0.6420 |
| Yamaguchi 2* | $-2$ | $1/2$ | −2.0000 | 0.5000 |
| sep sq well | $-2/3$ | $4(23\pi^4 - 1200\pi^2 + 9600)/3\pi^4$ | −0.6667 | −0.0427 |
| sep Yukawa | $-9/2$ | $5/2$ | −4.5000 | 2.5000 |

TABLE IV

Comparison of the series (up to $z^4$) and the exact values for the Yamaguchi potential.

| $\nu$ | $z$ | series | exact |
|---|---|---|---|
| 0.1 | 0.1322 | 1.0113 | 1.0114 |
| 0.1658 | 0.2032 | 1.0194 | 1.0195 |
| 0.2 | 0.2361 | 1.0217 | 1.0218 |
| 0.3 | 0.3195 | 1.0194 | 1.0201 |
| 0.4 | 0.3878 | 1.0051 | 1.0070 |
| 0.5 | 0.4444 | 0.9820 | 0.9859 |

We have also considered a new separable potential which has the form factor

$$f(p) = \beta_1\beta_2/\sqrt{(\beta_1^2 + p^2)(\beta_2^2 + p^2)}\,.$$

For $\beta_1 = \beta_2$, we retrieve the Yamaguchi form factor. For $\beta_2 >> \beta_1$, we get the Yukawa form factor (see, for example, Ref. [9]).

TABLE V

The two term separable potential.

| | |
|---|---|
| $P_0$ | $(-1/2)(1+\gamma)(4\gamma^2+8\gamma+1)(4\gamma+3)^{-3}$ |
| $P_1$ | $(-2)(1+\gamma)(2\gamma+3)(8\gamma^3+16\gamma^2+7\gamma+1)(4\gamma+3)^{-5}$ |
| $J_0$ | $2(28\gamma^5+80\gamma^4+96\gamma^3+95\gamma^2+58\gamma+15)(4\gamma+3)^{-3}(1+\gamma)^{-2}$ |
| $J_1$ | $16(48\gamma^7+266\gamma^6+586\gamma^5+750\gamma^4+646\gamma^3+347\gamma^2$ $+96\gamma+9)(4\gamma+3)^{-5}(1+\gamma)^{-2}$ |
| $d_3$ | $(-2)(64\gamma^5+128\gamma^4+72\gamma^3+117\gamma^2+114\gamma+37)(4\gamma+3)^{-3}(1+\gamma)^{-2}$ |
| $d_4$ | $(-4)(128\gamma^7+864\gamma^6+1568\gamma^5+1720\gamma^4+1840\gamma^3$ $+1105\gamma^2+178\gamma-39)(4\gamma+3)^{-5}(1+\gamma)^{-2}$ |

Finally, in Table V, we give our results for the rank two separable Yamaguchi potential, denoted here by Yamaguchi 2:

$$-f_1(p)f_1(p')+\eta f_2(p)f_2(p')$$

with $f_i(p)=\beta_i^2/(p^2+\beta_i^2)$. Our results are for the limit $\eta\to\infty$. Each of the parameters is a smooth function of $\gamma=\beta_1/\beta_2$ and in Tables I–III we give the values for the extreme cases $\gamma=0$ (which is equivalent to the rank one separable Yamaguchi potential) and $\gamma=\infty$ (Yamaguchi 2*). Other values of $\gamma$ produce intermediate values of the parameters.

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