ON RANDOMLY INTERRUPTED DIFFUSION* **

J. ŁUCZKA

Department of Theoretical Physics, Silesian University
Bankowa 14, 40-007 Katowice, Poland

(Received November 13, 1992)

Processes driven by randomly interrupted Gaussian white noise are considered. An evolution equation for single-event probability distributions is presented. Stationary states are considered as a solution of a second-order ordinary differential equation with two imposed conditions. A linear model is analyzed and its stationary distributions are explicitly given.

PACS numbers: 05.40. +j, 02.50. +s

1. Introduction

A standard diffusion process $y(t)$ is described by a second-order partial differential equation of parabolic type (therefore it is sometimes named parabolic diffusion [1]). For example, one of the possible form of such an equation can be written as

$$\frac{\partial}{\partial t}p(y,t) = -\frac{\partial}{\partial y}f(y)p(y,t) + D\frac{\partial}{\partial y}g(y)\frac{\partial}{\partial y}g(y)p(y,t), \qquad (1.1)$$

where $p(y,t)$ is a one-dimensional probability distribution of the process $y(t)$ or it can be a transition probability function for $y(t)$. The deterministic functions $f(y)$ and $g(y)$ are named drift and a diffusion function, respectively. The constant D is a diffusion coefficient. The stochastic process $y(t)$ is Markovian and in consequence its dynamics is known as far as one is able to solve (1.1) under some conditions. If so, then all finite-dimensional distributions, all multi-time correlation functions, and so on, are known.

---

* Presented at the V Symposium on Statistical Physics, Zakopane, Poland, September 21–30, 1992.

** The work was supported in part by the KBN Grant no 2.0387.91.01.

On the other hand, $y(t)$ can be considered to be a process described by the stochastic equation

$$\dot{y} = f(y) + g(y) \circ \eta(t), \quad y \in (y_1, y_2), \tag{1.2}$$

where $\eta(t)$ is Gaussian white noise with

$$\langle \eta(t) \rangle = 0, \quad \langle \eta(t)\eta(s) \rangle = 2D\delta(t-s), \quad D > 0. \tag{1.3}$$

The multiplication "$\circ$" in the expression $g(y)\circ\eta(t)$ denotes the Stratonovich interpretation [2] of Eq. (1.2) and in consequence the probability density $p(y, t)$ of the process (1.2) obeys Eq. (1.1).

Now, let us consider a situation when the second term in (1.2) is switched on and off at random instants:

— if it is switched off then the process is deterministic, in particular, it is Markovian;

— if it is switched on then the process is diffusional, in particular, it is Markovian.

The above process will be named *randomly interrupted diffusion.* It can be generated by the stochastic equation

$$\dot{x}_t = f(x_t) + g(x_t) \circ \eta(t)\Gamma(t), \quad x \in (x_1, x_2), \tag{1.4}$$

where $\Gamma(t)$ is a random process that can take two values $\{0, 1\}$ in a random way. Such a process can be constructed with the help of a well known symmetric two-state Markov process (telegraphic noise) $\xi(t)$ [3] as follows

$$\Gamma(t) = \tfrac{1}{2}[1 + \xi(t)] \tag{1.5}$$

and

$$\xi(t) = \{-1, 1\}, \quad \langle \xi(t) \rangle = 0, \quad \langle = \xi(t)\xi(s) \rangle = \exp(-2\nu|t-s|). \tag{1.6}$$

The multi-time correlation functions of $\xi(t)$ are given by the recurrence relation

$$\langle \xi(t_1)\xi(t_2)\xi(t_3)\ldots\xi(t_n) \rangle = \langle \xi(t_1)\xi(t_2) \rangle \langle \xi(t_3)\ldots\xi(t_n) \rangle \tag{1.7}$$

for $t_1 \geq t_2 \geq t_3 \geq \ldots \geq t_n$. The parameter $\nu \geq 0$ is the transition probability per unit time from one state to the other. The quantity $\tau_c = 1/2\nu$ is the correlation time of the process $\xi(t)$.

Now, a few words about an interpretation and possible applications of Eq. (1.4). If the variable t is interpreted as time then the second term on the right-hand side of Eq. (1.4) can be thought to be a Langevin force switched on and off in random moments. If t is interpreted as a spatial variable then (1.4) can describe diffusion in a stochastic two-layer medium: one layer is surroundings characterized by a diffusion coefficient $D_1 = 0$ (it is a vacuum) and the other is a medium with a diffusion coefficient $D_2 = D$ [4, 5]. This model of "nothing" ($D_1 = 0$) and "something" ($D_2 = D$) randomly distributed in space is a caricature of a random medium and can be a starting point of generalizations for N layers with different diffusion coefficients.

As possible applications one can mention:

(a) the problem of multiple scattering, where one considers the transit of particles through plates of matter separated by vacuum gaps; there are domains of a deterministic motion and domains of a diffusional motion [4];

(b) transport phenomena in porous media, in sponge and pumice-type structures;

(c) vapor transport through polymer membranes (separation of gaseous mixtures);

(d) wave propagation in random media.

Another applications are presented by van Kampen in Ref. [6]. The problem like (1.4) was considered in the literature [4–7]. Laskin [4] analyzed a case of $f(x) = 0$ and $g(x) = 1$. In Ref. [7] first-passage time problem was studied for dichotomic diffusion dynamics with the composite process $\eta(t)\xi(t)$. Here, we present exact results for full dynamics in terms of probability distributions.

2. Probability distributions

The process x_t in (1.4) is Markovian because the composite force $\eta(t)\Gamma(t)$ is uncorrelated. Indeed, the multitime correlations functions

$$\langle \eta(t_1)\Gamma(t_1)\eta(t_2)\Gamma(t_2)\ldots\eta(t_n)\Gamma(t_n)\rangle = 0 \tag{2.1}$$

for

$$t_1 \neq t_2 \neq \ldots \neq t_n \,. \tag{2.2}$$

Thus, any finite-dimensional probability distribution

$$P(x_1\,,\ t_1\,;\ x_2\,,\ t_2\,;\ldots;\ x_n\,,\ t_n)$$

is known by virtue of the equality

$$P(x_1\,,\ t_1\,;\ x_2\,,\ t_2\,;\ldots;\ x_n\,,\ t_n) = \Pi(x_1\,,\ t_1|x_2\,,\ t_2)$$

$$\times\Pi(x_2\,,\ t_2|x_3\,,\ t_3)\ldots\Pi(x_{n-1}\,,\ t_{n-1}|x_n\,,\ t_n)P(x_n\,,\ t_n)\,, \tag{2.3}$$

which is valid for all Markovian processes [2]. The distribution $\Pi(x,\ t|y,\ s)$ is a transition probability function for x_t. A one-dimensional probability distribution $P(x\ t)$ can be obtained from the relation

$$P(x,t) = \int_{-\infty}^{\infty} dy \Pi(x,t|y,0) P(y,0)\,, \tag{2.4}$$

where $P(x,0)$ is an initial distribution.

To obtain $P(x,t)$ or $\Pi(x,t|y,0)$, one can use the same method as in obtaining (1.1) for the process (1.2). *E.g.*, one can use the Furutsu–Novikov–Donsker formula [8] and then one obtains a similar equation as (1.1), namely,

$$\frac{\partial}{\partial t} F(x,t) = -\frac{\partial}{\partial x} f(x) F(x,t) + D(t) \frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x) F(x,t) \tag{2.5}$$

but now with the *stochastic diffusion coefficient* $D(t)$,

$$D(t) = D\Gamma(t)\Gamma(t)\,, \tag{2.6}$$

and $P(x,t)$ is connected with $F(x,t)$ *via* the relation

$$P(x,t) = \langle F(x,t) \rangle^{D(t)}\,, \tag{2.7}$$

where superscript $D(t)$ means averaging over all realizations of the stochastic function $D(t)$. The averaging can be performed [5] with the result

$$\begin{aligned} \frac{\partial}{\partial t} P(x,t) = &- \frac{\partial}{\partial x} f(x) P(x,t) + \frac{D}{2} \frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x) P(x,t) \\ &+ \frac{D^2}{4} \int_0^t ds e^{-2\nu(t-s)} \int_{-\infty}^{\infty} dy\, \mathbb{B}(x,t|y,s) \frac{\partial}{\partial y} g(y) \frac{\partial}{\partial y} g(y) P(y,s)\,, \end{aligned} \tag{2.8}$$

where the integral kernel $\mathbb{B}(x,t|y,s)$ is given by

$$\mathbb{B}(x,t|y,s) = \frac{\partial}{\partial x}\left[g(x) \frac{\partial}{\partial x} (g(x) R(x,t|y,s)) \right]\,, \tag{2.9}$$

and $R(x,t|y,s)$ is a transition probability distribution of a diffusion process (1.1) but with the diffusion coefficient $D/2$ instead of D, *i.e.* it is a solution of the problem [5]

$$\begin{aligned} \frac{\partial}{\partial t} R(x,t|y,s) = &- \frac{\partial}{\partial x} f(x) R(x,t|y,s) \\ & \frac{D}{2} \frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x) R(x,t|y,s), \quad t \geq s \geq 0\,, \end{aligned} \tag{2.10a}$$

$$\lim_{t \to s} R(x, t|y, s) = \delta(x - y) . \tag{2.10b}$$

Let us note that if $\nu \to \infty$ then the last term in (2.8) tends to zero and the process x_t tends to a corresponding diffusion process with a diffusion coefficient $D/2$. If as the initial condition one chooses

$$P(x, 0) = \delta(x - x_0) \tag{2.11}$$

then the solution of Eq. (2.8) is a transition probability $\Pi(x, t|x_0, 0)$.

3. Stationary states

Evolution equations for probability distributions of the parabolic diffusion and randomly interrupted diffusion are given by Eqs (1.1) and (2.8), respectively. A stationary state $p(y)$ (if exists) for the parabolic diffusion is determined by the ordinary differential equation of the first order,

$$Dg(y)\frac{d}{dy}g(y)p(y) - f(y)p(y) = j = \text{constant} \tag{3.1}$$

and can be easily solved (j is a stationary probability current). For the randomly interrupted diffusion, a stationary distribution $P(x)$ is determined by the *ordinary differential equation of the second order*,

$$a(x)\frac{d^2P(x)}{dx^2} + b(x)\frac{dP(x)}{dx} + c(x)P(x) = G(x) , \tag{3.2}$$

where an explicit form of the functions $a(x)$, $b(x)$, $c(x)$ and $G(x)$ is presented in Ref. [5].

For the case (3.1) one must determine one integration constant. It is determined by the normalization condition for $p(y)$. For the case (3.2) one must find two integration constants. The first condition is, as in the former case, the normalization of $P(x)$,

$$\langle 1 \rangle = \int_{x_1}^{x_2} P(x)dx = 1 . \tag{3.3}$$

The second condition takes the form

$$\langle W(x) \rangle = \int_{x_1}^{x_2} W(x)P(x)dx = C_0 , \tag{3.4}$$

where

$$W(x) = \frac{[f(x)g(x)]'}{u(x)} - 2\left[\frac{f(x)g(x)}{u(x)}\right]' - 2f^2(x)[Dg(x)u(x)]^{-1} \tag{3.5}$$

and

$$u(x) = 2\nu g(x) + f'(x)g(x) - f(x)g'(x) \tag{3.6}$$

(the prime denotes the derivative with respect to x).

The constant C_0 depends upon boundary conditions and the stationary probability current J_0 as follows

$$C_0 = 2[A(x_1) - A(x_2)] - \frac{2J_0}{D}\int_{x_1}^{x_2} \frac{f(x)}{g(x)u(x)}dx \tag{3.7}$$

with

$$A(x) = f(x)g(x)\frac{P(x)}{u(x)}. \tag{3.8}$$

In generic cases, the constant (3.7) is zero.

3. An example: a linear system with additive noise

For comparison, let us consider a special case of Eqs (1.2) and (1.4), namely

$$\dot{y} = -ay + \eta(t), \quad a > 0, \quad y \in (-\infty, \infty), \tag{4.1}$$

$$\dot{x}_t = -ax_t + \eta(t)\Gamma(t), \quad a > 0, \quad x \in (-\infty, \infty). \tag{4.2}$$

A process $y(t)$ described by (4.1) is a Markovian exponentially correlated Gaussian stochastic process (an Ornstein–Uhlenbeck process). The process x_t in (4.2) is non-Gaussian. Its dynamics can be found exactly [9]. Here we present its stationary distributions $P(x)$. Eq. (3.2) for the case (4.2) takes the form

$$xP''(x) + \left[2 - \frac{\nu}{a} + \frac{a}{D}x^2\right]P'(x) + \frac{a}{D}\left(3 - 2\frac{\nu}{a}\right)xP(x) = 0. \tag{4.3}$$

The condition (3.4) reduces to the form

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 P(x)dx = \frac{D}{2a} \tag{4.4}$$

and determines stationary fluctuations is the system (4.2). The substitution

$$z = -\frac{ax^2}{2D} \tag{4.5}$$

converts Eq. (4.3) into a confluent hypergeometric equation [10] which solutions are given by confluent hypergeometric functions. A solution that fulfills conditions (3.3) and (4.4) reads

$$P(x) = \left(\frac{a}{2D}\right)^{\frac{1}{2}} 2^{-2\mu} \frac{\Gamma(1-\mu)}{\pi} \left(\frac{ax^2}{2D}\right)^{-\mu} \exp\left(-\frac{ax^2}{2D}\right) U\left(\frac{1}{2}-2\mu, 1-\mu, \frac{ax^2}{2D}\right), \tag{4.6}$$

where $U(b, c, z)$ stands for a Tricomi (confluent hypergeometric) function, $\Gamma(1-\mu)$ is an Euler gamma function and

$$\mu = \frac{1}{2}\left(1 - \frac{\nu}{a}\right). \tag{4.7}$$

The solution (4.6) is valid for $\nu > 0$ (finite correlation time τ_c of the process $\xi(t)$ in (1.6)). If the correlation time τ_c is infinite ($\nu = 0$) then a solution of Eq. (4.3) is given by the singular distribution

$$P(x) = \frac{1}{2}\delta(x) + \frac{1}{2}\sqrt{\frac{a}{2\pi D}} \exp\left(-\frac{ax^2}{2D}\right). \tag{4.8}$$

It consists of a deterministic part and of a Gaussian part because in this case with probability $1/2$ the noise $\eta(t)$ is switched on and with the same probability it is switched off. If $\nu \in [0, a]$ then the distributions (4.6) diverge to infinity when x tends to zero. *E.g.*, for $\nu = a/2$ one obtains

$$P(x) = (2\pi)^{-1}\Gamma\left(\frac{3}{4}\right)\sqrt{\frac{a}{D}}\left(\frac{ax^2}{2D}\right)^{-\frac{1}{4}} \exp\left(-\frac{ax^2}{2D}\right), \tag{4.9}$$

which diverges as $|x|^{-1/2}$ when $x \to 0$. If $\nu \in (a, \infty)$ then the distributions (4.6) are finite for $x = 0$. *E.g.* for $\nu = 2a$, Eq. (4.6) takes the form (*cf.* [10] p. 287)

$$P(x) = \left(\frac{2a}{\pi D}\right)^{\frac{1}{2}} \exp\left(-\frac{ax^2}{4D}\right) D_{-2}\left(\sqrt{\frac{ax^2}{D}}\right), \tag{4.10}$$

where $D_v(z)$ is a parabolic cylinder function [10]. Since $D_{-2}(0) = 1$ (see [10] p. 324), so $P(x)$ in (4.10) is finite. If ν tends to infinity then (4.6) tends to a Gaussian distribution. For $\tau_c = 0$ ($\nu = \infty$) it has the form

$$P(x) = \left(\frac{a}{\pi D}\right)^{\frac{1}{2}} \exp\left(-\frac{ax^2}{D}\right). \tag{4.11}$$

Changing $D \to 2D$, we get a distribution of the process (4.1).

5. Final remarks

Processes with Gaussian white noise, which is randomly interrupted by an exponentially correlated two state Markovian process, are presented. Probability distributions of such processes are determined by partial integro-differential equations of the type (2.8). Eq. (2.8) contains a double integral operator over temporal and spatial variables. Additionally, the integral kernel $\mathbb{B}(x,t|y,s)$ is a solution of an auxiliary diffusion problem given by Eqs (2.10). In a general case, obviously, Eq. (2.8) cannot be solved. For linear models with additive and multiplicative noise, a solution of Eq. (2.8) can be obtained and is analyzed elsewhere [9]. In the paper, stationary properties of a linear model with additive noise are investigated. Stationary states are characterized by four classes of distributions: singular (4.8), diverging to infinity at zero (4.9), smooth at zero but non-Gaussian (4.10) and Gaussian (4.11) in dependence on the relation between correlation time $\tau_c = 1/2\nu$ of the noise $\xi(t)$ and the deterministic relaxation time $\tau_d = 1/a$.

I would like to thank Dr Adam Gadomski for discussions, Professor Peter Hänggi for informing me of his results and providing me with Ref. [7].

REFERENCES

[1] Z.J. Grzywna, J. Łuczka, *Acta Pharm. Jugosl.* **41**, 327 (1991).
[2] H. Risken, *The Fokker–Planck Equation*, Springer-Verlag, Berlin 1989.
[3] C.W. Gardiner, *Handbook of Stochastic Methods*, Springer-Verlag, Berlin 1985.
[4] N.V. Laskin, *Ukrain. Fiz. Zh.* **33**, 1429 (1988); *J. Phys. A: Math. Gen.* **22**, 1565 (1989).
[5] J. Łuczka, M. Niemiec, E. Piotrowski, *Phys. Lett.* **A167**, 475 (1992).
[6] N.G. van Kampen, *Physica* **96A**, 435 (1979).
[7] V. Balakrishnan, C. Van den Broeck, P. Hänggi, *Phys. Rev.* **A38**, 4213 (1988).
[8] K. Furutsu, *J. Res. Natl. Bur. Stand.* **D67**, 303 (1963); E.A. Novikov, *Zh. Eksp. Teor. Fiz.* **47**, 1919 (1964); M. Donsker, in: Proc. Conf. on The Theory and Applications of Analysis in Function Space, MIT Press, Cambridge, MA 1964.
[9] J. Łuczka, M. Niemiec, E. Piotrowski, to be published.
[10] W. Magnus, F. Oberhettinger, R.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* Springer-Verlag, Berlin 1966.