GAUSSIAN DISTRIBUTIONS IN QUANTUM OPTICS*

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Physical properties of some Gaussian distributions in quantum optics are considered. A useful definition for a temperature for a general Gaussian distribution is presented and used for analyzing the quantum optics version of Cramer's theorem.

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1. Introduction: "the classical quantum state"

For convenience and definiteness we shall consider in the following only the electromagnetic radiation field (although the results can be generalized considerably). More than 25 years ago Aharonov, Falkoff, Lerner and Pendleton [1] (AFLP) posed the following question: what quantum state(s) (if any) is such that the result of its splitting can be faithfully reproduced by two independent beams (states)? *i.e.*, given a beam (designated by 1) whose state is specified by a density matrix $\rho(1)$ and given (say) a semitransparent mirror, can we split this beam into two *independent* modes (designated by 3 and 4)

$$\rho(1) = \rho(3) \otimes \rho(4) \ ?$$

And if the answer is in the affirmative, what is the quantum state, $\rho(1)$, that allows this? This problem is interesting because, *classically*, this is always possible: Thus, given that two beams (3 and 4) are physically separated — their classical distribution function is a product. Whilst quantum mechanically, the frequent assumption to the contrary notwithstanding, this is, generally, not so. The qualitative reason is that classical physics, being free

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of the uncertainty principle, allows the exact reproduction of the beams (3 and 4) both in phase and amplitude by two independent sources (see Fig. 1). AFLP [1] were able to prove that the only quantum state satisfying the above criterion is Glauber's coherent state (CS):

$$|\alpha\rangle = D(\alpha)|0\rangle\,, \tag{1}$$

where

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)\,. \tag{2}$$

Here $a^\dagger$ is the creation operator for the mode in question, and $|0\rangle$ is the vacuum state. Since this state is unique in possessing the above classical property AFLP termed it the "most classical quantum state". It is possible to articulate this state as follows: the CS is the only (quantum) state leading, *via* a splitter, to a product state,

$$\rho(1) \to \rho(3) \otimes \rho(4)\,. \tag{3}$$

$\rho(i)$ is a density matrix for (the pure) state $i(= 1, 3, 4)$.

Now it can be shown that a (quantum) state which is a product state *cannot* lead to a violation of Bell's inequality [2]. All other states, upon being split, become entangled states and may lead to violation of Bell's inequality [2]. Thus, in this sense too, the CS's are the most classical quantum states [3]. This dual sense of classical attributes is our motivation for studying further this "splitter problem".

2. Bifactorizable density matrices

A natural generalization of the AFLP problem is: what initially uncorrelated states (beams) upon being (nontrivial) split (see Fig. 1(b)) will result in two independent beams 3 and 4.

We termed [4] density matrices having this property bifactorizable, *viz*, the density matrix $\rho(\cdot\,,\cdot)$ is "bifactorizable" if for some orthogonal modes 1 and 2,

$$\rho(\cdot\,,\cdot) = \rho(1) \otimes \rho(2)\,, \tag{4}$$

and

$$\rho(\cdot\,,\cdot) = \rho(3) \otimes \rho(4)\,, \tag{5}$$

where the orthogonal modes 3 and 4 are (using quantum mechanical notation) given by

$$a_1^\dagger = \mu a_3^\dagger + \nu a_4^\dagger\,, \tag{6}$$

$$a_2^\dagger = -\nu^* a_3^\dagger + \mu^* a_4^\dagger\,, \tag{7}$$

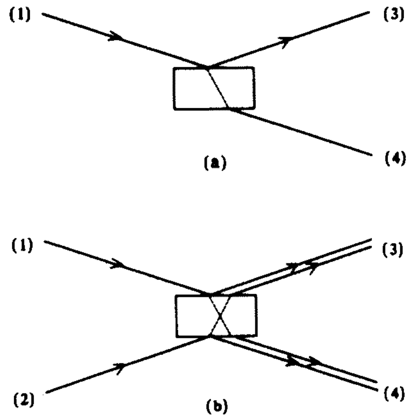

Fig. 1. Semi-transparent mirror giving rise to mode 3 and 4: (a) upon splitting of one mode 1; (b) Upon "splitting" two modes 1 and 2.

with

$$|\mu|^2 + |\nu|^2 = 1\,, \qquad |\mu\nu| \neq 0\,, \tag{8}$$

i.e., the modes 3 and 4 can be obtained from 1 and 2 *via* a "number conserving" splitter.

It was shown [4] that bifactorizable density matrices are Gaussian. (The converse is generally not true.)

3. General Gaussian density matrix (GGDM)

Using quantum field theoretical notation the GGDM can be defined by (referring to one mode only — generalization is trivial) its characteristic function

$$\begin{aligned} C(\lambda) &\equiv \mathrm{Tr}\,\rho \exp\left(\lambda a^\dagger - \lambda^* a\right) \\ &= \exp\left(\tfrac{1}{2}\langle(\lambda a^\dagger - \lambda^* a)^2\rangle\right). \end{aligned} \tag{9}$$

Thus, ρ for which Eq. (9) holds is a Gaussian density matrix, ρ_{GDM}.

A density matrix must be normalized, hermitian and positive definite. Imposing these conditions was shown [5] to imply that GGDM can be parameterized as

$$\begin{aligned} \rho_{\mathrm{GGDM}} \rightarrow &\rho(\alpha, \zeta, \kappa) \\ &= D(\alpha) S(\zeta) \frac{\exp(-\kappa a^\dagger a)}{Z} S^+(\zeta) D^+(\alpha)\,, \end{aligned} \tag{10}$$

with

$$S(\zeta) = \exp\left(\frac{\zeta a^{\dagger 2} - \zeta^* a^2}{2}\right), \tag{11}$$

$$Z = \frac{1}{1 - e^{-\kappa}}, \quad \kappa = \kappa^* \geq 0\,. \tag{12}$$

A state parameterized *via* Eq. (10) is termed "Thermal Squeezed State" [6], with κ identified as

$$\kappa = \beta\hbar\omega\,; \tag{13}$$

β is referred to as (inverse) temperature[1] . Thus we can associate a temperature with GGDM.

Returning to bifactorizable density matrices we now assert (this is proven in Ref. [4]) that in this case (of bifactorizability) all the density matrices are of equal temperature (also of equal $|\zeta|$). (Our point in this work, to be discussed later, is that the unequal temperatures are handled by Cramer's theorem.) An example of this is the AFLP [1] problem. This case, depicted in Fig. 1(a), can be viewed as a limiting case for Fig. 1(b) where the mode 2 is in its vacuum state, $|0\rangle$. The vacuum temperature now implies that all density matrices involved must be (*cf.* Eq. (10)) of the form

$$\rho = D(\alpha)|0\rangle\langle 0|D^\dagger(\alpha)\,, \tag{14}$$

i.e. we recover the AFLP result of having only the CS as a possible solution. Up to now we considered only uncorrelated beams. To study beams (3 and 4) correlated after the splitter (we take throughout the incoming beams 1 and 2 as independent) we will require some means to quantify the correlation. This is discussed in the following Section.

4. Correlated and "maximally correlated" modes

A measure of correlation between two systems (*e.g.* modes 3 and 4) that was considered by Zurek [7] and more recently by Barnett and Phoenix [8] is the so called index of correlation,

$$I_C^{3,4} \equiv S_3 + S_4 - S_{3,4}\,, \tag{15}$$

$$S_{3,4} = -\operatorname{Tr}\rho(3,4)\ln\rho(3,4)\,, \tag{16}$$

$$S_i = \operatorname{Tr}\rho_i \ln\rho_i \quad i = 3,4 \tag{17}$$

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[1] The identification $\kappa = \beta\hbar\omega$ is formal. Calling β inverse temperature (T^{-1}) is suggested by: (1) $T \geq 0$ and $T \to 0$ projects the vacuum (*i.e.* ground) state. (2) For the electromagnetic Hamiltonian, $H = \hbar\omega a^\dagger a$, $-(\partial/\partial\beta)\ln\operatorname{Tr}\rho$ gives the average energy.

with

$$\rho_i = \mathrm{Tr}_j\, \rho(\cdot\,,\cdot) \quad i \neq j = 3,4\,. \tag{18}$$

Let us consider some special cases:

(a) If ρ is separable in modes 3 and 4, *i.e.*,

$$\rho(3,4) = \rho_3(3) \otimes \rho_4(4)\,, \tag{19}$$

then we have trivially

$$S_{3,4} = S_3 + S_4\,, \tag{20}$$

and

$$I_c^{3,4} = 0\,. \tag{21}$$

Thus in the case of bifactorizable matrices we have

$$I_c^{1,2} = I_c^{3,4} = 0\,. \tag{22}$$

(b) What is $\rho(3,4)$ leading to maximal correlation, *i.e.* maximal value for $I_c^{3,4}$ under the constraint of fixed energy (all modes of equal frequency)? This problem was (posed and) solved by Barnett and Phoenix [8]. Their result is

$$\rho(3,4) = |\psi\rangle\langle\psi|\,, \tag{23}$$

with

$$|\psi\rangle = \exp\left(\theta a_3^\dagger a_4^\dagger - \theta^* a_4 a_3\right)|0\rangle\,. \tag{24}$$

This state is Gaussian.

5. Cramer's theorem and its field theoretic version

An interesting theorem in classical probability theory is Cramer's theorem [9]. The theorem states:

(a) Given two independent distributions for x_1 and x_2 respectively, $\rho_1(x_1)$ and $\rho_2(x_2)$, and

(b) given that the distribution for $x = x_1 + x_2$ is Gaussian, then it follows that $\rho_1(x_1)$ and $\rho_2(x_2)$ are Gaussian.

Hegerfeldt [10] showed that Cramer's theorem has a quantum version. The field theoretic (actually quantum optics) version is the following.

(a') Given two independent density matrices ρ_1 and ρ_2,

$$\rho(1,2) = \rho_1(a_1^\dagger, a_1) \otimes \rho_2(a_2^\dagger, a_2)\,, \tag{25}$$

and,

(b') Given that

$$\rho_4 = \mathrm{Tr}_3\, \rho(3,4) = \bar{\rho}(a_4^\dagger, a_4) \tag{26}$$

is GGDM (here $a_1^\dagger a_2^\dagger, a_3^\dagger, a_4^\dagger$ satisfy Eqs (6–8)),

then $\rho_1(a_1^\dagger, a_1)$ and $\rho_2(a_2^\dagger, a_2)$ are GGDM. The proof of this is given in Ref. [5]. Recalling that to every GGDM we can assign a temperature [5, 6] we conjecture that $T_4 \geq T_1, T_2$, *i.e.*, the temperature of mode 4 (or 3) is higher than that of 1 and 2. We now consider the case where modes 1 and 2 are in pure states (*i.e.*, $T_1 = T_2 = 0$). This will illustrate a particularly curious (and wholly quantum) case that is accommodated by Cramer's theorem. Let the initial modes be $(i = 1, 2)$.

$$|\psi_i\rangle = S_i(\zeta)|0\rangle\,, \tag{27}$$

$$S_i(\zeta) = \exp\left(\frac{\left(\zeta a_i^{\dagger 2} - \zeta^* a_i^2\right)^2}{2}\right)\,. \tag{28}$$

With the splitter parameters chosen as

$$\mu = \frac{1}{\sqrt{2}} e^{i\varphi_\mu}\,, \tag{29}$$

$$\nu = \frac{1}{\sqrt{2}} e^{i\varphi_\nu}\,, \tag{30}$$

with $\varphi_\mu + \varphi_\nu = \pi/2$, we are led to

$$|\psi(3,4)\rangle = \exp\left(\theta a_3^\dagger a_4^\dagger - \theta^* a_4 a_3\right)|0\rangle\,, \tag{31}$$

$$\theta = i\frac{\zeta}{2}\,. \tag{32}$$

Thus for this particular choice of the splitter parameters we get a "maximally correlated state". It is straightforward [8] to show that

$$\mathrm{Tr}_3\, |\psi(3,4)\rangle\langle\psi(3,4)| = Z^{-1} \sum_{n_4} e^{-n_4 \beta\omega} |n_4\rangle\langle n_4|\,, \tag{33}$$

with $(\hbar = 1)$

$$e^{-\beta\omega} = \tanh|\theta|\,, \tag{34}$$

$$Z = \frac{1}{1 - e^{-\beta\omega}}\,, \tag{35}$$

i.e. mode 4 is thermal upon tracing out mode 3. The above "occurs" in Hawking's black hole radiation [11] — there mode 3 refers to antiparticles falling into the black hole.

6. Summary

We considered the old characterization of coherent state (CS), as the unique state leading to two independent modes upon being split, as a special case of thermal coherent states (TCS) where two independent modes upon being split (by arbitrary, complex, splitter) lead to two independent modes: in the old case one of the initial independent modes was the vacuum. The general case requires equal temperatures for all the modes. These states cannot lead to violation of Bell's inequality upon splitting.

A generalization of the above, *viz.*, two independent modes leading upon splitting and partial tracing over one mode to a Gaussian was identified as the quantum version of Cramer's theorem. This theorem was shown to accommodate the case, purely quantum, of two pure states leading, upon passing through a splitter, to a thermal state. The relation of this to Hawking's black hole radiation was pointed out.

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