AN EXACT SOLUTION TO THE GENERAL TWO-BODY HAMILTONIAN*

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Dedicated to Janusz Dąbrowski in honour of his 65th birthday

A method of finding the exact solutions to the general two-body nuclear Hamiltonian is developed. It is based on the theory of Lie algebras of special orthogonal group. The algebraic structure of the model is discussed in details and the simplifications carried by the group theoretical approach are pointed out.

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1. Introduction

It is known fact that many essential features of rotating nuclei are connected with the alignment processes of the orbitals carrying large single-particle angular momentum. These orbitals are known

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** On leave of absence from Institute of Theoretical Physics, University of Warsaw, Poland to be the intruders (unique parity) among the normal parity states. Thus, in the first approximation, they do not mix with surrounding normal parity states and, to large extent, are characterized by their angular momentum j.

In this context it appears to be very fruitful to limit the single-particle space to the single high-j subshell. The models based on this assumption appeared to be very powerful in predicting the basic features of rotating nuclei. For example, direct application of different nuclear models and methods in this simplified shell model space allowed to study (predict) in a very straightforward way the oscillatory behaviour of yrast-yrare interaction (see *e.g.* Refs [1, 2]), many properties of electromagnetic transitions (see *e.g.* Refs [3, 4]), the distribution of diabolical points and oscillatory behaviour of pair transfer matrix elements (see *e.g.* Refs [5, 6]) and many other effects.

The nuclear methods usually provide the approximate solutions. It is interesting from the theoretical point of view to compare this approximate treatment to the rigorously exact solutions for these models in spite of the fact that it is hard to judge which type of treatment better simulates the reality. There have been many attempts in this direction. One of the method involves the direct diagonalisation in the spherical basis — jm-scheme (see *e.g.* Refs [7–9] and Refs quoted therein). Another class of methods is based on the Lie algebras of group theory [10, 11]. For example, in the Ref. [11] we discussed the advantages of group theoretical treatment over any standard approach when it is applied to the nuclear cranking Hamiltonian.

In the present paper we develop a formalism to find the exact solutions to any two-body Hamiltonian. There are in general two ways of proceeding. One can express any two-body Hamiltonian using only $N_{\alpha,\beta} = c^{\dagger}_{\alpha}c_{\beta}$ operators. It is known that $N_{\alpha,\beta}$ operators are the generators of Lie algebra of unitary group and, in a case of single j–shell, it is the Lie algebra of $\mathrm{U}(2j+1)$ group. This algebra was described in details in our previous publication [11]. One should point out, however, the basic difference between the realization of $u(2j+1)$ algebra described in the Ref. [11] and the present one. In the present realization we can attack any two-body Hamiltonian. However, it is straightforward to show, that the irreducible representations in this case are labelled by a number of particles. In the previous realization the limitations superimposed on the Hamiltonian (see the explicit form of generators given in Ref. [11]) led to appearance of a new quantum number — representation label — and substantial reduction of dimensionality of the problem.

An alternative way is to express the nuclear Hamiltonian using, beyond $N_{\alpha,\beta} = c^{\dagger}_{\alpha}c_{\beta}$ operators, also $B^{\dagger}_{\alpha,\beta} = c^{\dagger}_{\alpha}c^{\dagger}_{\beta}$ and $B_{\alpha,\beta} = c_{\alpha}c_{\beta}$ operators. As was shown in Ref. [10] these operators (more precisely their special combinations) form the structure of Lie algebra of SO group. In particular, for single j–shell, it is the $so(4j+2)$ algebra. One should stress that this method allows also for the studies of particle number breaking Hamiltonians like, very often used in nuclear structure, mean field approximation to the pairing Hamiltonian $H_{\rm pair} \propto \Delta(P^{\dagger}+P)$. Here, Δ stands for pairing gap and $P^{\dagger}$ (P) are the pair creation (annihilation) operators.

2. The algebraic structure of the model

As was already mentioned above the main building blocks of $so(4j+2)$ algebra are the following operators:

$$N_{\alpha,\beta} = c^{\dagger}_{\alpha}c_{\beta}\,, \qquad \text{where} \quad \alpha,\beta = 1,2,\ldots,2n\,, \tag{1}$$

$$B^{\dagger}_{\alpha,\beta} = c^{\dagger}_{\alpha}c^{\dagger}_{\beta} = (B_{\alpha,\beta})^{\dagger}\,, \quad \text{where} \quad \alpha < \beta = 1,2,\ldots,2n\,, \tag{2}$$

where $c^{\dagger}_{\alpha}$ and c_{α} are the creation and annihilation single-particle operators in the single j–shell model space. The single-particle states are enumerated in the following way: If the single-particle levels are split into $n = j + {}^{1}/_{2}$ pairs by *e.g.* prolate quadrupole mean field then odd and even indices $\alpha = 2i-1$ and $\alpha = 2i$ $(i = 1,2\ldots,n)$ label the time-reversed states $|2i\rangle = T|2i-1\rangle$ and the lowest pair of states corresponds to $m = \pm{}^{1}/_{2}$ and $i = n$, next lowest pair has $m = \pm{}^{3}/_{2}$ and $i = n-1$ and so on.

It has been shown in Ref. [10] that the following linear combinations of the operators (1) and (2):

$$L_{2\alpha-1,2\beta-1} = -\tfrac{i}{2}\left(N_{\alpha,\beta} - N_{\beta,\alpha} - B^{\dagger}_{\alpha,\beta} + B_{\alpha,\beta}\right)\,, \tag{3}$$

$$L_{2\alpha-1,2\beta} = \tfrac{1}{2}\left(N_{\alpha,\beta} + N_{\beta,\alpha} + B^{\dagger}_{\alpha,\beta} + B_{\alpha,\beta}\right)\,, \tag{4}$$

$$L_{2\alpha,2\beta-1} = -\tfrac{1}{2}\left(N_{\alpha,\beta} + N_{\beta,\alpha} - B^{\dagger}_{\alpha,\beta} - B_{\alpha,\beta}\right)\,, \tag{5}$$

$$L_{2\alpha,2\beta} = -\tfrac{i}{2}\left(N_{\alpha,\beta} - N_{\beta,\alpha} + B^{\dagger}_{\alpha,\beta} - B_{\alpha,\beta}\right)\,, \tag{6}$$

$$L_{2\alpha-1,2\alpha} = N_{\alpha,\alpha} - \tfrac{1}{2}\,, \tag{7}$$

where $\alpha < \beta = 1,2,3\ldots2n$, obey the standard commutation relations for SO group:

$$[L_{\alpha,\beta}, L_{\gamma,\delta}] = i\left(\delta_{\alpha,\gamma}L_{\beta,\delta} - \delta_{\alpha,\delta}L_{\beta,\gamma} - \delta_{\beta,\gamma}L_{\alpha,\delta} + \delta_{\beta,\delta}L_{\alpha,\gamma}\right)\,. \tag{8}$$

One should also notice that the generators defined through Eqs (3)–(7) are hermitian:

$$L_{\alpha,\beta} = -L_{\beta,\alpha} = L^{\dagger}_{\alpha,\beta} \tag{9}$$

and, what is clearly visible, this transformation is reversible. It means that any two-body Hamiltonian can be rewritten in terms of generators $L_{\alpha,\beta}$.

2.1. The Gel'fand-Tsetlin representation

The underlying special orthogonal symmetry offers the possibility of explicit construction of basis states. We use here the method developed by Gel'fand and Tsetlin [12]. The method is based on the following decomposition

$$\mathrm{SO}(n) \supset \mathrm{SO}(n-1) \ldots \supset \mathrm{SO}(3) \supset \mathrm{SO}(2)\,. \tag{10}$$

According to this method the basis state — Gel'fand–Tsetlin (GT) pattern — is the set of all integer (or all half integer) numbers which, for the groups of even dimension $l = 2k+2$, (we deal with even dimensions only) has the following form:

$$\begin{pmatrix} m_1 & m_2 & \cdots & m_{k-1} & m_k & m_{k+1} \\ m_{2k,1} & m_{2k,2} & \cdots & m_{2k,k-1} & m_{2k,k} & \\ m_{2k-1,1} & m_{2k-1,2} & \cdots & m_{2k-1,k-1} & m_{2k-1,k} & \\ m_{2k-2,1} & m_{2k-2,2} & \cdots & m_{2k-2,k-1} & & \\ m_{2k-3,1} & m_{2k-3,2} & \cdots & m_{2k-3,k-1} & & \\ \cdots & \cdots & \cdots & \cdots & & \\ \vdots & \vdots & & & & \\ m_{4,1} & m_{4,2} & & & & \\ m_{3,1} & m_{3,2} & & & & \\ m_{2,1} & & & & & \\ m_{1,1} & & & & & \end{pmatrix}. \tag{11}$$

The numbers $m_{i,j}$ are restricted to obey the following relations:

$$\begin{aligned} m_{2p+1,i+1} \le m_{2p,i} \le m_{2p+1,i}\,, &\quad \text{where} \quad i = 1,2,\ldots,p\,, \\ m_{2p,i+1} \le m_{2p-1,i} \le m_{2p,i}\,, &\quad \text{where} \quad i = 1,2,\ldots,p-1\,, \\ -m_{2p,p} \le m_{2p-1,p} \le m_{2p,p}\,. & \end{aligned} \tag{12}$$

The first row of the GT pattern (11) is the irreducible representation (*irrep*) label (is fixed) and $m_1 \ge m_2 \ge \ldots m_k \ge m_{k+1}$. In other words the *irreps* are labelled by highest weights associated with the eigenvalues of the Cartan subalgebra. In the case considered here the Cartan subalgebra can be constructed by $L_{2\alpha-1,2\alpha}$ (7) generators. They are, up to additive constant, the number operators for given single particle levels. Consequently, for fermionic systems (completely antisymmetric representations), the highest weights can consist $^1/_2$or $-^1/_2$ only (see Eq. (7)) what simplifies the structure of the model. Generally the highest weights have the following form: ($n = j + {}^1/_2$ is the number of doubly degenerated levels):

$$\left(\tfrac{1}{2}\right)^k \left(-\tfrac{1}{2}\right)^{2n-k} \quad \text{where} \quad k = 1, 2, \ldots, 2n. \tag{13}$$

The simplified notation used here have the following meaning: the highest weight is built by $m_1 = m_2 = \ldots = m_k = {}^1/_2$ followed by $m_{k+1} = m_{k+2} = \ldots = m_{2n} = -^1/_2$. However one can easily proof that the permissible *irreps* have $k = 2n$ or $k = 2n - 1$ only.

From didactical point of view is very instructive to consider the SO(8) case as an example ($j = {}^3/_2$ model — very frequently discussed in the literature in the case of Ω-fold degenerated $j = {}^3/_2$ multiplet). For the $(^1/_2)^4$ *irrep* the allowed GT patterns have the following form:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \\ \frac{1}{2} & \frac{1}{2} & \pm\frac{1}{2} & \\ \frac{1}{2} & \frac{1}{2} & & \\ \frac{1}{2} & \pm\frac{1}{2} & & \\ \frac{1}{2} & & & \\ \pm\frac{1}{2} & & & \end{pmatrix}, \tag{14}$$

while for the $(^1/_2)^3(-^1/_2)$ *irrep* we get:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \\ \frac{1}{2} & \frac{1}{2} & \pm\frac{1}{2} & \\ \frac{1}{2} & \frac{1}{2} & & \\ \frac{1}{2} & \pm\frac{1}{2} & & \\ \frac{1}{2} & & & \\ \pm\frac{1}{2} & & & \end{pmatrix}. \tag{15}$$

The numbers of states in both representations are equal and the total number of states is 2^4 (2^{2n}) spanning the full Fock space. One can immediately generalize this conclusion for arbitrary n. Indeed, $k = 2n$

and $k = 2n-1$ *irreps* contain 2^{2n-1} states each and together they span full Fock space.

One should also notice, that the $m_{2k-1,k}$ numbers are the only *active* quantum numbers for the problem. All the remaining $m_{i,j}$ are completely *passive*. It is important when one defines the action of Lie algebra generators onto the basis states related to the given GT pattern.

One can associate the basis vector to any Gel'fand–Tsetlin pattern. Thus, the representation is uniquely defined by the action of the generators on the basis vector. The representation given by Gel'fand and Tsetlin [12] is appropriate, however, only for antihermitian generators. We thus choose the following definition for hermitian generators[1]:

$$L_{2p+1,2p}\xi(\alpha) = \tfrac{1}{2}A(l_{2p-1,p})\xi^{+}(\alpha^{p}_{2p-1}) + \tfrac{1}{2}A(l_{2p-1,p}-1)\xi^{-}(\alpha^{p}_{2p-1}), \tag{16}$$

$$L_{2p+2,2p+1}\xi(\alpha) = C_{2p}\xi(\alpha). \tag{17}$$

Where we are using (in agreement with original notation) the following symbols: $\xi^{\pm}(\alpha^{p}_{2p-1})$ denotes the wave function for the GT pattern built by replacing m^{p}_{2p-1} by $m^{p}_{2p-1}\pm 1$ and $l_{2p-1,p}\equiv m_{2p-1,p}$. The coefficients A and C are equal in our case:

$$C_{2p} = 2m_{2p-1,p}m_{2(p+1)-1,p+1}, \tag{18}$$

$$A(l_{2p-1,p}) = \delta(m_{2p-1,p}, -\tfrac{1}{2}), \tag{19}$$

$$A(l_{2p-1,p}-1) = \delta(m_{2p-1,p}, \tfrac{1}{2}). \tag{20}$$

As it was mentioned earlier, the only active quantum numbers are $m_{2k-1,k}$. Also the matrix elements defined through the Eqs (16)–(20) and the selection rules depend only on the active quantum numbers. It is thus very convenient to use simplified notation for the GT pat-

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[1] We adopt here the phase convention which is equivalent to the standard Condon–Shortley phase convention for SO(3) group if: $L_{21}\equiv J_z$, $L_{32}\equiv J_x$ and $L_{31}\equiv J_y$. One should also note an error in the original definition of A coefficients resulting in a factor 1/2 in Eq. (16). Moreover, the coefficients B appearing in the original paper by Gel'fand and Tsetlin [12] vanish for completely antisymmetric representations.

erns (11), namely:

$$\begin{bmatrix} m_{2n} \\ m_{2(2n-1)-1,2n-1} \\ m_{2(2n-2)-1,2n-2} \\ \vdots \\ m_{3,2} \\ m_{1,1} \end{bmatrix} \equiv \begin{bmatrix} \nu_{2n} \\ \nu_{2n-1} \\ \nu_{2n-2} \\ \vdots \\ \nu_2 \\ \nu_1 \end{bmatrix} , \tag{21}$$

vhere we use integer $\nu_j \equiv 2m_{2j-1,j} = \pm 1$ instead of the original half nteger numbers $m_{i,j}$.

2. Additional symmetries — particle number symmetry and signature

The irreducible representations in this case mix the states with lifferent particle numbers. It happens because the particle number operator

$$\hat{N} = \sum_{\alpha=1}^{2n} \hat{N}_{\alpha,\alpha} = \sum_{\alpha=1}^{2n} L_{2\alpha-1,2\alpha} + n \tag{22}$$

loes not commute with all the generators of the Lie algebra. In the ase of the particle number symmetry conserving Hamiltonians one an, however, quite easily select good particle number subspaces. As s seen from the Eq. (17), any GT basis vector is the eigenvector of he particle number operator (22). The eigenvalues are equal[2]:

$$2N = 2n - \sum_{k=1}^{2n} \nu_{k-1}\nu_k . \tag{23}$$

One can show that all the states belonging to $k = 2n$ *irrep* correspond o even particle number subspace, while the $k = 2n - 1$ *irrep* contains odd-N states. Eq. (23) allows to select a given particle number subpace.

Also the angular momentum projection onto symmetry axis is a good quantum number. Indeed, expressing the J_z operator in terms of the generators one gets:

$$\begin{aligned} \hat{J}_z &= \sum_{i=1}^{n} \Omega_i \left(\hat{N}_{2i-1,2i-1} - \hat{N}_{2i,2i}\right) \\ &= -\sum_{i=1}^{n} \Omega_i \left(L_{4i-2,4i-3} - L_{4i,4i-1}\right) , \end{aligned} \tag{24}$$

[2] Here, and in the following, we adopt the convention $\nu_0 = 1$.

where $\Omega_i \equiv j + 1 - i$. The corresponding eigenvalues are:

$$K = -\tfrac{1}{2} \sum_{i=1}^{n} \Omega_i \nu_{2i-1} \left(\nu_{2i-2} - \nu_{2i}\right) . \tag{25}$$

This property of GT states is especially useful when calculating the matrix elements for the components of spherical tensors.

It is worth to note that any GT vector is an eigenstate of number operator for single particle level. It allows to rewrite the GT vectors using the representation of the occupation quantum numbers ($n_i = 0, 1$):

$$\xi(\alpha) \equiv \begin{bmatrix} n_1 & n_2 \\ n_3 & n_4 \\ \vdots & \vdots \\ n_{2n-3} & n_{2n-2} \\ n_{2n-1} & n_{2n} \end{bmatrix} . \tag{26}$$

One can regard the states having the opposite occupation numbers as signature (time reversal) coupled. The problem appears how to calculate the correct phase (or alternatively signature quantum number, r_x, for given GT state)

$$\hat{R}_x \begin{bmatrix} n_1 & n_2 \\ n_3 & n_4 \\ \vdots & \vdots \\ n_{2n-3} & n_{2n-2} \\ n_{2n-1} & n_{2n} \end{bmatrix} = r_x \begin{bmatrix} n_2 & n_1 \\ n_4 & n_3 \\ \vdots & \vdots \\ n_{2n-2} & n_{2n-3} \\ n_{2n} & n_{2n-1} \end{bmatrix} . \tag{27}$$

Using the following phase convention:

$$\hat{T}|jm\rangle \equiv \overline{|jm\rangle} \equiv (-)^{j+m}|j-m\rangle \tag{28}$$

and consequently

$$\hat{R}_x|jm\rangle = i(-)^{m-1/2}\overline{|jm\rangle} , \tag{29}$$

$$\hat{R}_x\overline{|jm\rangle} = i(-)^{m-1/2}|jm\rangle , \tag{30}$$

one ends up with the following phase factor:

$$r_x = (-)^{\sigma}(-)^{\sum_{i=1}^{2n} |\Omega_i| n_i} . \tag{31}$$

Here, σ counts the paired states for given GT scheme and n_i are the occupation factors.

After selection of "signature coupled" states (see Eqs (26) and (27) and using the above defined phase relations one can easily construct good signature states as:

$$
\begin{aligned}
|K,+\rangle &= \tfrac{1}{\sqrt{2}}\left(|K\rangle + \hat{R}_x|K\rangle\right), \\
|K,-\rangle &= \tfrac{1}{\sqrt{2}}\left(|K\rangle - \hat{R}_x|K\rangle\right),
\end{aligned} \tag{32}
$$

for even systems, and

$$
\begin{aligned}
|K,+\rangle &= \tfrac{1}{\sqrt{2}}\left(|K\rangle - i\hat{R}_x|K\rangle\right), \\
|K,-\rangle &= \tfrac{1}{\sqrt{2}}\left(|K\rangle + i\hat{R}_x|K\rangle\right),
\end{aligned} \tag{33}
$$

for odd systems.

2.3. Matrix elements and selection rules for so(4j + 2) generators

The advantage of group theoretical approach is the simple form of matrix elements for $so(4j+2)$ algebra generators. As a consequence one gets very straightforward selection rules when calculating matrix elements of nuclear Hamiltonian. Indeed, it is easy to proof the following formulas:

$$L_{2k+1,2l}\xi(\alpha) = \tfrac{1}{2}\nu_l\nu_k\xi_k^l(\alpha), \qquad \text{where} \quad k \geq l \tag{34}$$

$$L_{2l+2,2l+1}\xi(\alpha) = \tfrac{1}{2}\nu_l\nu_{l+1}\xi(\alpha), \tag{35}$$

$$L_{2k,2l+1}\xi(\alpha) = \tfrac{1}{2}\nu_l\nu_k\xi_{k-1}^{l+1}(\alpha), \quad \text{where} \quad k > l+1, \tag{36}$$

$$L_{2k,2l}\xi(\alpha) = -\tfrac{i}{2}\nu_l\nu_k\xi_{k-1}^{l}(\alpha), \quad \text{where} \quad k > l, \tag{37}$$

$$L_{2k+1,2l+1}\xi(\alpha) = \tfrac{i}{2}\nu_l\nu_k\xi_k^{l+1}(\alpha), \quad \text{where} \quad k > l, \tag{38}$$

where $\xi_k^l(\alpha)$ denotes GT vector obtained from original one, $\xi(\alpha)$, by replacing $\{\nu_l, \nu_{l+1}, \cdots, \nu_k\}$ by $\{-\nu_l, -\nu_{l+1}, \cdots, -\nu_k\}$, see Eq. (21).

Using Eqs (34)–(38) one can easily calculate matrix elements for any nuclear Hamiltonian. In the appendix we will give the explicit formulas for several selected operators frequently used in the nuclear structure.

3. Summary

The method for finding the exact solutions to the fermionic systems has been developed. The method is based on the theory of Lie algebra of SO group. In particular, we applied the well known techniques based on the Gel'fand–Tsetlin scheme to find the base functions and the matrix elements of the Lie algebra generators. The resulting matrix elements appear to be very simple with straightforward selection rules showing the advantage of the group theoretical treatment over any other method used so far. Using these formulas we calculated the explicit form of the matrix elements for several operators frequently used in the nuclear structure physics.

The special orthogonal symmetry allows to study arbitrary two-body Hamiltonian including also the particle number breaking Hamiltonians. However, in the cases of the Hamiltonians possessing special symmetries like the particle number symmetry, signature or axial symmetry, we developed the methods which allows to extract, from the full space, the subspace of a given symmetry. These methods appear to be again rather straightforward in the practical applications.

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Appendix

A.1. Matrix elements of quadrupole tensor

A.1.1. Matrix elements of Y_{20} component

The single particle matrix elements of Y_{20} component of quadrupole tensor are equal (see [13] p. 363):

$$\langle j\mu|Y_{20}|j\mu\rangle = -\frac{1}{4}\sqrt{\frac{5}{4\pi}}\frac{3\mu^2 - j(j+1)}{j(j+1)}. \tag{39}$$

Introducing the following quantities:

$$\eta = -\frac{1}{4}\sqrt{\frac{5}{4\pi}} \tag{40}$$

and

$$q_i^{(0)} = 3\mu_i^2 - j(j+1), \tag{41}$$

where, $\mu_i = n + 1/2 - i$, one can write the Y_{20} field as:

$$Y_{20} = \frac{\eta}{j(j+1)} \sum_{i=1}^{n} q_i^{(0)} \left(N_{2i-1,2i-1} + N_{2i,2i}\right) . \tag{42}$$

It is straightforward to express the Y_{20} field (42) in terms of $so(4j+2)$ generators (see Eq. (7)):

$$Y_{20} = -\frac{\eta}{j(j+1)} \sum_{i=1}^{n} q_i^{(0)} \left(L_{2(2i-2)+2,2(2i-2)+1} + L_{2(2i-1)+2,2(2i-1)+1}\right) . \tag{43}$$

In the derivation of Eq. (43) the following identity has been used:

$$\sum_{i=1}^{n} q_i \equiv 0 . \tag{44}$$

Finally, using the Eq. (35) one ends up with the explicit formula for the matrix elements of Y_{20} field:

$$\langle \xi(\alpha)|Y_{20}|\xi(\alpha)\rangle = -\frac{1}{2}\frac{\eta}{j(j+1)} \sum_{i=1}^{n} q_i^{(0)} \nu_{2i-1}(\nu_{2i-2} + \nu_{2i}) . \tag{45}$$

A. 1. 2. Matrix elements of $Y_{21} \pm Y_{2-1}$ components

The matrix elements for $Y_{2\pm1}$ components of quadrupole spherical tensor in the spherical basis are equal (see [13] p. 363):

$$\langle j\mu|Y_{2\pm1}|j\mu \mp 1\rangle = -\frac{1}{8}\sqrt{\frac{5}{4\pi}}\frac{(1 \mp 2\mu)\sqrt{6(j \mp \mu + 1)(j \pm \mu)}}{j(j+1)} . \tag{46}$$

Note, that the matrix elements between time reversed states vanish. The $Y_{21} \pm Y_{2-1}$ operators could be represented as:

$$Y_{21} \pm Y_{2-1} = \frac{1}{2}\frac{\eta}{j(j+1)} \times \sum_{i=1}^{n-1} q_i^{(1)} \left\{N_{2i-1,2i+1} \mp N_{2i+1,2i-1} + N_{2i+2,2i} \mp N_{2i,2i+2}\right\} , \tag{47}$$

where

$$q_i^{(1)} = 2(i-n)\sqrt{6i(2n-i)} . \tag{48}$$

Rewriting Eq. (47) in terms of generators one gets:

$$Y_{21} + Y_{2-1} = \frac{i}{2}\frac{\eta}{j(j+1)} \times \sum_{i=1}^{n-1} q_i^{(1)} \left\{ L_{4i+3,4i-1} - L_{4i+1,4i-3} + L_{4i+4,4i} - L_{4i+2,4i-2} \right\}, \quad (49)$$

$$Y_{21} - Y_{2-1} = \frac{1}{2}\frac{\eta}{j(j+1)} \times \sum_{i=1}^{n-2} q_i^{(1)} \left\{ L_{4i+3,4i} + L_{4i+1,4i-2} - L_{4i+4,4i-1} - L_{4i+2,4i-3} \right\}. \quad (50)$$

Finally, using the Eqs (34)–(38), one can easily calculate the matrix elements of $Y_{21} \pm Y_{2-1}$ fields in the GT basis:

$$(Y_{21}+Y_{2-1})\xi(\alpha) = -\frac{1}{4}\frac{\eta}{j(j+1)} \sum_{i=1}^{n-1} q_i^{(1)} \{ (\nu_{2i-1}\nu_{2i+1} - \nu_{2i-2}\nu_{2i})\xi_{2i}^{2i-1}(\alpha) + (\nu_{2i-1}\nu_{2i+1} - \nu_{2i}\nu_{2i+2})\xi_{2i+1}^{2i}(\alpha) \}, \quad (51)$$

$$(Y_{21}-Y_{2-1})\xi(\alpha) = -\frac{1}{4}\frac{\eta}{j(j+1)} \sum_{i=1}^{n-1} q_i^{(1)} \{ (\nu_{2i-2}\nu_{2i+1} - \nu_{2i-1}\nu_{2i})\xi_{2i}^{2i-1}(\alpha) + (\nu_{2i-1}\nu_{2i+2} - \nu_{2i}\nu_{2i+1})\xi_{2i+1}^{2i}(\alpha) \}. \quad (52)$$

As is seen form the Eqs (51) and (52) both terms have the same selection rule: the states coupled by $Y_{21} \pm Y_{2-1}$ fields have to differ by two consecutive quantum numbers.

A.1.3. Matrix elements of $Y_{22} \pm Y_{2-2}$ components

The matrix elements for $Y_{2\pm 2}$ components of quadrupole spherical tensor in the spherical basis are equal:

$$\langle j\mu | Y_{2\pm 2} | j\mu \mp 2\rangle = -\frac{1}{8}\sqrt{\frac{5}{4\pi}} \frac{\sqrt{6(j\pm\mu-1)(j\pm\mu)(j\mp\mu+1)(j\mp\mu+2)}}{j(j+1)}. \quad (53)$$

Let us consider, first, these parts of $Y_{22} \pm Y_{2-2}$ fields which produce the couplings between the time reversed states. Introducing the quantity:

$$q_0^{(2)} = (-1)^n n\sqrt{6(n+1)(n-1)}, \quad (54)$$

they could be written as:

$$(Y_{22} \pm Y_{2-2})^{(1)} = \frac{1}{2}\frac{\eta}{j(j+1)} q_0^{(2)} \{ N_{2n-3,2n} \pm N_{2n,2n-3} \mp N_{2n-2,2n-1} - N_{2n-1,2n-2} \} . \quad (55)$$

Expressing the $(Y_{22} \pm Y_{2-2})^{(1)}$ fields by means of generators we get:

$$(Y_{22} + Y_{2-2})^{(1)} = -\frac{1}{2}\frac{\eta}{j(j+1)} q_0^{(2)} \{L_{4n-3,4n-4} + L_{4n,4n-7} - L_{4n-1,4n-6} - L_{4n-2,4n-5}\} , \quad (56)$$

$$(Y_{22} - Y_{2-2})^{(1)} = -\frac{i}{2}\frac{\eta}{j(j+1)} q_0^{(2)} \{L_{4n-1,4n-7} + L_{4n,4n-6} + L_{4n-3,4n-5} + L_{4n-2,4n-4}\} . \quad (57)$$

Finally, acting by $(Y_{22} \pm Y_{2-2})^{(1)}$ operators on the GT vector we get (see Eqs (34)–(38)):

$$(Y_{22}+Y_{2-2})^{(1)}\xi(\alpha) = -\frac{1}{4}\frac{\eta}{j(j+1)} q_0^{(2)} \Big((\nu_{2n}\nu_{2n-4} - \nu_{2n-1}\nu_{2n-3})\xi_{2n-1}^{2n-3}(\alpha) + (1 - \nu_{2n-1}\nu_{2n-3})\xi_{2n-2}^{2n-2}(\alpha) \Big) , \quad (58)$$

$$(Y_{22}-Y_{2-2})^{(1)}\xi(\alpha) = -\frac{1}{4}\frac{\eta}{j(j+1)} q_0^{(2)} \Big((\nu_{2n}\nu_{2n-3} - \nu_{2n-1}\nu_{2n-4})\xi_{2n-1}^{2n-3}(\alpha) + (\nu_{2n-1}\nu_{2n-2} - \nu_{2n-2}\nu_{2n-3})\xi_{2n-2}^{2n-2}(\alpha) \Big) . \quad (59)$$

Both terms have the same selection rule: the matrix elements do not vanish between the GT states which differ either by one quantum number ν_{2n-2} or by three quantum numbers $\nu_{2n-3}, \nu_{2n-2}, \nu_{2n-1}$.

The remaining parts of $Y_{22} \pm Y_{2-2}$ fields are written as:

$$(Y_{22} \pm Y_{2-2})^{(2)} = \frac{1}{2}\frac{\eta}{j(j+1)} \sum_{i=1}^{n-2} q_i^{(2)} \{N_{2i-1,2i+3} \pm N_{2i+3,2i-1} \pm N_{2i,2i+4} + N_{2i+4,2i}\} , \quad (60)$$

where

$$q_i^{(2)} = \sqrt{6(2n-i-1)(2n-i)i(i+1)} . \quad (61)$$

Thus, expressing both fields by means of the generators we have:

$$(Y_{22}+Y_{2-2})^{(2)} = -\frac{1}{2}\frac{\eta}{j(j+1)}\sum_{i=1}^{n-2} q_i^{(2)} \times \left\{L_{4i+3,4i-3} - L_{4i+5,4i-2} + L_{4i+8,4i-1} - L_{4i+7,4i}\right\}, \tag{62}$$

$$(Y_{22}-Y_{2-2})^{(2)} = -\frac{i}{2}\frac{\eta}{j(j+1)}\sum_{i=1}^{n-2} q_i^{(2)} \times \left\{L_{4i+5,4i-3} + L_{4i+6,4i-2} - L_{4i+7,4i-1} - L_{4i+8,4i}\right\}. \tag{63}$$

Finally, acting by $(Y_{22} \pm Y_{2-2})^{(2)}$ fields on the GT state we obtain:

$$(Y_{22}+Y_{2-2})^{(2)}\xi(\alpha) = -\frac{1}{4}\frac{\eta}{j(j+1)}\sum_{i=1}^{n-2} q_i^{(2)}\Big\{(\nu_{2i-2}\nu_{2i+3} - \nu_{2i-1}\nu_{2i+2})\xi_{2i+2}^{2i-1}(\alpha) + (\nu_{2i-1}\nu_{2i+4} - \nu_{2i}\nu_{2i+3})\xi_{2i+3}^{2i}(\alpha)\Big\}, \tag{64}$$

$$(Y_{22}-Y_{2-2})^{(2)}\xi(\alpha) = -\frac{1}{4}\frac{\eta}{j(j+1)}\sum_{i=1}^{n-2} q_i^{(2)}\Big\{(\nu_{2i-1}\nu_{2i+3} - \nu_{2i-2}\nu_{2i+2})\xi_{2i+2}^{2i-1}(\alpha) + (\nu_{2i-1}\nu_{2i+3} - \nu_{2i}\nu_{2i+4})\xi_{2i+3}^{2i}(\alpha)\Big\} \tag{65}$$

with the following selection rule for both terms: the GT states should differ by four consecutive quantum numbers to be coupled by $(Y_{22} \pm Y_{2-2})^{(2)}$ fields.

A.2. Matrix elements of quadrupole deformed mean field

The quadrupole deformed mean field Hamiltonian (Nilsson potential) has the following form [14]:

$$V_{\rm sp} = -\tfrac{1}{3}\hbar\omega_0\varepsilon\sqrt{\frac{4\pi}{5}}\left(\cos\gamma Y_{20} - \frac{\sin\gamma}{\sqrt{2}}(Y_{22}+Y_{2-2})\right). \tag{66}$$

Introducing the following quantity (energy unit), see *e.g.* [15]

$$\kappa_j = \tfrac{1}{12}\hbar\omega_0\langle r^2\rangle_j\varepsilon\,. \tag{67}$$

It is easy to observe that the matrix elements of the axially deformed part of Nilsson potential can be obtained by formal replacement of

$$\eta \to \kappa_j \cos\gamma\,, \tag{68}$$

in Eq. (45), while for nonaxial part of Nilsson potential we should substitute

$$\eta \to -\frac{1}{\sqrt{2}}\kappa_j \sin\gamma\,, \tag{69}$$

in Eqs (58) and (64).

A.3. Matrix elements and selection rules for J_x and J_y operators

The J_+ operator is explicitly given as

$$\hat{J}_+ = \sum_{i,i'=1}^{n} \{f_{ii'}^{(n)}\left(N_{2i'-1,2i-1} - N_{2i,2i'}\right) + g_{ii'}^{(n)}\left(N_{2i'-1,2i} + N_{2i-1,2i'}\right)\} \tag{70}$$

with

$$f_{ii'}^{(n)} = \sqrt{(i-1)(2n-i+1)}\delta_{i',i-1}\,, \tag{71}$$

$$g_{ii'}^{(n)} = \tfrac{1}{2}(-)^n n\delta_{i,n}\delta_{i',n}\,. \tag{72}$$

Using this formula one can easily construct J_x and J_y operators and rewrite them in terms of $so(4j+2)$ algebra generators. The explicit formulas are given below.

Let consider, first, J_x operator. It is convenient to split J_x operator into two parts:

$$\hat{J}_x = \hat{J}_x^{(1)} + \hat{J}_x^{(2)}\,, \tag{73}$$

where

$$\hat{J}_x^{(1)} = \tfrac{1}{2}(-)^n n\left(L_{4n-1,4n-2} - L_{4n,4n-3}\right)\,, \tag{74}$$

$$\hat{J}_x^{(2)} = \tfrac{1}{2}\sum_{2}^{n}\sqrt{(i-1)(2n-i+1)}\{L_{2(2i-1)-1,2(2i-1)-4} - L_{2(2i-1),2(2i-1)-5} - L_{2(2i)-1,2(2i)-4} + L_{2(2i),2(2i)-5}\}\,. \tag{75}$$

Acting on basis state $\xi(\alpha)$ the $J_x^{(1)}$ operator gives:

$$\hat{J}_x^{(1)}\xi(\alpha) = \tfrac{1}{4}(-)^n n\left(1 - \nu_{2n}\nu_{2n-2}\right)\xi_{2n-1}^{2n-1}(\alpha)\,. \tag{76}$$

One just obtains very straightforward *selection rule* for this operator: The states coupled through $\hat{J}_x^{(1)}$ should differ by one number ν_{2n-1} only.

Using the formulas given by Eqs (34)–(38) it is easy to calculate the action of $J_x^{(2)}$ operator. Namely

$$J_x^{(2)}\xi(\alpha) = \tfrac{1}{4}\sum_{i=2}^{n}\sqrt{(i-1)(2n-i+1)} \times \Big((\nu_{2i-3}\nu_{2i-2} - \nu_{2i-4}\nu_{2i-1})\xi_{2i-2}^{2i-3}(\alpha) - (\nu_{2i-2}\nu_{2i-1} - \nu_{2i-3}\nu_{2i})\xi_{2i-1}^{2i-2}(\alpha)\Big) \tag{77}$$

with the following *selection rule*: the states coupled via $J_x^{(2)}$ should differ by two consecutive quantum numbers.

By analogy, the J_y operator can be represented as:

$$\hat{J}_y = \hat{J}_y^{(1)} + \hat{J}_y^{(2)}, \tag{78}$$

where

$$\hat{J}_y^{(1)} = \tfrac{1}{2}(-)^n n\left(L_{4n-3,4n-1} + L_{4n-2,4n}\right), \tag{79}$$

$$\hat{J}_y^{(2)} = -\tfrac{1}{2}\sum_{2}^{n}\sqrt{(i-1)(2n-i+1)}\{L_{2(2i-1)-1,2(2i-1)-5} + L_{2(2i-1),2(2i-1)-4} - L_{2(2i)-1,2(2i)-5} + L_{2(2i),2(2i)-4}\}. \tag{80}$$

Acting on basis state $\xi(\alpha)$ the $J_y^{(1)}$ operator gives:

$$\hat{J}_y^{(1)}\xi(\alpha) = -\tfrac{i}{4}(-)^n n\nu_{2n-1}\left(\nu_{2n-2} - \nu_{2n}\right)\xi_{2n-1}^{2n-1}(\alpha), \tag{81}$$

while for $J_y^{(2)}$ operator we obtain:

$$J_y^{(2)}\xi(\alpha) = -\tfrac{i}{4}\sum_{i=2}^{n}\sqrt{(i-1)(2n-i+1)} \times ((\nu_{2i-4}\nu_{2i-2} - \nu_{2i-3}\nu_{2i-1})\xi_{2i-2}^{2i-3}(\alpha) - (\nu_{2i-3}\nu_{2i-1} - \nu_{2i-2}\nu_{2i})\xi_{2i-1}^{2i-2}(\alpha)). \tag{82}$$

As is visible from the structure of the Eqs (76)–(77) and the Eqs (81)–(82) the J_x and J_y operators have the same selection rules.

A.4. Matrix elements and selection rules for monopole pairing Hamiltonian

The monopole pairing Hamiltonian has the following form:

$$-GP^{\dagger}P = -G\sum_{k,l=1}^{n} B^{\dagger}_{2k-1,2k}B_{2l-1,2l}\,. \qquad (83)$$

Using the following formula,

$$B^{\dagger}_{2k-1,2k} = (B_{2k-1,2k})^{\dagger} = \tfrac{1}{2}\{iL_{4k-1,4k-3} - L_{4k,4k-3} - L_{4k-1,4k-2} - iL_{4k,4k-2}\}\,, \qquad (84)$$

it is straightforward to express the Hamiltonian (83) as a bilinear form in terms of $so(4j+2)$ generators. Finally, using the Eqs (34)–(38), one can calculate the matrix elements of the pairing Hamiltonian:

$$\begin{aligned}-GP^{\dagger}P\xi(\alpha) = &-\tfrac{1}{4}G\sum_{k=1}^{n}(\nu_{2k}+\nu_{2k-2})(\nu_{2k}-\nu_{2k-1})\xi(\alpha)\\ &-\tfrac{1}{8}G\sum_{k>l=1}^{n}\{(\nu_{2k}+\nu_{2k-2})(\nu_{2l}+\nu_{2l-2})\\ &\times(\nu_{2k-2}\nu_{2l-2}-\nu_{2k-1}\nu_{2l-1})\}\tilde{\xi}^{2k-1}_{2l-1}(\alpha)\,.\end{aligned} \qquad (85)$$

Here, $\tilde{\xi}^{k}_{l}(\alpha)$ denotes the GT vector obtained from the original $\xi(\alpha)$ by replacement of ν_k and ν_l by $-\nu_k$ and $-\nu_l$, respectively.

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