LANGEVIN EQUATIONS WITH MULTIPLICATIVE NOISE: APPLICATION TO DOMAIN GROWTH *

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Langevin Equations of Ginzburg–Landau form, with multiplicative noise, are proposed to study the effects of fluctuations in domain growth. These equations are derived from a coarse-grained methodology. The Cahn–Hilliard–Cook linear stability analysis predicts some effects in the transitory regime. We also derive numerical algorithms for the computer simulation of these equations. The numerical results corroborate the analytical predictions of the linear analysis. We also present simulation results for spinodal decomposition at large times.

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1. Introduction

It has been pointed out in the interpretation of recent experiments that noise plays an important role in dynamic processes like pattern formation. Examples of such effects have been observed in the generation of sidebranching in dendritic growth [1], cells in Rayleigh–Benard convection [2] and William domains in the electrohydrodynamic instability of nematic liquid crystals [3]. Ref. [4] gives an overview of this field. In some of these

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experiments [1, 2], the origin of the noise is not clearly stablished, and the very small thermal additive noise seems not to be enough to explain the results. We think that a new type of modeling of fluctuations may be needed to explain them. In other experiments in liquid crystals, the noise has been deliberately superimposed to the AC voltage [3]. The results imply a strong effect on the response of the system, like changes in the threshold of the instability points. A clear evidence of this fact has been presented recently for a Swift–Hohenberg model [5]. A modeling of the last situations is given in terms of Langevin equations with multiplicative noise, for which the noise appears multiplying a function of the relevant variables. Then, the effects of the noise depend on the state of the system and due to the coupling they are more important than those induced by simple additive noise. We will consider here the situation in which an internal noise could also appear in a multiplicative way. We derive Langevin equations with multiplicative noise from a mesoscopic derivation using a coarse-grained procedure [6]. In this way, it would be easier to give an interpretation of the multiplicative noise equations. In general, the study of these type of equations would have relevance in domain growth in phase separation dynamics [7], pattern formation [5, 8], polymers [9], *etc.*

Here, we consider a system of two components, like a binary liquid or alloy, which could undergo phase separation [7]. The system is suddenly quenched from a one-phase region inside its coexistence region. Then, the homogeneous region becomes unstable and domains of the new stable phases start growing. This mechanism is called spinodal decomposition. The deterministic evolution of such a system was studied when a variable dependent diffusion coefficient was taken into account [10]. This assumption has been considered to model deep quenching [11] or to take into account the presence of an external field, like gravity [8]. We find that the assumption of a concentration dependent diffusion coefficient implies multiplicative thermal fluctuations.

In Section 2 , we present the models that we are interested. Section 3 is devoted to the coarse-grained procedure to derive Langevin equations with multiplicative internal noise. In Section 4 the derivation of the numerical algorithm to simulate these equation is presented. In Section 5, we present the study of domain growth in spinodal decomposition with emphasis in the multiplicative noise effects. A summary of conclusions is presented in Section 6. In the Appendix, technical details of the derivation of a Fokker–Planck equation from a Langevin equations are presented.

2. A model

In this section we will derive a Ginzburg–Landau type of Langevin equation with multiplicative noise. This equation will be used in the study of

phase separation dynamics of a binary mixture. The concentration variable, $c(\vec{r},t)$, is a conserved quantity and obeys the following Langevin equation,

$$\frac{\partial c(\vec{r},\tau)}{\partial \tau} = \nabla M \nabla \frac{\delta F}{\delta c} - \frac{\beta^{-1}}{2}\nabla(\nabla \frac{\delta}{\delta c})M + \nabla^i m \xi^i(\vec{r},\tau)\,, \tag{2.1}$$

where $F[c]$ is the Ginzburg–Landau free energy functional:

$$F(\{c\}) = \tfrac{1}{2}\int d\vec{r}(-\frac{c^2}{2} + \frac{c^4}{4} + \frac{(\nabla c)^2}{2}) \tag{2.2}$$

and $M(c) = m^2(c)$ is the concentration dependent diffusion coefficient. The noise is a d-dimensional vector with a correlation:

$$\langle \xi^i(\vec{r},\tau)\xi^j(\vec{r}',\tau')\rangle = 2\beta^{-1}\delta_{ij}\delta(\vec{r}-\vec{r}')\delta(\tau-\tau')\,, \tag{2.3}$$

β^{-1} is the intensity of the gaussian white noise. A common assumption regarding the dependence of M on the concentration has been obtained by phenomenological arguments [12]. That is:

$$M(c) = 1 - ac^2\,, \tag{2.4}$$

where a is a parameter related to temperature.

When M is independent of the field variable we obtain the usual model B of phase separation dynamics with additive noise [7]. When M does depend on the field variable, apart from the multiplicative noise term, we find a spurious term, the second term on the right-hand side (r.h.s.) of Eq. (2.1), of stochastic origin. This spurious term ensures the evolution of the system to the correct equilibrium solution, in terms of the Boltzmann expression with the energy functional (2.2). We show that both terms of stochastic origin give new relevant contributions even in a linear stability analysis, that is to the standard Cahn–Hilliard–Cook theory. However, nonlinear effects of such models are difficult to study analytically. The numerical results indicate that the multiplicative noise induces a delay in the short time behavior of the domain growth dynamics, in accordance with the linear analysis. We also study the correlation and structure functions, and the domain growth effective exponents for large times.

3. A coarse-grained derivation

In the standard coarse-grained procedure [6], one divides the lattice into regular cells of volume Δx^d containing N sites and defines the concentration of the binary mixture at the cell α, c_α by:

$$c_\alpha = \frac{1}{N}\sum_{k\epsilon\alpha}\sigma_k\,, \tag{3.1}$$

where $\sigma_k = 1, -1$ indicates a site occupied by a particle A and B, respectively. Then, it is assumed that a markovian master equation is obeyed by the probability $P(\{c\}, t)$ of the configuration of cells, $\{c\} = \{c_1, c_2, \ldots\}$,

$$\partial_t P(\{c\}, t) = \sum_{\alpha i \epsilon} \Big(W(\{c\}^{\alpha i} \to \{c\}) P(\{c\}^{\alpha i}, t) - W(\{c\} \to \{c\}^{\alpha i}) P(\{c\}, t) \Big) , \quad (3.2)$$

the indexes α and i numerate the cells and their nearest neighbors in the positive direction, respectively. $W(\{c\}^{\alpha i} \to \{c\})$ is the transition probability between the initial configuration $\{c\}^{\alpha i} = \{c_1, c_2, \ldots, c_\alpha - \epsilon, c_{\alpha+i} + \epsilon, \ldots\}$ and the final one $\{c\} = \{c_1, c_2, \ldots, c_\alpha, c_{\alpha+i}, \ldots\}$. ϵ is the concentration interchanged in an elementary step of the evolution.

We consider situations for which the system evolves to an equilibrium state and the detailed balance condition is fulfilled:

$$\frac{W(\{c\}^{\alpha i} \to \{c\})}{W(\{c\} \to \{c\}^{\alpha i})} = \frac{P_{\rm st}(\{c\})}{P_{\rm st}(\{c\}^{\alpha i})} . \quad (3.3)$$

The steady state distribution, $P_{\rm st}(\{c\})$, is expressed in terms of a coarse-grained free energy $F(\{c\})$:

$$P_{\rm st}(\{c\}) \sim {\rm e}^{-\beta F(\{c\})} . \quad (3.4)$$

Then, we write for the transition probabilities [14]

$$\begin{aligned} W(\{c\}^{\alpha i} \to \{c\}) &= M(\{c\}^{\alpha i}, \{c\}) {\rm e}^{\beta \frac{\Delta F}{2}} , \\ W(\{c\} \to \{c\}^{\alpha i}) &= M(\{c\}, \{c\}^{\alpha i}) {\rm e}^{-\beta \frac{\Delta F}{2}} \end{aligned} \quad (3.5)$$

which, in this way, are compatible with Eq. (3.3), provided M is symmetric by interchange of the initial and final states. In the usual derivation of the field model, with constant diffusion coefficient and additive noise, no dependence of M on the configurations is considered, and it is assumed that $M(\{c\}^{\alpha i}, \{c\}) = P(\epsilon)$, where $P(\epsilon)$ is a sharp function around $\epsilon = 0$. The generalization that we present here will give us a model with a variable dependent diffusion coefficient and a multiplicative noise. By assuming that ϵ is a small quantity, we can expand the different terms of the right hand side of Eq. (3.2) in power series of ϵ:

$${\rm e}^{\pm\beta \frac{\Delta F}{2}} = 1 \pm \beta \frac{\epsilon}{2} K_{\alpha i} F \pm \beta \left(\frac{\epsilon}{2}\right)^2 K_{\alpha i}^2 F + \frac{\beta^2}{2} \left(\frac{\epsilon}{2}\right)^2 (K_{\alpha i} F)^2 \cdots , \quad (3.6)$$

$$P(\{c\}^{\alpha i}, t) = \left(1 + \epsilon K_{\alpha i} + \tfrac{1}{2} \epsilon^2 K_{\alpha i}^2 \cdots\right) P(\{c\}, t) , \quad (3.7)$$

$$M(\{c\}^{\alpha i},\{c\}) = P(\epsilon)\left(1+\frac{\epsilon}{2}K_{\alpha i}+\tfrac{1}{2}\left(\frac{\epsilon}{2}\right)^2 K_{\alpha i}^2\cdots\right)M(\{c\}), \tag{3.8}$$

where the operator $K_{\alpha i}$ is given by

$$K_{\alpha i} = \frac{\partial}{\partial c_{\alpha+i}} - \frac{\partial}{\partial c_\alpha} \tag{3.9}$$

and in (3.8) we have made use of the symmetry condition on $M(\{c\}^{\alpha i},\{c\})$. By substituting Eqs (3.5)–(3.8) into Eq. (3.2) we get to the lowest order:

$$\frac{\partial P(\{c\},t)}{\partial t} = \Gamma\sum_{\alpha i} K_{\alpha i}M_{\alpha i}\left((K_{\alpha i}F)+\beta^{-1}K_{\alpha i}\right)P(\{c\},t), \tag{3.10}$$

where $\Gamma = \langle\epsilon^2\rangle\beta/2$ and $\langle\epsilon^2\rangle$ is the second moment of $P(\epsilon)$. Eq. (3.10) can be written in more useful form (see Appendix for details),

$$\frac{\partial P}{\partial t} = -\Delta x^2\Gamma\frac{\partial}{\partial c_\alpha}(\nabla_L^i)_{\alpha\beta}M_{\beta i}(\nabla_R^i)_{\beta\sigma}\left\{\frac{\partial F}{\partial c_\sigma}+\beta^{-1}\frac{\partial}{\partial c_\sigma}\right\}P, \tag{3.11}$$

where ∇_L^i and ∇_R^i are the left and right discrete versions of the gradient operators, Eq. (A.1). Summation over indexes is understood.

The Langevin equation, in the Stratonovich interpretation, associated with the Fokker–Planck Eq. (3.11) is given by

$$\begin{aligned}\dot{c}_\alpha =& \Gamma\Delta x^2(\nabla_L^i)_{\alpha\beta}M_{\beta i}(\nabla_R^i)_{\beta\sigma}\frac{\partial F}{\partial c_\sigma}\\ &-\frac{\Gamma}{2}\beta^{-1}\Delta x^2(\nabla_L^i)_{\alpha\beta}(\nabla_R^i)_{\beta\sigma}\frac{\partial M_{\beta i}}{\partial c_\sigma}+(\nabla_L^i)_{\alpha\beta}m_{\beta i}\xi_\beta^i(t),\end{aligned} \tag{3.12}$$

where $\xi_\beta^i(t)$ is a gaussian white noise of zero mean and correlation

$$\langle\xi_\alpha^i(t)\xi_\beta^j(t')\rangle = 2\Delta x^2\Gamma\beta^{-1}\delta_{ij}\delta_{\alpha\beta}\delta(t-t') \tag{3.13}$$

and $m_{\beta i}$ is defined by

$$M_{\beta i}(\{c\}) = \left(m_{\beta i}(\{c\})\right)^2. \tag{3.14}$$

The way to prove that the Langevin Eq. (3.12) corresponds to the Fokker–Planck Eq. (3.11) is to derive the latter from the former. In Appendix we presents the details of this derivation.

At this point, Eqs (3.11) and (3.12) are formal and general equations in which the expressions of $M_{\alpha i}(\{c\})$ need to be specified for each particular

model. Furthermore, these equations are given in terms of the cell variables and we are also interested in finding the corresponding equations in the continuous spatial limit. Then, we need to make some assumptions on the form of $M_{\alpha i}(\{c\})$. In general, $M_{\alpha i}(\{c\})$ is a function that depends on the concentration values of all the lattice points of a given configuration. In the continuous spatial limit, this gives rise to a functional expression of $M(c)$. In order to obtain a local mobility function $M(c)$ like the one considered in the macroscopic model, [11–13] given by Eq. (2.4), we take into account that the transition probabilities, Eq. (3.5), only involve interchanges of matter between nearest neighbour cells α, $\alpha + i$ at each elementary step. Then, we restrict ourselves to functions $M_{\alpha i}(\{c\})$ that only depend on the concentration values of the cells α, $\alpha + i$ and on a limited number, n, of cells in the vicinity of α and $\alpha + i$. Therefore, we propose the following simple generic expression of $M_{\alpha i}(\{c\})$:

$$M_{\alpha i} = \sum_{\beta} Q^{i}_{\alpha\beta} f(c_\beta)\,, \tag{3.15}$$

where $f(c_\beta)$ is a function of only one variable c_β and the matrix elements $Q^{i}_{\alpha\beta}$ are different from zero only when the index corresponds to α, $\alpha + i$ or the n cells in the vicinity of this couple. By taking into account the normalization condition:

$$\sum_{\beta} Q^{i}_{\alpha\beta} = 1\,, \tag{3.16}$$

we obtain that the continuous limit of $M_{\alpha i}(\{c\})$ is given by $M(c) = f(c)$.

To simplify the model, we take that $Q^{i}_{\alpha\alpha} = Q^{i}_{\alpha\alpha+i} = Q_0$ and, we can see that Q_0 is of order $n^{-1/d}$. Then the transition probabilities, Eq. (3.5), only depend on the concentration values of cells that are closer than a distance of the order of $n^{1/d}\Delta x$. Then, to characterize the size of this region, we define a new parameter R by [14]:

$$R = \Delta x Q_0^{-1/d}\,, \tag{3.17}$$

which is precisely of the order of $n^{1/d}\Delta x$ and represents the mesoscopic length scale of the model.

An explicit example of $M_{\alpha i}(\{c\})$, Eq. (3.15), which corresponds to Eq. (2.4) in the continuous limit, is given by:

$$M_{\alpha i}(\{c\}) = Q_0\{(1 - ac^2_\alpha) + (1 - ac^2_{\alpha+i})\} + Q_0 \sum_{\beta}(1 - ac^2_\beta)\,, \tag{3.18}$$

which depends on c_α, $c_{\alpha+i}$ and on all the nearest neighbors β of this couple.

The dynamics represented by Eq. (3.10) or Eqs (3.11)–(3.12) involves not only the function $M_{\alpha i}$ but also derivatives in terms of the operator $K_{\alpha i}$, Eq. (3.9), like $K_{\alpha i}M_{\alpha i}$. The result of the action of $K_{\alpha i}$ on the mobility Eq. (3.15), or, in particular, on the example given by Eq. (3.18), could be written as:

$$\frac{1}{\Delta x^{d+1}}K_{\alpha i}M_{\alpha i} = (\nabla^i_R)_{\alpha\beta}\frac{1}{\Delta x^d}\frac{\partial M_{\alpha i}}{\partial c_\beta} = \frac{1}{R^d}(\nabla^i_R)_{\alpha\beta}\frac{df(c_\beta)}{dc_\beta}\,. \tag{3.19}$$

Now, the statistical properties of the mesoscopic model considered in this paragraph are completely specified by Eqs (3.15) and (3.19) and the corresponding Fokker–Planck or Langevin equations, Eqs (3.11) or (3.12), respectively.

At this point, it would be of interest to give a version of this mesoscopic model in the continuous spatial limit, as is in Eqs (2.1)–(2.4). Then, to write Eq. (3.19) in the continuous limit, we make the following identification:

$$\frac{1}{\Delta x^{d+1}}K_{\alpha i}M_{\alpha i} \to \nabla\frac{\delta}{\delta c}M(c) = \frac{1}{R^d}\nabla\frac{df(c)}{dc}\,, \tag{3.20}$$

where $M(c)$ is the continuous limit of Eq. (3.15), $M(c) = f(c)$. It is interesting to notice that Eq. (3.20) involves the mesoscopic parameter R and it could be considered as a definition of the functional derivative of $M(c)$, which otherwise would be ill-defined. This would be a reminiscence of the fact that a characteristic mesoscopic parameter R is required in the complete specification of the model.

Now, for the generic form of $M_{\alpha i}$, Eq. (3.15), we can write the Fokker–Planck Eq. (3.11) in the continuous limit:

$$\frac{\partial P}{\partial \tau} = -\int d\vec{r}\frac{\delta}{\delta c(\vec{r})}\nabla M\nabla\left[\frac{\delta F}{\delta c(\vec{r})} + \beta^{-1}\frac{\delta}{\delta c(\vec{r})}\right]P\,, \tag{3.21}$$

where the new time scale is

$$\tau = t\Gamma\Delta x^{2+d}\,. \tag{3.22}$$

Analogously, Eq. (2.1) is the continuous Langevin equation corresponding to the Fokker–Planck Eq. (3.21).

From the Fokker–Planck or the Langevin, Eqs (3.11) and (3.12), respectively, it is possible to derive the equation for the moments. For example, at the continuous limit the first moment obeys the following equation (see Appendix):

$$\frac{\partial\langle c(\vec{r},\tau)\rangle}{\partial\tau} = \nabla\langle M\nabla\frac{\delta F}{\delta c}\rangle - \beta^{-1}\nabla\langle\nabla\frac{\delta}{\delta c}M\rangle\,, \tag{3.23}$$

where the functional derivative included in the last term is evaluated in accordance with the prescription given by Eq. (3.20).

The most interesting aspect of the Langevin equation with multiplicative noise is the presence of terms like the last one in Eq. (3.23). In Section 5 we will see its effects.

4. The numerical algorithm

In order to incorporate cases like Eq. (2.1) in a general formalism, we write a general Langevin field equation with multiplicative noise in the following discrete form

$$\dot{\psi}_\mu(t) = v_\mu(\psi) + g_{\mu\alpha}(\psi)\xi_\alpha(t)\,, \tag{4.1}$$

where we have used now the notation $\psi_\mu(t) = c(\vec{r}_\mu, t)$. Eq. (4.1) is a set of coupled stochastic differential equations for the variables $\psi_\mu(t)$, defined in a d-dimensional lattice of total volume V and cubic cells of volume $\Delta V = (\Delta x)^d$. The noise $\xi_\alpha(t)$ has the gaussian white-noise correlation

$$\langle \xi_\alpha(t)\xi_\beta(t')\rangle = 2D\delta_{\alpha\beta}\delta(t-t')\,, \tag{4.2}$$

where the intensity of the discrete noise is now $D = \beta^{-1}\Delta x^{-d}$. $g_{\mu\nu}(\psi)$ is the multiplicative function that couples the variable to the noise. This the standard procedure as far as the relevant lengths of our system are between Δx and $V^{1/d}$. Then, the problem is to integrate a finite number of coupled multiplicative Langevin equations.

The algorithm for one variable Langevin equations with multiplicative noise can be easily generalized to the multivariable case, Eq. (4.1), when either $g_{\mu\alpha}$ is constant (additive noise) or it is diagonal. When $g_{\mu\alpha}$ is neither constant nor diagonal (as in Eq. (2.1) for a conserved variable) some problems appear that should be studied carefully [15].

We will shortly review now the standard derivation of the algorithm and the problems involved in it, and then we will consider our proposal for the resolution of the problem.

The numerical algorithm is obtained from the formal integration of Eq. (4.1) during a time step Δ:

$$\psi_\mu(t+\Delta) = \psi_\mu(t) + \int_t^{t+\Delta} \{v_\mu(\psi(t')) + g_{\mu\nu}(\psi(t'))\xi_\alpha(t')\}dt'\,. \tag{4.3}$$

In the standard procedure, one expands the arguments of the integrals in Eq. (4.3)

$$v_\mu(t') = v_\mu(t) + \left.\frac{\partial v_\mu}{\partial \psi_\nu}\right|_{\psi(t)} \left(\psi_\nu(t') - \psi_\nu(t)\right) + \cdots , \tag{4.4}$$

$$g_{\mu\alpha}(t') = g_{\mu\alpha}(t) + \left.\frac{\partial g_{\mu\alpha}}{\partial \psi_\nu}\right|_{\psi(t)} \left(\psi_\nu(t') - \psi_\nu(t)\right) + \cdots . \tag{4.5}$$

By taking into account that, to the lowest order, we have:

$$\psi_\mu(t+\Delta) - \psi_\mu(t) = g_{\mu\alpha}(\psi(t)) \int_t^{t+\Delta} \xi_\alpha(t')dt' + o(\Delta) \tag{4.6}$$

and then, substituting Eqs (4.4)–(4.6) into Eq. (4.3) one gets an algorithm valid to first order in Δ:

$$\begin{aligned}\psi_\mu(t+\Delta) =& \psi_\mu(t) + v_\mu(\psi(t))\Delta + g_{\mu\alpha}(\psi(t))X_\alpha(t) \\ &+ \left.\frac{\partial g_{\mu\alpha}}{\partial \psi_\nu}\right|_{\psi(t)} g_{\nu\beta}(\psi(t))Y_{\alpha\beta}(t) + o(\Delta^{3/2}) ,\end{aligned} \tag{4.7}$$

where

$$X_\alpha(t) = \int_t^{t+\Delta} \xi_\alpha(t')dt' , \tag{4.8}$$

$$Y_{\alpha\beta}(t) = \int_t^{t+\Delta} \xi_\alpha(t')dt' \int_t^{t'} \xi_\beta(t'')dt'' . \tag{4.9}$$

$X_\alpha(t)$ is a gaussian random variable of order $\Delta^{1/2}$ with zero mean and variance $2D\Delta$. It can be simulated easily as:

$$X_\alpha(t) = \sqrt{2D\Delta}\eta_\alpha , \tag{4.10}$$

where η_α are gaussian random numbers of zero mean and variance equal to one. However, $Y_{\alpha\beta}(t)$ is a non-gaussian process of order Δ. Its statistical properties can be calculated, but there is no way to simulate it exactly [25, 26]. A possible strategy consists in separating $Y_{\alpha\beta}(t)$ in two parts [27, 28]

$$Y_{\alpha\beta}(t) = \tfrac{1}{2}S_{\alpha\beta}(t) + \tfrac{1}{2}A_{\alpha\beta}(t) , \tag{4.11}$$

where $S_{\alpha\beta}(t)$ is the symmetric part of the integral (4.9)

$$S_{\alpha\beta}(t) = Y_{\alpha\beta} + Y_{\beta\alpha} = \int_t^{t+\Delta} \xi_\alpha(t')dt' \int_t^{t+\Delta} \xi_\beta(t'')dt'' = X_\alpha(t)X_\beta(t)\,, \quad (4.12)$$

which can be simulated using Eq. (4.10). $A_{\alpha\beta}(t)$ is the antisymmetric part of Eq. (3.7) and some kind of approximation should be done to simulate it [27, 28].

Here, we take a different attitude. We do not try to follow the previous procedures to simulate $A_{\alpha\beta}$ but to use from the very beginning the interpretation of the stochastic integral implicit in Eq. (4.3)[1]. This type of equation was obtained in Section 2 in the context of the Stratonovich interpretation. The prescription stablishes that a stochastic integral, like the one that appears in Eq. (4.3), should be interpret as [17, 20]

$$\int_t^{t+\Delta} dt' g_{\mu\alpha}(\psi(t'))\xi_\alpha(t') = g_{\mu\alpha}\left(\frac{\psi(t)+\psi(t+\Delta)}{2}\right) \int_t^{t+\Delta} dt'\xi_\alpha(t') + o(\Delta^{3/2}) \quad (4.13)$$

and now $g_{\mu\alpha}$ is expanded as

$$g_{\mu\alpha}\left(\frac{\psi(t)+\psi(t+\Delta)}{2}\right) = g_{\mu\alpha}(\psi(t)) + \frac{1}{2}\frac{\partial g_{\mu\alpha}(\psi(t))}{\partial\psi_\beta(t)} g_{\beta\gamma}(\psi(t))X_\gamma(t) + o(\Delta)\,. \quad (4.14)$$

Substituting now Eqs. (4.13-14) into Eq. (4.3) we get up to first order in Δ:

$$\psi_\mu(t+\Delta) = \psi_\mu(t) + v_\mu(\psi(t))\Delta + g_{\mu\alpha}(\psi(t))X_\alpha(t) + \frac{1}{2}\frac{\partial g_{\mu\alpha}(\psi(t))}{\partial\psi_\nu} g_{\nu\beta}(\psi(t))X_\alpha(t)X_\beta(t) + o(\Delta^{3/2})\,. \quad (4.15)$$

The fact that the algorithm could now be given in terms of only symmetric processes $S_{\alpha\beta}$ does not mean that the antisymmetric term $A_{\alpha\beta}$ in Eq. (4.11) is zero in the Stratonovich interpretation. Eq. (4.15) is an alternative equation of Eq. (4.7), both containing the same statistical information, but they could not be compared term by term.

The algorithm (4.15) can present some technical difficulties when one tries to make the corresponding computer program, so it would be worth to

[1] This is neither necessary in the single variable case nor in the multivariable case when $g_{\mu\nu}$ is diagonal, because for these situations the tensor that multiplies $Y_{\mu\nu}$ in Eq. (4.7) is symmetric, and the antisymmetric term $A_{\mu\nu}$ does not appears. These are the cases in which the standard derivation works.

simplify it. The idea is to substitute Eq. (4.15) by other algorithm with the same statistical properties. These properties are contained in the probability density for the field, $P[\psi]$, so this substitution has to maintain the same dynamics for $P[\psi]$. The well known Fokker–Planck equation obeyed by $P[\psi]$ corresponding to the stochastic differential equation (4.1)–(4.2) could be written, in the Stratonovich interpretation [17], as

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \psi_\mu}(v_\mu + D\frac{\partial g_{\mu\alpha}}{\partial \psi_\nu} g_{\nu\alpha})P + D\frac{\partial}{\partial \psi_\mu}\frac{\partial}{\partial \psi_\nu} g_{\mu\alpha} g_{\nu\alpha} P\,. \tag{4.16}$$

It can be proved [14] that the same Fokker–Planck equation is obtained if Eq. (4.15) is substituted by the more simple algorithm

$$\psi_\mu(t+\Delta) = \psi_\mu(t) + (v_\mu + D\frac{\partial g_{\mu\alpha}}{\partial \psi_\nu} g_{\nu\alpha})\Delta + g_{\mu\alpha} X_\alpha(t)\,, \tag{4.17}$$

where we have replaced the last term in Eq. (4.15) by its mean value. Therefore both algorithms, of first order in Δ, are stochastically equivalent in the sense that they correspond to the same Fokker–Planck equation (4.16). We call Eq. (4.17) the Minimum Algorithm for Multiplicative Langevin Equations (MAMLE).

5. Application to domain growth

In this section, the algorithm derived previously is applied to the particular model described by Eqs (2.1)–(2.4). The simulation will take place in a two dimensional lattice of $L \times L$ cells of volume Δx^2. Then, the Eqs (2.1)–(2.4) have to be expressed in terms of the discrete variables. Following Section 5, we interpret the continuous model described by Eqs (2.1)–(2.4) in terms of a family of mesoscopic models which are specified by the discrete Langevin equation (3.12) and a family of functions $m_{\alpha i}(\{c\})$ given in Eqs (3.15)–(3.18). The noise correlation is given by Eq. (3.13).

As it is explained in Ref. [14], one of the advantages of this interpretation is that the dynamics obtained from the continuous and discrete models are equivalent term by term according to the standard rules of functional calculus. The different models of the family differ in the number of neighbors considered and this introduces a mesoscopic lenght. Here, in order to consider a simple version for the computer simulation, we have taken only a dependence on one couple α, $\alpha + i$. Then, we take the following expression for $M_{\alpha i}(c)$:

$$M_{\alpha i}(\{c\}) = \tfrac{1}{2}\big(f(c_\alpha) + f(c_{\alpha+i})\big)\,. \tag{5.1}$$

This choice implicitly defines a characteristic mesoscopic length scale, $R = \sqrt{2}\Delta x$.

Now, the MAMLE is obtained by substituting the particular expressions of Eqs (2.1)–(2.4) into the general equation (4.17). The explicit calculation of the third term in the right hand side of Eq. (4.17) is:

$$\begin{aligned}\frac{g_{\mu\alpha}}{c_\nu} g_{\nu\alpha} &= (\nabla^i_L)_{\mu\alpha} \tfrac{1}{2} \left(f'(c_\alpha)\delta_{\alpha\nu} + f'(c_{\alpha+i})\delta_{\alpha+i\ \nu} \right) (\nabla^i_L)_{\nu\alpha} m_{\alpha i} \\ &= \tfrac{1}{2} (\nabla^i_L)_{\mu\alpha} m_{\alpha i} (\nabla^i_R)_{\alpha\nu} f'(c_\nu) \,. \end{aligned} \tag{5.2}$$

With this result, the MAMLE finally is:

$$\begin{aligned} c_\mu(\tau + \Delta) = c_\mu(\tau) + \Big(&-\tfrac{1}{2} (\nabla^i_L)_{\mu\nu} m^2_{\nu i} (\nabla^i_R)_{\nu\sigma} (c - c^3 + (\nabla)^2 c)_\sigma \\ &-2\beta^{-1} (\nabla^i_L)_{\mu\nu} m_{\nu i} (\nabla^i_R)_{\nu\sigma} f'(c_\sigma) \Big) \Delta + (\nabla^i_L)_{\mu\beta} m_{\beta i} X^i_\beta(\tau) \,, \end{aligned} \tag{5.3}$$

where the gaussian numbers $X_\mu(\tau)$ are computed from Eq. (4.10).

Now one can obtain the equation of motion for the first moment of the Fourier transformed variable $c_k(t)$ in the linear approximation:

$$\langle \dot{c}_k \rangle = -\tfrac{1}{2} k^2 \left(k^2 - 1 + \frac{4a\beta^{-1}}{R^2} \right) \langle c_k \rangle \,. \tag{5.4}$$

As the structure function, $S(\vec{k}, t)$ gives a better idea of the growth process we present also the study of this quantity which is the Fourier transform of the pair correlation function, $G(\vec{r}, t)$

$$G(\vec{r}, t) = \frac{1}{L^2} \sum_{\vec{r}\,'} \langle c(\vec{r} + \vec{r}\,', t) c(\vec{r}\,', t) \rangle \,. \tag{5.5}$$

The evolution of the spherically averaged structure function $S(k, t)$ in the linear regime is [15]

$$\begin{aligned} \frac{d}{dt} S(k,t) = -k^2 \left(k^2 - 1 + \frac{4a\beta^{-1}}{R^2} \right) S(k,t) + 2\beta^{-1} k^2 \\ -2\beta^{-1} a k^2 \frac{1}{(2\pi)^d} \int d\vec{q} S(q,t) \,. \end{aligned} \tag{5.6}$$

From Eqs (5.4) and (5.6) one can see that only the modes with $k < k_c = 1 - 4a\beta^{-1}/R^2$ are unstable, and they will grow in the early stages of the evolution. On the contrary, modes with $k > k_c$ remains stable during the linear regime. So, one can expect that, for initial times, with a smaller

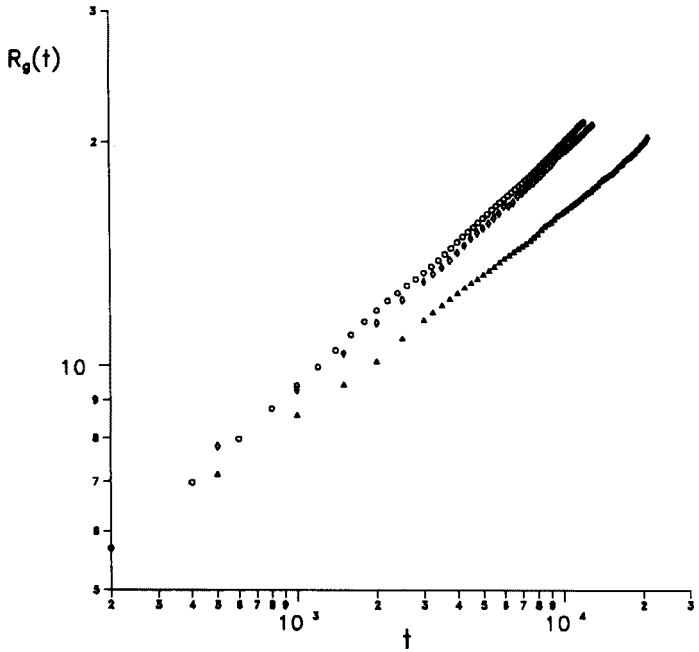

Fig. 1. log-log plot of the time evolution of $R_g(t)$ for a deterministic system (○) and for a system with additive noise (◇) and with multiplicative noise (△).

number of modes growing, the domain growth would be different than in the additive noise case $a = 0$.

A simulation of this system with different values of the parameters has been performed. We have taken different values of the intensity of the noise below the critical point, and two values of the constant a. We have considered a system of 120×120 points with a spacial mesh $\Delta x = 1$, a time step $\Delta = 0.025$ and periodic boundary conditions. We start from an homogeneous initial state $c = 0$, and the system is let to evolve until a time $\tau = 20000$. Each data results from the statistical average of ten independent runs.

The results from the simulation show that at very early times, the domain growth is faster for larger values of the intensity of the noise, and for smaller values of the parameter a. This indicate that fluctuations facilitate the phase separation, specially in the additive case ($a = 0$). For large times, the situation is quite different. Fig. 1 shows the growth of the relevant length $R_g(t)$ defined as the smallest value of r for which the correlation function becomes zero. We present the results for an additive noise ($\beta^{-1} = 0.1$, $a = 0$) and a multiplicative noise ($\beta^{-1} = 0.1$, $a = 0.5$) and we compare these with the result obtained for the deterministic case ($\beta^{-1} = 0$, $a = 0$).

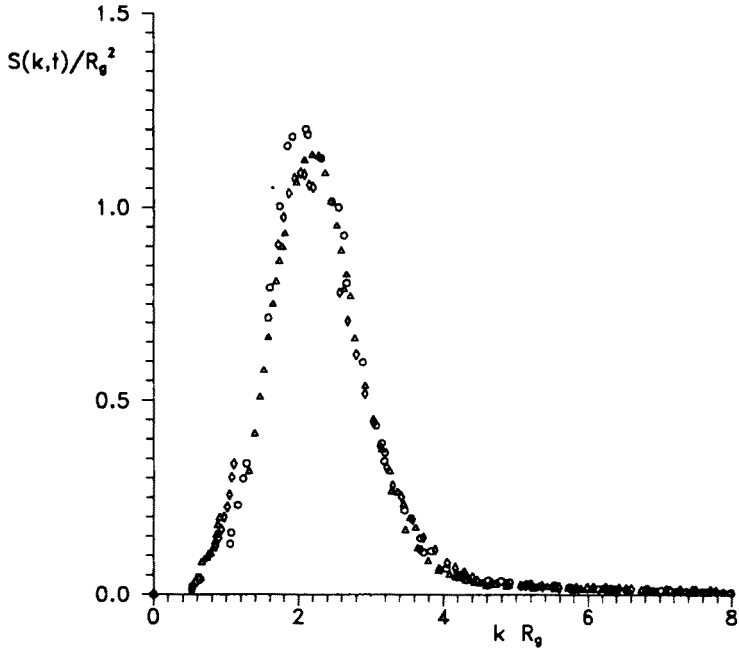

Fig. 2. Scaled radial correlation function, Eq. (5.7), vs the scaled variable $kR_g(t)$ for the same cases of Fig. 1.

We can see that, as occurs in the deterministic problem [10], an increase in the parameter a produces a delay in the dynamics. For the additive noise, the characteristic length is slightly smaller than in the deterministic case, but it growths with the same $^1/_3$ power law. For the case with multiplicative noise, R_g presents a crossover from a $^1/_4$ to $^1/_3$ power law as a function of time, just as in the deterministic case with $a \neq 0$.

In order to treat to see a new aspects coming from the multiplicative noise, and not due to the presence of a in the deterministic part of Eq. (5.3), we have explored the possible scaling properties of this model for very large times. Fig. 2 shows the scaled structure function

$$s(kR_g(t)) = (R_g(t))^{-2}\, S(k,t)\,, \tag{5.7}$$

as a function of the scaled variable kR_g. For each case, the form of this function is the same for the different times showed, and it is very similar for the three cases. Similar results can be obtained for the scaled correlation function, so we conclude that at late stage, multiplicative noise does not seem relevant.

6. Conclusions

In this paper we have presented a derivation of Ginzburg–Landau equations with multiplicative noise for conserved order parameter with concentration-dependent mobility and multiplicative noise, which could be relevant in the context of phase separation and domain growth dynamics. This equation incorporates new terms of stochastic origin that give new contributions to the evolution of statistical properties. In particular, we have obtained new contributions to the Cahn–Hilliard–Cook theory. Our derivations of the Fokker–Planck equations are based on coarse-grained procedures in a discrete lattice, and the corresponding Langevin equations are obtained by standard techniques of stochastic processes. A mesoscopic model, which contains a characteristic mesoscopic length, is introduced. An expression of the mesoscopic model is also done in the continuous spatial limit. The presence of a mesoscopic length in the model avoids possible misinterpretations, of, for example, functional derivatives of the mobility function. In this respect, we would like to remark that the dynamics depend on the characteristics of the particular mesoscopic model and it would be of interest to dedicate some effort to obtain such models from first principles.

We have also derived a new algorithm of simulation of general multivariable Langevin equations with multiplicative noise. This algorithm does not involve the evaluation of non-gaussian quantities that cannot be simulated exactly. Furthermore, we have also introduced a simplified version of the algorithm which still reproduces the statistical properties of the original Langevin equation. We call it the Minimum Algorithm for Multiplicative Langevin Equations (MAMLE). Both algorithms are specially suitable for models with a conserved order parameter, since the nontrivial coupling between the noise and the field made the application of standard algorithms problematic.

As an application, we have made use of the MAMLE in the integration of stochastic equations of a concentration field in the context of domain growth. The simulation of a Ginzburgh–Landau equation with multiplicative noise to study spinodal decomposition has confirmed the role of the multiplicative noise that is predicted from a linear analysis of the model. For large times we see that the growth mechanism dominate and the effects of the multiplicative noise are not relevant.

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Appendix A

Here we present the mathematical details of the equivalence between the Fokker–Planck Eq. (3.11) and the Langevin Eq. (3.12). Although the derivation of the Fokker–Planck equation corresponding to a Langevin equation is standard, the case of the model derived in Section 3 presents special aspects owing to the presence of the gradient operators and the dependence of M on the concentrations of the whole configuration.

To write Eq. (3.10) as Eq. (3.11), we make use of the right and left discrete gradients, defined by:

$$(\nabla_{\mathrm{R}}^{i})_{\alpha\beta} = \frac{1}{\Delta x}(\delta_{\alpha+i\,\beta} - \delta_{\alpha\beta}),$$
$$(\nabla_{L}^{i})_{\alpha\beta} = \frac{1}{\Delta x}(\delta_{\alpha\beta} - \delta_{\alpha-i\,\beta}). \tag{A.1}$$

Then, the operator $K_{\alpha i}$ can be expressed as

$$K_{\alpha i} = \Delta x(\nabla_{R}^{i})_{\alpha\beta}\frac{\partial}{\partial c_{\beta}}. \tag{A.2}$$

Furthermore, to obtain the Langevin equation corresponding to the Fokker–Planck Eq. (3.10), we write Eq. (3.11) in a more familiar form

$$\frac{\partial P(\{c\},t)}{\partial t} = -\Delta x^{2}\Gamma\frac{\partial}{\partial c_{\alpha}}(\nabla_{L}^{i})_{\alpha\beta}\left\{M_{\beta i}(\nabla_{R}^{i})_{\beta\sigma}\frac{\partial F}{\partial c_{\sigma}}\right.$$
$$\left.-\beta^{-1}m_{\beta i}(\nabla_{R}^{i})_{\beta\sigma}\left(\frac{\partial m_{\beta i}}{\partial c_{\sigma}}\right) + \beta^{-1}m_{\beta i}(\nabla_{R}^{i})_{\beta\sigma}\frac{\partial}{\partial c_{\sigma}}m_{\beta i}\right\}P(\{c\},t), \tag{A.3}$$

where the function $m(\{c\})$ was defined in Eq. (3.14). In Eq. (A.3) we have made use of the property $(\nabla_{R}^{i})_{\alpha\beta} = -(\nabla_{L}^{i})_{\beta\alpha}$.

In this Fokker–Planck equation the first term is the deterministic drift, the second term is the "Stratonovich" spurious drift and the last one is the Stratonovich-like diffusion term. From Eq. (A.3), one can conjecture that the corresponding Langevin equation is given by Eq. (3.12). This is a stochastic differential equation in the Stratonovich interpretation, with a gaussian white noise of zero mean and correlation given by Eq. (3.13).

Now, to prove that the Langevin Eq. (2.12) corresponds to the Fokker–Planck Eq. (2.11), we derive Eq. (2.11) from Eq. (2.12). According to van Kampen lemma [20]

$$P(\{c\},t) = \langle\prod_{\alpha}\delta(c_{\alpha}(t) - c_{\alpha})\rangle = \langle\rho(\{c\},t)\rangle \tag{A.4}$$

and averaging the stochastic Liouville equation for $\rho(\{c\},t)$ we obtain

$$\frac{\partial P}{\partial t} = -\Delta x^2 \Gamma \frac{\partial}{\partial c_\alpha}(\nabla^i_L)_{\alpha\beta}\left\{M_{\beta i}(\nabla^i_R)_{\beta\sigma}\frac{\partial F}{\partial c_\sigma} - \beta^{-1} m_{\beta i}(\nabla^i_R)_{\beta\sigma}\frac{\partial m_{\beta i}}{\partial c_\sigma}\right\} P$$
$$- \frac{\partial}{\partial c_\alpha}(\nabla^i_L)_{\alpha\beta} m_{\beta i}\langle \xi^i_\beta(t)\rho\rangle \,. \tag{A.5}$$

Now, the average in Eq. (A.5) is worked out with the aid of Novikov theorem [21]

$$\langle \xi^i_\beta(t)\rho\rangle = -\Delta x^2 \beta^{-1}\Gamma \frac{\partial}{\partial c_\sigma}\left\langle \left.\frac{\delta c_\sigma(t)}{\delta \xi^i_\beta(t')}\right|_{t=t'} \rho \right\rangle . \tag{A.6}$$

The response function $\delta c/\delta\xi$ is obtained from Eq. (3.12):

$$\left.\frac{\delta c_\sigma(t)}{\delta \xi^i_\beta(t')}\right|_{t=t'} = (\nabla^i_R)_{\sigma\beta} m_\beta \,. \tag{A.7}$$

By substituting Eqs (A.6) and (A.7) into Eq. (A.5), we recover the Fokker–Planck equation (3.11).

Both equations (3.11) and (3.12) can be employed indistinctly to derive the statistical properties of the system. As an example, we obtain the equation of the first moment. This could be done by averaging the Langevin equation (3.12) and making use of the Novikov theorem to deal with the noise term. The result can also be obtained directly from the Fokker–Planck equation (A.3)

$$\frac{\partial}{\partial t}\langle c_\mu(t)\rangle = \Delta x^2 \Gamma \langle (\nabla^i_L)_{\mu\beta} M_\beta (\nabla^i_R)_{\beta\sigma}\frac{\partial F}{\partial c_\sigma}\rangle$$
$$- \beta^{-1}\Gamma \Delta x^2 \langle (\nabla^i_L)_{\mu\beta}(\nabla^i_R)_{\beta\sigma}\frac{\partial M_\beta}{\partial c_\sigma}\rangle \,, \tag{A.8}$$

where the last term accounts for the fact we have a multiplicative noise.

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