# A SIMPLIFIED RELATIVISTIC APPROACH FOR THE STUDY OF THE SINGLE-PARTICLE ENERGIES IN NUCLEI AND HYPERNUCLEI

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*Dedicated to Janusz Dąbrowski in honour of his 65th birthday*

An account is given of a simplified relativistic approach for the study of single-particle energies for neutrons in nuclei and for a Λ-particle in hypernuclei and also for their average variation with the mass number. The analysis is based on the Dirac equation with scalar potential and fourth component of vector potential, which are assumed to be of rectangular shapes with the same radius. The energy eigenvalue equation is obtained analytically for every bound state, as well as the large and small component of the wave function. Attempts are also made to derive in certain cases approximate analytic expressions for the single particle energies as functions of the mass number. Numerical results, mainly for hypernuclei, are finally given and discussed.

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## 1. Introduction

As it is well known, there is substantial experimental information on single nucleon energies [1], in spite of existing ambiguities. This information comes from various nuclear reactions like (p,2p), (e,e'p) and one nucleon transfer reactions.

Much progress has also been made by experimentalists in the past and recent years for the measurement of Λ-particle energies in hypernuclei by means of emulsion techniques, the strangeness exchange reaction $(K^-, \pi^-)$ and the associated production reaction $(\pi^+, K^+)$ [2]. These experimental

advances have stimulated considerable theoretical work (see Ref. [3] and references therein).

The purpose of this paper is to give an account of a simplified relativistic approach for the study of the single particle energies for neutrons in nuclei and for a Λ-particle in hypernuclei and for their average variation with mass number. Most of its content has been published in recent years or is in the process of publication. More details can be found in the relevant references.

Our analysis is based on the Dirac equation with scalar potential $U_s(r)$ and fourth component of vector potential $U_v(r)$ . As it is well known, the advantage in such a general approach is that the spin-orbit potential springs out naturally from the theory and one does not have to introduce it "ad hoc" as in the non relativistic Shrödinger equation. The latter equation could be also used instead, but there is no much complication in the numerical work by using the Dirac equation.

The potentials $U_S(r)$ and $U_V(r)$ are assumed for our purposes to be of rectangular shape and of the same radius. This is an oversimplification of the actual Λ-nucleus interaction and in this sense the present approach is less satisfactory than other more sophisticated ones. It has the advantage, however, that it leads to an exact energy eigenvalue equation and to an analytic expression for the wave function. The eigenvalue equation may be used subsequently in deriving in certain cases approximate analytic expressions for the single particle energies as functions of the mass number. These expressions play the role of elementary "semi-empirical mass formulae". It should be also noted that the assumed potential model gives rather satisfactory single particle energies for low-lying states in sufficiently heavy hypernuclei.

In Section 2 the energy eigenvalue equation for the ground and excited states is given, as well as the analytic expressions for the large and small component of the wave function. In Section 3 the problem of the derivation in certain cases of explicit expressions of the single particle energies in terms of the mass number is discussed and a number of such expressions are given. The final Section is devoted to numerical results and comments.

## 2. The eigenvalue equation and the wave function for the ground and excited states

As it was mentioned in the introduction the Dirac equation we are using is of the form [4]:

$$\left(c\vec{\alpha}\vec{p} + \beta\mu c^2 + \beta U_S(r) + U_V(r)\right)\Psi = E\Psi \tag{1}$$

which has attracted much interest in the past years (see Ref. [5] and references therein). In this equation $\vec{\alpha}$ and $\beta$ are the usual Dirac matrices, $\Psi$ the Dirac spinor and $E$ the total energy $E = \varepsilon + \mu c^2$.

By expressing the Dirac spinors in terms of the large (G) and small (F) component:

$$\Psi = \Psi_{Nljm} = \begin{pmatrix} \frac{iG_{Nlj}(r)}{r} \\ \frac{F_{Nlj}(r)\vec{\sigma}\vec{r}}{r} \end{pmatrix} \varphi_{ljm}\,, \tag{2}$$

where

$$\varphi_{ljm} = (Y_l^{m_l} \otimes \chi_{1/2}^{m_s})_{jm} \tag{3}$$

and $\chi_{1/2}^{m_s}$ are the Pauli spinors, one may derive from (1) the following Schrödinger-type equation (for central potentials) [6, 7]

$$g''(r) - \left(\frac{l(l+1)}{r^2} + \frac{2\mu}{\hbar^2}(V_{\text{centr}} + V_{\text{s.o.}} - \varepsilon)\right) g(r) = 0\,, \tag{4}$$

where

$$\begin{aligned} g(r) &= D^{-1/2}(r)G(r)\,, \\ D(r) &= \frac{1}{\hbar c}(2\mu c^2 + \varepsilon + U_-(r))\,, \end{aligned} \tag{5}$$

$G$ is the large component of the Dirac wave function and

$$\begin{aligned} V_{\text{centr}}(r,\varepsilon) = U_+(r) + \frac{\hbar^2}{2\mu}\Big(&\frac{1}{\hbar^2 c^2}\big(U_+(r) - \varepsilon\big)\big(U_-(r) + \varepsilon\big) \\ &- D^{-1}(r)D'(r)r^{-1} - \big(2D(r)\big)^{-1}D''(r) + \frac{3}{4}D^{-2}(r)\big(D'(r)\big)^2\Big)\,, \end{aligned} \tag{6}$$

$$V_{\text{s.o}}(r,\varepsilon) = -\frac{\hbar}{2\mu}\,\frac{1}{2\mu c^2 + \varepsilon + U_-(r)}\,\frac{1}{r}\,\frac{dU_-(r)}{dr}\vec{l}\vec{\sigma}\,. \tag{7}$$

The potentials $U_\pm(r)$ are defined as follows

$$U_\pm(r) = U_S(r) \pm U_V(r) \tag{8}$$

We consider the case in which $U_+(r)$ and $U_-(r)$ are rectangular-wells with the same radius $R$ and depths $D_+$ and $D_-$ respectively *i.e.*

$$U_\pm(r) = -D_\pm\big(1 - \Theta(r - R)\big)\,, \tag{9}$$

where $\Theta$ is the unit step function and $R = r_0 A^{1/3}$ [8]. $A = A_c$ is the mass number of the core system.

The generalized Dirac equation with the rectangular potentials we are discussing may be solved analytically, for every bound state [9]. The expressions for G and F are given in terms of spherical Bessel functions $j_l$ and spherical Hankel functions of the first kind $h_l^{(1)}$ and are the following

$$G(r) = \widetilde{N} nr \left( (1 - \Theta(r - R)) j_\ell(nr) + \Theta(r - R) \frac{j_\ell(nR)}{h_\ell^{(1)}(in_0 R)} h_\ell^{(1)}(in_0 r) \right), \tag{10}$$

$$F(r) = \widetilde{N} nc\hbar \Big( (1 - \Theta(r - R)) \frac{1}{\varepsilon + 2\mu c^2 - D_-} (nr j_{\ell-1}(nr) + (k - \ell) j_\ell(nr)) + \Theta(r - R) \frac{1}{\varepsilon + 2\mu c^2} \frac{j_\ell(nR)}{h_\ell^{(1)}(in_0 R)} \left( in_0 r h_{\ell-1}^{(1)}(in_0 r) + (k - \ell) h_\ell^{(1)}(in_0 r) \right) \Big), \tag{11}$$

while the energy eigenvalue equation is [9]

$$\left( 1 - \frac{D_-}{2\mu c^2 + \varepsilon} \right) \frac{in_0 R h_{l-1}^{(1)}(in_0 R)}{h_l^{(1)}(in_0 R)} = \frac{(k - l) D_-}{2\mu c^2 + \varepsilon} + \frac{nR j_{l-1}(nR)}{j_l(nR)}. \tag{12}$$

In these expressions the quantities $n$ and $n_0$ are defined as follows:

$$n = \left( \frac{2\mu}{\hbar^2} (D_+ + \varepsilon) \left( 1 - (D_- - \varepsilon)(2\mu c^2)^{-1} \right) \right)^{1/2}, \tag{13}$$

$$n_0 = \left( \frac{2\mu}{\hbar^2} \left( - \varepsilon (1 + \varepsilon (2\mu c^2)^{-1}) \right) \right)^{1/2} \tag{14}$$

and $k = \pm(j + 1/2)$ for $j = (l \mp 1/2)$. The quantum numbers in $G$, $F$, $\varepsilon$ and $\widetilde{N}$ have been suppressed.

## 3. Approximate analytic expressions for the single particle energie

The disadvantage with eigenvalue equation (12 ) is that in general it can not be solved explicitly for $\varepsilon$ and thus it is not possible to have an explicit expression of the energy in terms of the potential parameters. We may show, however, that in certain cases this can be achieved approximately in a rather satisfactory way. Thus, one is led to fairly simple approximate analytic expressions for $\varepsilon$. Before proceeding we recall that usually in practice $D_-$ is quite smaller than $2\mu c^2$ (at least for hypernuclei) and $D_+$ much smaller than $D_-$.

Let us consider the case in which $nR$ and $n_0R$ are sufficiently larger than $l/2(l+1)$ (see also Ref. [10]) so that we may use the following asymptotic form for $j_l(nR)$ and $h_l^{(1)}(in_0R)$

$$j_l(nR) \simeq \frac{1}{nR}\sin(nR-\frac{l\pi}{2}), \quad h_l^{(1)}(in_0R) \simeq \frac{-1}{n_0R}\exp\left(-n_0R - il\frac{\pi}{2}\right). \quad (15)$$

Thus, Eq. (12) may be written in the form:

$$n_0R + nR\cot(nR - \frac{l\pi}{2}) = \frac{D_-}{2\mu c^2+\varepsilon}[n_0R - (k-l)]. \quad (16)$$

It is seen immediately that for the ground state: $1s_{\frac{1}{2}}$, $(l=0, k=-1)$ this equation coincides with the exact eigenvalue equation for this state (see appendix of Ref. [8]). For the excited states, however, it is approximate.

Eq. (16) may be written in the following form which is suitable for our treatment:

$$n^2R^2 = \frac{2\mu}{\hbar^2}D_+R^2(1+C_R)\sin^2[(2N+l)\frac{\pi}{2} - nR], \quad (17)$$

where $N = 1, 2, 3, \ldots$, $l = 0, 1, 2, \ldots$ and

$$C_R = \left(\frac{\varepsilon}{D_-} - \frac{\varepsilon}{D_+} - 1\right)D_-(2\mu c^2)^{-1} + \frac{1}{\frac{2\mu}{\hbar^2}D_+R^2}\left(-(n_0R)^2 + \left(n_0R(1-\frac{D_-}{2\mu c^2+\varepsilon}) + \frac{(k-l)D_-}{2\mu c^2+\varepsilon}\right)^2\right). \quad (18)$$

We are interested in the case of a well of sufficiently large depth $D_+$ and radius $R$. Thus if we write Eq. (17) in terms of the arcsin of the small quantity

$$x = nR\left(\left(\frac{2\mu}{\hbar^2}\right)D_+R^2(1+C_R)\right)^{-1/2},$$

we may keep only the leading term in the expansion of $\arcsin x$ and obtain the eigenvalue equation in the approximate form

$$(nR)^2 = \frac{(2N+l)^2\pi^2}{4(1+\lambda_R)^2}, \quad (19)$$

where

$$\lambda_R = \lambda_0(1+C_R)^{-1/2} \quad (a), \qquad \lambda_0 = \left(\frac{2\mu}{\hbar^2}D_+R^2\right)^{-1/2} \quad (b). \quad (20)$$

It is interesting to note that if we neglect completely the terms of order $(2\mu c^2)^{-1}$ the eigenvalue equation (12) goes over to the corresponding non relativistic one. In this case Eq. (19) may be written in the form

$$\varepsilon_{NR_1} = -D_+ + \frac{\hbar^2}{8\mu}\frac{(2N+l)^2\pi^2}{(1+\lambda_0)^2R^2}\,. \tag{21}$$

This is an approximate expression of the non-relativistic energy for a particle of mass $\mu$ in a rectangular-well potential of depth $D_+$ and radius $R$. In the case of the $s$-states the above expression reduces to a known approximate expression [11] derived from the corresponding Schrödinger eigenvalue equation.

An approximate relativistic expression for the energy may be derived from expression (19) if the unknown energy which appears in terms which are expected to be small is estimated by means of (21). In this way we obtain the following approximate expression for the relativistic energy which we shall denote by $\varepsilon_{R1}$

$$\varepsilon_{R_1} = -D_+ + \frac{\hbar^2}{8\mu^*_{g_1}}\frac{(2N+l)^2\pi^2}{(1+\lambda_{R1})^2R^2}\,, \tag{22}$$

where $\mu^*_{g_1}$ is a sort of "effective mass" given by

$$\mu^*_{g_1} = \mu\left(1 + (\varepsilon_{NR1} - D_-)(2\mu c^2)^{-1}\right) \tag{23}$$

and $\lambda_{R1}$ is given by (20a) in which the energy in the expression of $C_R$ has been substituted by $\varepsilon_{NR1}$ (expression (21)). The above procedure may be iterated. Thus we may use as expressions for the energy in $\mu^*_{g_1}$ and $\lambda_{R_1}$ which appear in (22) and (23) the ones obtained in the previously described way and so on.

An improved expression for $\varepsilon(\varepsilon = \varepsilon_{R_2})$ may be obtained if instead of retaining only the leading term $x$ in the expansion of $\arcsin x$ :

$$\begin{aligned}\arcsin x &= x + \frac{x^3}{2\cdot 3} + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 + \ldots \\ &= x(1+F(x))\,,\end{aligned} \tag{24}$$

that is, instead of setting $F(x) = 0$ , we write in the above expression $F(x) \approx F(x^{ap})$ where $x^{ap}$ is the expression resulting from the eigenvalue equation solved approximately with $\arcsin x \approx x$. The relevant results are given in Ref. [12].

We discuss also the case of states for which in addition $|\varepsilon| \ll D_+$. This condition is satisfied for loosely bound states in a sufficiently deep well. In

this case it may be seen from Eq. (16), in analogy with the corresponding non-relativistic treatment [13], that the cotangent should be close to zero. Thus, we arrive at the following approximate expression

$$\varepsilon_{R_3} = 2D_+ \left(-1 + \tilde{\lambda}_0(2N + l - 1)\frac{\pi}{2}\right) , \tag{25}$$

where

$$\tilde{\lambda}_0 = \lambda_0 \left(\frac{\mu}{\mu^*_{g3}}\right)^{1/2} \tag{26}$$

and $\mu^*_{g3}$ is calculated from the expression of

$$\mu^*_g = \mu\big(1 + (\varepsilon - D_-)(2\mu c^2)^{-1}\big) , \tag{27}$$

by using for the energy the corresponding non-relativistic expression [13]

$$\varepsilon_{NR_3} = 2D_+ \left(-1 + \lambda_0(2N + l - 1)\frac{\pi}{2}\right) . \tag{28}$$

Expressions (25) and (28) gave poor results in the cases we studied. However, an alternative expression which gives considerably improved results in a variety of cases (see next section , Table III) may be derived by treating approximately expression (17) on the basis of expansion 4.4.42 of Ref. [14]. Such a procedure has not been followed, to our knowledge, even in the non relativistic case. The final result for the energy eigenvalue is

$$\varepsilon_{R_4} = -D_+ + \frac{\hbar^2}{2\mu^*_g}\frac{\theta^2_{Nl}}{R^2} , \tag{29}$$

where

$$\theta_{Nl} = \frac{4\lambda_R - 3\alpha_{Nl}}{\lambda_R^2 - 3}\left(1 + \left(1 - \frac{(\lambda_R^2 - 3)(7 - 3\alpha^2_{Nl})}{(4\lambda_R - 3\alpha_{Nl})^2}\right)^{\frac{1}{2}}\right) \tag{30}$$

with $\alpha_{Nl} = (2N + l - 1)\pi/2$.

The energies entering in $\mu^*_g$ and $\lambda_R$ may be evaluated by using for $\varepsilon$ the $\varepsilon_{NR_4}$ which is of the same structure as (29) but with $\mu$ instead of $\mu^*_g$ and with $\lambda_0$ instead of $\lambda_R$ in the expression of $\theta_{Nl}$.

## 4. Numerical results and comments

In this Section we give numerical results we have obtained so far for the single particle energies by using the formalism exhibited in the previous two Sections. Further work for possible improvements and extensions is under way.

We give first the results regarding the Λ-particle energies in hypernuclei. The eigenvalue equation for the ground state (Eq. (12) with $l = 0$ and $k = -1$) was solved numerically by using the following values of the potential parameters $D_+ = 30.55$ MeV, $D_- = 300$ MeV and $r_0 = 1.01$ fm. These were obtained by choosing approximately $D_- = 300$ MeV [7], for $\mu$ the reduced mass of the Λ-core system and determining $D_+$ and $r_0$ by (unweighted) least squares fitting [8] of the ground state energies of the Λ to the corresponding experimental values. This is somehow analogous to the procedure used by Davis *et al.* [15] (see also the discussion of Ref. [16]) for the asymptotic expression of the binding energy $B_\Lambda = -\varepsilon$ with the rectangular-well in the non-relativistic case. If the value $D_- = 443$ Mev [17] is chosen, for the fitting with the present model, the results obtained for $B_\Lambda$ are very similar to those of Table I. It should be noted that $D_-$ could also be determined by the least squares fitting but it is doubtful whether the value obtained in such a way is reliable.

TABLE I

Binding energy eigenvalues of various states obtained with $D_+ = 30.55$ MeV, $D_- = 300$ MeV, $r_0 = 1.01$ fm ($R = r_0 A_{\text{core}}^{1/3}$) and various hypernuclei, solving numerically the eigenvalue Eq. (12).

| $A_{\text{core}}$ | $s_{1/2}$ MeV | $p_{3/2}$ MeV | $p_{1/2}$ MeV | $p$ MeV | $d_{5/2}$ MeV | $d_{3/2}$ MeV | $d$ MeV |
|---|---|---|---|---|---|---|---|
| 12 | 11.2 | | | | | | |
| 15 | 13.1 | | | | | | |
| 27 | 17.5 | 5.6 | 4.5 | 5.2 | | | |
| 31 | 18.4 | 7.1 | 6.1 | 6.8 | | | |
| 39 | 19.8 | 9.5 | 8.7 | 9.2 | | | |
| 50 | 21.1 | 11.9 | 11.3 | 11.7 | 1.6 | 0.2 | 1.0 |
| 88 | 23.6 | 16.7 | 16.3 | 16.6 | 8.4 | 7.5 | 8.0 |
| 137 | 25.1 | 19.7 | 19.4 | 19.6 | 13.0 | 12.4 | 12.8 |
| 207 | 26.3 | 21.9 | 21.8 | 21.9 | 16.5 | 16.2 | 16.4 |

The values of the binding energies of the $\Lambda$ in the ground and the excited states $p_{3/2}, p_{1/2}, d_{5/2}, d_{3/2}$, obtained for a number of hypernuclei are given in Table I. In the same table the binding energies of the unsplitted states $p, d, f$ are given. These were obtained by means of the usual weighted average of the splitted states. The numerical results displayed in this table may be compared with the experimental values and with the results of previous calculations as for example with the more realistic Woods–Saxon [WS] potential [18, 19] and thus we may have an idea of the limitations of the potential model used and make a first assessment of its physical significance in reproducing the $\Lambda$ particle energies in hypernuclei. It is realized (see Fig. 1 of Ref. [9]) that the WS results are closer to the experimental values, as one should expect in view of its more realistic shape in the surface region, while those with the rectangular-well usually underestimate the binding energies. In certain cases, however, the results obtained with the rectangular-well potential are fairly satisfactory. Firstly, the fit to experimental ground state binding energies of the $\Lambda$ is very good. Also the $\Lambda$-binding energies for the low lying excited states compare rather satisfactorily with corresponding experimental ones for the heavier hypernuclei. It is further noted that an improved description of the $\Lambda$ energies in the excited states should be achieved with the rectangular model if state dependence of the potential parameters is allowed. If this is so, the limitations of the model are restricted. For the Woods–Saxon and the symmetrized Woods–-Saxon potentials, however, the state dependence of the parameters is quite weak [18, 20].

We also note that certain differences occur in the results if different experimental energies are used for determining the potential parameters. If the experimental results of P.H. Pile *et al.* Ref. [2] are used together with the value $17.5 \pm 0.5$ MeV for ${}^{32}_{\Lambda}S$ from the $(K^-, \pi^-)$ reaction, the best fit values of the (weighted) least squares fitting obtained with $D_- = 443$ MeV are $D_+ = 28.23$ MeV and $r_0 = 1.125$ fm. Furthermore, if the energies of the $\Lambda$ in the 3 lighter elements are exempted from the fitting, the best fit values become $D_+ = 31.3$ MeV and $r_0 = 0.93$ fm.

In order to test the accuracy of the approximate expressions $\varepsilon_{R_1}$ and $\varepsilon_{R_2}$, numerical calculations have been performed by using the first set of parameters and the results were compared with those obtained by solving numerically the eigenvalue equation (12). The results obtained for the $1s_{1/2}$ and $1p_{3/2}$ states and for various hypernuclei are displayed in Table II. In each case the values of the "exact" relativistic energy $\varepsilon_{ex}$ for the rectangular potentials, that is the one obtained by solving numerically the eigenvalue equation (12) and the approximate ones $\varepsilon_{R_1}$ and $\varepsilon_{R_2}$ are shown. In addition, the quantities $nR$ and $n_0R$ calculated with $\varepsilon_{ex}$ are also displayed. It is seen from the results of this table (see also Ref. [12] for more details ) that for the

$1s$ and $1p$ states and mainly for the larger values of $R$ which correspond also to larger values of $nR$ and $n_0R$ the approximate expressions $\varepsilon_{R_1}$ and $\varepsilon_{R_2}$ are good approximations to $\varepsilon_{ex}$. From the same table it is also seen that the results with $\varepsilon_{R_2}$ are better compared to those obtained with $\varepsilon_{R_1}$. In some cases the improvement is considerable. In Table III the results obtained with $\varepsilon_{R_4}$ are displayed for various states and various hypernuclei and are compared with those obtained with the numerical solution of the eigenvalue equation. It is seen that for the ground state the accuracy of $\varepsilon_{R_4}$ is very good for the smaller as well as for the larger values of $R$. We further observe from both tables and from the tables of Ref. [12] that for the higher states the accuracy of the various approximate expressions is deteriorating quite rapidly. It should be noted, however, that the accuracy depends on the values of $D_+$ and $R$. If in a physical problem the values of these quantities were larger, the accuracy for each state should have been improved.

TABLE II

Binding energies (in MeV) of various states obtained with $D_+ = 30.55$ MeV, $D_- = 300$ MeV and $r_0 = 1.01$ fm, ($R = r_0 A_{\text{core}}^{1/3}$) and various hypernuclei. The values obtained with the numerical solution of the eigenvalue equation (12) are denoted by $\varepsilon_{ex}$ while those obtained with the approximate expressions by $\varepsilon_{R_1}$ and $\varepsilon_{R_2}$ (see text and Ref. [12] for these expressions and also for the expressions of $nR$ and $n_0R$).

| $A_{\text{core}}$ | $s_{1/2}$ $nR$ | $s_{1/2}$ $n_0R$ | $s_{1/2}$ $-\varepsilon_{ex}$ | $s_{1/2}$ $-\varepsilon_{R_1}$ | $s_{1/2}$ $-\varepsilon_{R_2}$ | $p_{3/2}$ $nR$ | $p_{3/2}$ $n_0R$ | $p_{3/2}$ $-\varepsilon_{ex}$ | $p_{3/2}$ $-\varepsilon_{R_1}$ | $p_{3/2}$ $-\varepsilon_{R_2}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 12 | 2.1 | 1.8 | 11.2 | 9.1 | 11.8 | | | | | |
| 15 | 2.2 | 2.1 | 13.1 | 11.6 | 13.4 | | | | | |
| 39 | 2.5 | 3.6 | 19.8 | 19.4 | 19.8 | 3.4 | 2.5 | 9.5 | 5.5 | 9.0 |
| 50 | 2.5 | 4.0 | 21.1 | 20.9 | 21.1 | 3.5 | 3.0 | 11.9 | 8.8 | 10.9 |
| 88 | 2.6 | 5.2 | 23.6 | 23.5 | 23.6 | 3.7 | 4.3 | 16.7 | 14.8 | 15.6 |
| 137 | 2.7 | 6.2 | 25.1 | 25.1 | 25.1 | 3.8 | 5.5 | 19.7 | 18.3 | 18.7 |
| 207 | 2.7 | 7.3 | 26.3 | 26.2 | 26.3 | 3.9 | 6.6 | 21.9 | 21.0 | 21.1 |

We may conclude on the basis of the above mentioned results that the analytic expressions $\varepsilon_{R_1}$ and $\varepsilon_{R_2}$ give, in a number of cases, single particle energy values which are fairly close to those obtained from the numerical solution of the eigenvalue equation ,derived by means of the Dirac equation with potentials $U_S$ and $U_V$ of rectangular shape and of the same radius. Expression $\varepsilon_{R_4}$ gives in some of these cases better results than those obtained with the above mentioned expressions and in particular with $\varepsilon_{R_1}$.

TABLE III

Binding energies (in MeV) of various states obtained with $D_+ = 30.55$ MeV, $D_- = 300$ MeV and $r_0 = 1.01$ fm ($R = r_0 A_{\rm core}^{1/3}$) and various hypernuclei.The values obtained with the numerical solution of the eigenvalue equation (12) are denoted by $\varepsilon_{\rm ex}$ while those obtained with the approximate expression by $\varepsilon_{R_4}$ (see text).

| $A_{\rm core}$ | $s_{1/2}$ $-\varepsilon_{R_4}$ | $s_{1/2}$ $-\varepsilon_{\rm ex}$ | $p_{3/2}$ $-\varepsilon_{R_4}$ | $p_{3/2}$ $-\varepsilon_{\rm ex}$ | $p_{1/2}$ $-\varepsilon_{R_4}$ | $p_{1/2}$ $-\varepsilon_{\rm ex}$ | $d_{5/2}$ $-\varepsilon_{R_4}$ | $d_{5/2}$ $-\varepsilon_{\rm ex}$ | $d_{3/2}$ $-\varepsilon_{R_4}$ | $d_{3/2}$ $-\varepsilon_{\rm ex}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 12 | 11.3 | 11.2 | | | | | | | | |
| 15 | 13.2 | 13.1 | | | | | | | | |
| 19 | 15.1 | 15.0 | 1.4 | 1.8 | 0.3 | 0.5 | | | | |
| 27 | 17.6 | 17.5 | 4.4 | 5.6 | 3.2 | 4.5 | | | | |
| 31 | 18.5 | 18.4 | 5.8 | 7.1 | 4.7 | 6.1 | | | | |
| 39 | 19.8 | 19.8 | 8.1 | 9.5 | 7.3 | 8.7 | | | | |
| 50 | 21.1 | 21.1 | 10.6 | 11.9 | 9.9 | 11.3 | | | | |
| 88 | 23.6 | 23.6 | 15.5 | 16.7 | 15.2 | 16.3 | 5.5 | 8.4 | 4.5 | 7.5 |
| 137 | 25.2 | 25.1 | 18.7 | 19.7 | 18.5 | 19.4 | 10.3 | 13.0 | 9.6 | 12.4 |
| 207 | 26.3 | 26.3 | 21.1 | 21.9 | 21.0 | 21.8 | 14.2 | 16.5 | 13.8 | 16.2 |

Finally we mention that a least squares fit to the $1s_{1/2}$ neutron energies was made using the nuclei of Table I of Ref. [20]. These energies were determined from known experimental values of proton energies [1] by subtracting approximate values of the Coulomb energies as in Refs [20] and [21]. The energies were computed by solving the eigenvalue equation (12) numerically. The $D_+$ and $r_0$ were used as adjustable parameters and for $D_-$ an approximate fixed value was assumed ($D_- = 698$ MeV, [17]). It was found that the best fit values are $D_+ = 99.2$ MeV and $r_0 = 0.70$ fm and give a fairly satisfactory overall fit.

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