SYMMETRIES AND SPECTRA OF MAPS*

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The symmetry decomposition of the Frobenius–Perron operator and the associated Zeta functions is worked out for the case of reflection-symmetric 1-d maps.

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1. Introduction

The symmetry decomposition of linear operators is an important and well known method in quantum mechanics. What is less well appreciated is that similar considerations apply to the case of classical evolution equations, such as the Liouville equation or the Frobenius Perron equation. As soon as matrix representations of these operators are used, one is again in the framework familiar from quantum mechanics and the same mathematical methods can be applied. The purpose of these notes is to discuss in some detail how the symmetry decomposition can be achieved on the level of periodic orbit theory. The relevant trace formulas have been discussed in [1] for flows and [2] for maps.

The key to the problem lies in the observation that the symmetries of the equations of motion are reflected in the periodic orbits as well. Operating with a symmetry on an orbit either reproduces the orbit (so-called symmetric orbit) or yields a new one of the same shape (non-symmetric orbit). A symmetric orbit may be decomposed further into irreducible segments such

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that the properties of the full orbit (periods, actions, instabilities, Maslov phases *etc.*) can be obtained from the properties of the segments. Thus the desymmetrization of trace formulas builts on a desymmetrization of periodic orbits.

The general principles of this symmetry decomposition have been discussed by Cvitanović and Eckhardt [3], Lauritzen [4] and Robbins [5], particular examples have been given by Gutzwiller [6] and Hönig and Wintgen [7]. Here the case of 1-d maps will be worked out in detail. The merits of this calculation and the example are that it can be carried out all the way to the end without numerical computations. In addition, the effect of boundary orbits, *i.e.*, fixed points invariant under symmetries, comes out rather clearly.

The outline of the paper is as follows. In the next section, the trace formula for 1-d maps is derived. The symmetry decomposition is given in Section 3. Then the full spectrum and the symmetry reduced spectra of an example are worked out in Sections 4 and 5. Some concluding remarks are collected in Section 6.

2. Spectra of 1-d maps

Consider a 1-d map $x_{n+1} = f(x_n)$ on an interval I. The time evolution of a function $\rho(x)$ on I over k steps is described by the Frobenius–Perron equation

$$\rho_{n+k}(y) = \int_I \mathcal{L}_k(y,x)\rho_n(x)\,dx \tag{1}$$

with the kernel

$$\mathcal{L}_k(y,x) = \delta\left(y - f^{(k)}(x)\right) , \tag{2}$$

where $f^{(k)}$ denotes the k'th iterate of the map. The kernel of the one step propagator will often be written $\mathcal{L}$, without the index. Suppose that $\mathcal{L}$ has at least formally a decomposition into eigenvalues λ_ν and projections onto eigendistributions P_ν. Then

$$\operatorname{tr}\mathcal{L}_k = \int_I dx\,\mathcal{L}_k(x,x) = \sum_\nu \lambda_\nu^k . \tag{3}$$

The Laplace transform

$$\Omega(z) = \sum_{k=1}^{\infty} z^k \operatorname{tr}\mathcal{L}_k = \sum_\nu \sum_{k=1}^{\infty} (z\lambda_\nu)^k = \sum_\nu \frac{z\lambda_\nu}{1 - z\lambda_\nu} , \tag{4}$$

thus has poles at the values $z = 1/\lambda_\nu$, the inverses of the eigenvalues. It is related to the Fredholm determinant $Z(z) = \det(1 - z\mathcal{L}) = \prod(1 - z\lambda_\nu)$ by

$$\Omega(z) = -z\frac{\partial}{\partial z} \ln Z(z) . \tag{5}$$

Periodic orbits enter through the evaluation of

$$\operatorname{tr} \mathcal{L}_k = \int_I dx\, \delta\left(x - f^{(k)}(x)\right) . \tag{6}$$

Obviously, the integral picks up a contribution from every point x_P which returns after k iterations, *i.e.*, is periodic with period k. Denote the set of all such points by $Fix(k)$. Undoing the δ-function one finds a weight for each such point of the form $1/|1 - Df^{(k)}(x_P)|$, where $Df^{(k)}$ denotes the derivative of the k-th iterate of f at x_P. Thus

$$\operatorname{tr} \mathcal{L}_k = \sum_{x_P \in Fix(k)} \frac{1}{|1 - Df^{(k)}(x_P)|} \tag{7}$$

and

$$\Omega(z) = \sum_{k=1}^{\infty} \sum_{x_P \in Fix(k)} \frac{z^k}{|1 - Df^{(k)}(x_P)|} . \tag{8}$$

The primitive period of an orbit is that number n_p for which $f^{(n_p)}(x_P) = x_P$ but $f^{(k)}(x_P) \neq x_P$ for all $k < n_p$. Primitive periodic orbits will be labelled by p. A primitive periodic orbit of period n_p has n_p different points in phase space, but all points have the same derivative $\Lambda_p := Df^{(n_p)}(x_P)$. The sums in the above expression for $\Omega(z)$ also extend over multiple traversals of primitive periodic orbits (*i.e.*, $k = rn_p$, $r > 1$); they have n_p different points and derivatives Λ_p^r. Therefore, one can replace the sums in (8) by sums over all primitive periodic orbits p and their multiple traversals r,

$$\Omega(z) = \sum_p \sum_{r=1}^{\infty} n_p \frac{z^{n_p r}}{|1 - \Lambda_p^r|} . \tag{9}$$

Using

$$|1 - \Lambda_p^r|^{-1} = |\Lambda_p|^{-r}(1 - \Lambda_p^{-r})^{-1} = |\Lambda_p|^{-r} \sum_{j=0}^{\infty} \Lambda_p^{-rj} \tag{10}$$

and the abbreviation

$$t_p = \frac{z^{n_p}}{|\Lambda_p|} \qquad z\frac{\partial}{\partial z} t_p = n_p t_p \tag{11}$$

one finds

$$
\begin{aligned}
\Omega(z) &= \sum_p \sum_{j=0}^{\infty} \sum_{r=1}^{\infty} n_p \left(z^{n_p}|\Lambda_p|^{-1}\Lambda_p^{-j}\right)^r \\
&= \sum_p \sum_{j=0}^{\infty} \frac{n_p t_p \Lambda_p^{-j}}{1 - t_p \Lambda_p^{-j}} \\
&= \sum_p \sum_{j=0}^{\infty} \left(-z\frac{\partial}{\partial z}\right) \ln\left(1 - t_p\Lambda_p^{-j}\right) \\
&= \left(-z\frac{\partial}{\partial z}\right) \ln \prod_p \prod_{j=0}^{\infty} \left(1 - t_p\Lambda_p^{-j}\right) \\
&= \left(-z\frac{\partial}{\partial z}\right) \ln Z(z) = -z\frac{Z'(z)}{Z(z)} .
\end{aligned} \tag{12}
$$

Comparing with Eq. (5) for the Fredholm determinant one concludes that $Z(z)$ has two representations: one involving the eigenvalues of the operator in the form of a Fredholm determinant det $(1 - z\mathcal{L})$ [8], the other involving the periodic orbits in a form reminiscent of the Selberg Zeta functions in the theory of geodesic motion on surfaces of constant negative curvature [9, 10]:

$$
Z(z) = \prod_\nu (1 - z\lambda_\nu) = \prod_{j=0}^{\infty} \prod_p \left(1 - t_p\Lambda_p^{-j}\right) . \tag{13}
$$

Thus the inverse eigenvalues can be computed as the zeros of the infinite product over periodic orbits. It often is convenient to separate the j-product and to consider $Z(z) = \prod_j \zeta_j^{-1}$ as a product of dynamical zeta functions $\zeta_j^{-1} = \prod_p (1 - t_p\Lambda_p^{-j})$. An explicitly computable example will be given in Sect. 4. We now turn to the symmetry decomposition.

3. Symmetry reduction

Let g denote a reflection, *i.e.*, $g(x) = -x$. Suppose that the function f acts on a symmetric interval, $gI \equiv I$, and satisfies $f(-x) = -f(x)$. Then g is a symmetry of the map in the sense that if $\{x_n\}$ is one trajectory, then $\{gx_n\}$ is another one since

$$
gx_{n+1} = gf(x_n) = f(gx_n) . \tag{14}
$$

It is thus possible to restrict the dynamics to one half of the original interval. Every time a trajectory leaves this interval, it can be mapped back using g. This interval is an example of a fundamental domain [3].

The reflection g together with the identity e form a two element group which has as irreducible representations a symmetric and a antisymmetric one, denoted by indices + and −. The corresponding projections

$$P_+ = \tfrac{1}{2}(e+g)\,, \tag{15}$$
$$P_- = \tfrac{1}{2}(e-g) \tag{16}$$

are a decomposition of unity, $e = P_+ + P_-$. The decomposition of the evolution kernel $\mathcal{L}_1$ is given by

$$\begin{aligned} e\mathcal{L}e &= (P_+ + P_-)\mathcal{L}(P_+ + P_-) \\ &= P_+\mathcal{L}P_+ + P_-\mathcal{L}P_- \\ &= \quad \mathcal{L}_+ \quad + \quad \mathcal{L}_-\,. \end{aligned} \tag{17}$$

The last equation defines the evolution operators restricted to the symmetric and antisymmetric subspaces; terms with mixed projectors ($P_+\mathcal{L}P_-$ and $P_+\mathcal{L}P_-$) vanish. For the symmetric subspaces one now finds with equations (15) and (17),

$$\mathcal{L}_{k,+}(y,x) = \tfrac{1}{4}\left(\mathcal{L}_k(y,x) + \mathcal{L}_k(y,gx) + \mathcal{L}_k(gy,x) + \mathcal{L}_k(gy,gx)\right)\,, \tag{18}$$

which with the explicit form of $\mathcal{L}$, Eq. (2), becomes

$$\begin{aligned} \mathcal{L}_{k,+}(y,x) =& \tfrac{1}{4}\Big(\delta(y - f^{(k)}(x)) + \delta(y + f^{(k)}(x)) \\ &+ \delta(-y - f^{(k)}(x)) + \delta(-y + f^{(k)}(x))\Big)\,. \end{aligned} \tag{19}$$

The two middle terms and the first and last terms coincide, so that the terms relevant for the trace in the symmetric subspace become

$$\mathcal{L}_{k,+}(x,x) = \tfrac{1}{2}\Big(\delta(x - f^{(k)}(x)) + \delta(x + f^{(k)}(x))\Big)\,. \tag{20}$$

Similarly, in the antisymmetric subspace, one finds

$$\mathcal{L}_{k,-}(x,x) = \tfrac{1}{2}\Big(\delta(x - f^{(k)}(x)) - \delta(x + f^{(k)}(x))\Big)\,. \tag{21}$$

To determine the spectrum in the subspaces, one again computes the Laplace transforms $\Omega_{+,-}$ and evaluates traces using periodic orbits. Because of the presence of the second term, it is useful to distinguish three cases.

Non-symmetric cycles: For non-symmetric orbits, the sets of points

$\{gx_P\} = \emptyset$. Thus g generates a second orbit with the same number of points and the same stability properties. Both orbits only contribute to the first term in (20), (21), and their being two of them cancels the $^1/_2$. Thus, if $\{a\}$ labels one representative from these non-symmetric orbits, their contributions to $\Omega_{+,-}(z)$ are

$$\frac{1}{2}\sum_a \sum_{r=1}^{\infty} 2n_a \frac{z^{n_a r}}{|1-\Lambda_a^r|} = \left(-z\frac{\partial}{\partial z}\right) \ln \prod_a \prod_{j=0}^{\infty} \left(1 - z^{n_a}|\Lambda_a|^{-1}\Lambda_a^{-j}\right) . \quad (22)$$

Symmetric cycles: Operating with g on the set of points of a symmetric orbit reproduces the set. Periods for symmetric orbits are always even ($2n_s$) and they satisfy $f^{(n_s)}(x_P) = gx_P$ for all points of the trajectory. These orbits contribute to the trace through $\delta(x - f^{(k)}(x))$ for even traversals, $k = 2rn_s$, and through $\delta(x + f^{(k)}(x))$ for odd traversals, $k = (2r-1)n_s$. The number of phase space points is always $2n_s$, since the trace is to be computed over the full phase space. Let Λ_s denote the Lyapunov exponent computed for a segment of length n_s. The sign difference in the δ-functions for even and odd traversals can easily be incorporated by defining an effective $\tilde{\Lambda}_s = -\Lambda_s$. Then the contribution of symmetric orbits to $\Omega_+(z)$ can be computed as follows (with t_s as defined in Eq. (11)):

$$\begin{aligned}
&\frac{1}{2}\sum_s \sum_{r=1}^{\infty} 2n_s \frac{z^{2rn_s}}{|1-\Lambda_s^{2r}|} + \frac{1}{2}\sum_s \sum_{r=1}^{\infty} 2n_s \frac{z^{(2r-1)n_s}}{|1+\Lambda_s^{2r-1}|} \\
&= \sum_s \sum_{r=1}^{\infty} \sum_{j=0}^{\infty} n_s \left(1 + \frac{1}{t_s \tilde{\Lambda}_s^{-j}}\right) \left(t_s^2 \tilde{\Lambda}_s^{-2j}\right)^r \\
&= \sum_s \sum_{j=0}^{\infty} n_s \left(\frac{t_s \tilde{\Lambda}_s^{-j} + 1}{(1 - t_s \tilde{\Lambda}_s^{-j})(1 + t_s \tilde{\Lambda}_s^{-j})}\right) t_s \tilde{\Lambda}_s^{-j} \\
&= \sum_s \sum_{j=0}^{\infty} \left(n_s \frac{t_s \tilde{\Lambda}_s^{-j}}{1 - t_s \tilde{\Lambda}_s^{-j}}\right) \\
&= \left(-z\frac{\partial}{\partial z}\right) \ln \prod_s \prod_{j=0}^{\infty} \left(1 - t_s \tilde{\Lambda}_s^{-j}\right) .
\end{aligned} \quad (23)$$

From an analogous calculation for the antisymmetric subspace, one finds for the contribution of symmetric orbits

$$\tfrac{1}{2}\sum_s\sum_{r=1}^{\infty}2n_s\frac{z^{2rn_s}}{|1-\tilde{\Lambda}_s^{2r}|}-\tfrac{1}{2}\sum_s\sum_{r=1}^{\infty}2n_s\frac{z^{(2r-1)n_s}}{|1+\tilde{\Lambda}_s^{2r-1}|}$$
$$=\left(-z\frac{\partial}{\partial z}\right)\ln\prod_s\prod_{j=0}^{\infty}\left(1+t_s\tilde{\Lambda}_s^{-j}\right)\,. \quad (24)$$

Boundary orbits: These two cases cover all but one orbit of the system, the fixed point at the origin. It is special, since it sits on the boundary of the fundamental domain and thus contributes to both $\delta(x-f^{(k)}(x))$ and $\delta(x+f^{(k)}(x))$ at the same time. Taking traces, one finds for the contribution of a boundary orbit b.o. to $\Omega_+(z)$:

$$\begin{aligned}&\tfrac{1}{2}\sum_{r=1}^{\infty}\left(\frac{z^r}{|1-\Lambda_{\mathrm{b.o.}}^r|}+\frac{z^r}{|1+\Lambda_{\mathrm{b.o.}}^r|}\right)\\&=\tfrac{1}{2}\sum_{r=1}^{\infty}\sum_{j=0}^{\infty}z^r|\Lambda_{\mathrm{b.o.}}|^{-r}(\Lambda_{\mathrm{b.o.}}^{-rj}+(-1)^j\Lambda_{\mathrm{b.o.}}^{-rj})\\&=\sum_{r=1}^{\infty}\sum_{j=0,\mathrm{even}}^{\infty}z^r|\Lambda_{\mathrm{b.o.}}|^{-r}\Lambda_{\mathrm{b.o.}}^{-rj}\\&=\left(-z\frac{\partial}{\partial z}\right)\ln\prod_{j=0}^{\infty}\left(1-z|\Lambda_{\mathrm{b.o.}}|^{-1}\Lambda_{\mathrm{b.o.}}^{-2j}\right)\,.\end{aligned} \quad (25)$$

The contribution to $\Omega_-(z)$ contains the odd j's,

$$\tfrac{1}{2}\sum_{r=1}^{\infty}\left(\frac{z^r}{|1-\Lambda_{\mathrm{b.o.}}^r|}-\frac{z^r}{|1+\Lambda_{\mathrm{b.o.}}^r|}\right)$$
$$=\left(-z\frac{\partial}{\partial z}\right)\ln\prod_{j=0}^{\infty}\left(1-z|\Lambda_{\mathrm{b.o.}}|^{-1}\Lambda_{\mathrm{b.o.}}^{-2j+1}\right)\,. \quad (26)$$

That they contribute only to dynamical zeta functions for certain j is typical of boundary orbits. In the semiclassical context, Hönig and Wintgen [7] explained this behaviour with the symmetry properties of harmonic oscillator wavefunctions, which enter when the motion perpendicular to the orbit is harmonically approximated. As the preceding calculation shows, it can also be obtained from straightforward manipulations in the purely classical case.

Collecting contributions, one has the following expressions for the symmetry reduced zeta functions:

$$Z_+(z) = \prod_a \prod_{j=0}^{\infty} (1 - t_a \Lambda_a^{-j}) \prod_s \prod_{j=0}^{\infty} (1 - t_s \tilde{\Lambda}_s^{-j}) \prod_{j=0}^{\infty} (1 - z|\Lambda_{\text{b.o.}}|^{-1} \Lambda_{\text{b.o.}}^{-2j}) \tag{27}$$

and

$$Z_-(z) = \prod_a \prod_{j=0}^{\infty} (1 - t_a \Lambda_a^{-j}) \prod_s \prod_{j=0}^{\infty} (1 + t_s \tilde{\Lambda}_s^{-j}) \prod_{j=0}^{\infty} (1 - z|\Lambda_{\text{b.o.}}|^{-1} \Lambda_{\text{b.o.}}^{-2j+1}) . \tag{28}$$

These equations are the main result of the present section. When put together, one again finds the zeta function for the full product, $Z(z) = Z_+(z)Z_-(z)$. In the next sections, we will study an example and work out all three zeta functions explicitly.

4. An example

Consider the specific one parameter family of piecewise linear bimodal maps

$$f_s(x) = \begin{cases} -1 + \frac{2}{1-s}(1+x) & -1 \le x \le -s \\ -\frac{1}{s}x & -s \le x \le s \\ +1 - \frac{2}{1-s}(1-x) & s \le x \le 1 \end{cases} , \tag{29}$$

with $0 < s < 1$ on the interval $I = [-1, +1]$ (see Fig. 1). Since each one of the three subintervals in the definition of f_s is mapped onto the full interval, one can set up a symbolic coding with an alphabet of three symbols, say Left, Center and Right, depending on the interval in which the point comes to lie. Any string of these three letters can be realized as a trajectory, whence we say the dynamics has a complete ternary coding. Evidently, there are three fixed points,

$$\begin{aligned} x_{\text{L}} &= -1 , & \Lambda_{\text{L}} &= \frac{2}{1-s} > 0 , \\ x_{\text{C}} &= \ \ 0 , & \Lambda_{\text{C}} &= -\frac{1}{s} < 0 , \\ x_{\text{R}} &= +1 , & \Lambda_{\text{R}} &= \frac{2}{1-s} > 0 . \end{aligned} \tag{30}$$

Because of the linearity of the map, the derivative Λ_p for any orbit factorizes into

$$\Lambda_p = \Lambda_{\text{L}}^{n_{\text{L}}} \Lambda_{\text{C}}^{n_{\text{C}}} \Lambda_{\text{R}}^{n_{\text{R}}} , \tag{31}$$

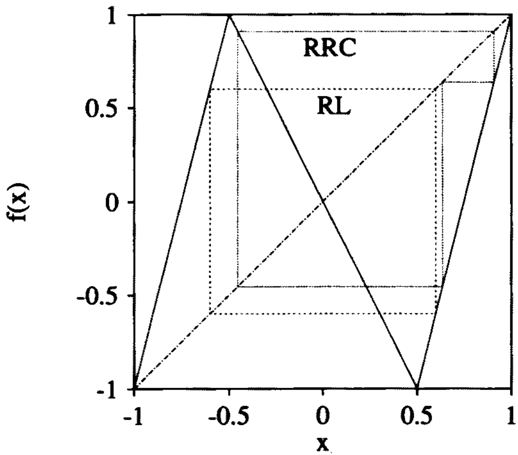

Fig. 1. The symmetric map for $s = 1/2$ and two cycles: the symmetric orbit RL (dashed) and the non-symmetric orbit RRC (dotted).

where $n_{\rm L}$, $n_{\rm C}$ and $n_{\rm R}$ are respectively the number of symbols L, C and R in the symbol string for p.

Consider the dynamical zeta function $1/\zeta_j$ entering in the definition of the Fredholm determinant. With $t_p = |\Lambda_p|^{-1}\Lambda_p^{-j}$ the weights associated with a periodic orbit p, they become

$$1/\zeta_j = \prod_p (1 - n_p z^{n_p} |\Lambda_p|^{-1}\Lambda_p^{-j}) \tag{32}$$

$$\begin{aligned} =&(1 - zt_L)(1 - zt_C)(1 - zt_R)(1 - z^2 t_{LR})(1 - z^2 t_{LC})(1 - z^2 t_{CR}) \\ &(1 - z^3 t_{LLR})(1 - z^3 t_{LLC})(1 - z^3 t_{LRC})\cdots \\ =&1 - z(t_L + t_C + t_R) - z^2(t_{LR} - t_L t_R) \\ &- z^2(t_{LC} - t_L t_C) - z^2(t_{CR} - t_C t_R)\cdots . \end{aligned} \tag{33}$$

When expanded in powers of z, the contributions from all orbits with periods larger than 1 come in groups which may be interpreted as corrections to the actual weights of the long orbits as compared to what one extrapolates from short orbits [11–14]. For the linear map (29), one finds, as a consequence of the factorization (31), that the products in higher orders of z cancel exactly. The zeta function simply becomes

$$1/\zeta_j = 1 - z\left(|\Lambda_{\rm L}|^{-1}\Lambda_{\rm L}^{-j} + |\Lambda_{\rm C}|^{-1}\Lambda_{\rm C}^{-j} + |\Lambda_{\rm R}|^{-1}\Lambda_{\rm R}^{-j}\right) . \tag{34}$$

Substituting the data for the fixed points one finds

$$Z_s(z) = \prod_{j=0}^{\infty}\left[1 - z\left(2\left(\frac{1-s}{2}\right)^{j+1} + s^{j+1}(-1)^j\right)\right] , \tag{35}$$

from which one can read off the eigenvalues

$$\lambda_\nu = 2\left(\frac{1-s}{2}\right)^{\nu+1} + s^{\nu+1}(-1)^\nu \tag{36}$$

or

$$\lambda_0 = 1\,, \tag{37}$$
$$\lambda_1 = (1-2s-s^2)/2\,, \tag{38}$$
$$\lambda_2 = (1-3s+3s^2+3s^3)/4\,, \tag{39}$$
$$\vdots$$

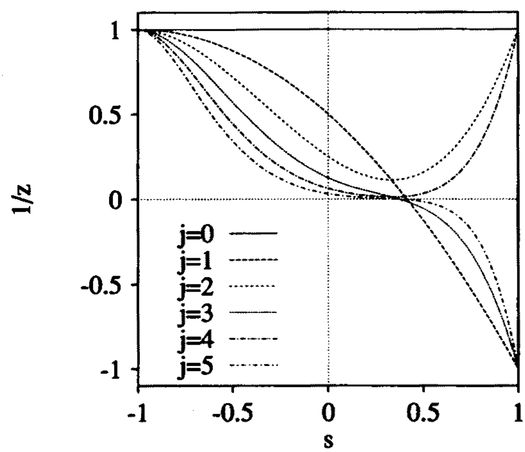

Fig. 2. The eigenvalues of the map as a function of the parameter s.

The first six eigenvalues are shown in Fig. 2. The eigenvalue $\lambda_0 = 1$ corresponds to the invariant density, the higher ones determine the decay of correlations. The gap between 1 and the absolute values of the other eigenvalues is largest at the crossing of the $j = 1$ and $j = 2$ eigenvalues at $s = {}^1/_3$ where $\lambda_1 = \lambda_2 = {}^1/_9$. There the slopes of the three segments are ± 3 and all three intervals have the same size. For $s = 0$ the map is, except for a shift and scale change the discontinuous modulo map $y_{n+1} = 2y \bmod 1$ with a spectrum given by $\lambda_j = 2^{-j}$.

5. Symmetry decomposition of the example

As mentioned before, the symmetry of the map allows one to reduce the dynamics to a fundamental domain by mapping points back into it using the reflection g. If this g is taken as a new symbol in defining trajectories, then the symbol L becomes redundant (if the fundamental domain is taken to be [0, 1] no trajectory actually every reaches the left interval). Whenever an L appears in a symbolic string, one has to reflect and to take into account that all symbols following are to be reflected as well. It then is best to introduce a new set of symbols: Let R be replaced by 1, the central region by 0 and the reflection by 2. Then the translation of any {L, C, R} string into an {0, 1, 2} string proceeds in three steps: first, all C's are replace by 0's. Next the string is shifted so as to start with an R; all other L's and R's are studied in relation to this R. Every R is replaced by a 1 until the first L occurs. Then a 2 is recorded and all subsequent L's and R's are exchanged. Then one continues as described. Alternatively, one can say that 1's are denoted for every R following an R and every L following an L; the symbol 2 is recorded only if two symbols (ignoring 0's) differ, *i.e.*, if an R follows an L or vice versa. For instance, the string R R C L C L R R is thus mapped into 1 1 0 2 0 1 2 1.

With this new coding, all symmetric orbits are easily identified: they have an odd number of 2's. In the full interval, every symbol 2 indicates a swing between the two fundamental domains [0, 1] and [−1, 0]. A trajectory with an odd number of 2's is periodic in the fundamental domain, but not in the full interval; it has to be traversed twice before returning to the starting point in the full interval.

The primitive, period one periodic orbits in the fundamental domain are

$$\begin{aligned} x_0 &= 0\,, & \Lambda_0 &= -\frac{1}{s}\,, \\ x_1 &= 1\,, & \Lambda_1 &= \frac{2}{1-s}\,, \\ x_2 &= \frac{1+s}{3-s}\,, & \Lambda_2 &= \frac{2}{1-s}\,, \end{aligned} \tag{40}$$

In the full domain, x_2 and $-x_2$ are the two points of the period two orbit RL.

After these preparations, one can now write out the zeta functions in the symmetry reduced subspaces. Even and odd n_2 correspond to symmetric and non-symmetric orbits, thus $\tilde{\Lambda}_2 = -\Lambda_2$. The next step is to include all powers of Λ_0 in the last term; the odd ones necessarily have to be divided out again. Then

$$\begin{aligned} Z_+(z) &= \prod_p \prod_{j=0}^{\infty} (1 - t_p \Lambda_p^{-j}) \prod_{j=0}^{\infty} (1 - t_0 \Lambda_0^{-2j}) \frac{(1 - t_0 \Lambda_0^{-2j+1})}{(1 - t_0 \Lambda_0^{-2j+1})} \\ &= \frac{\prod_{j=0}^{\infty} \prod_p (1 - t_p \Lambda_p^{-j})}{\prod_{j=0}^{\infty} (1 - t_0 \Lambda_0^{-2j+1})}\,, \end{aligned} \tag{41}$$

where in the first case the product over the cycles p is on all strings of 0, 1 and 2 with eigenvalues Λ_0, Λ_1 and $\tilde{\Lambda}_2$, without any restriction. As in case of (33), the product can be calculated, yielding

$$\prod_p \left(1 - z^{n_p}|\Lambda|^{-1}\Lambda^{-j}\right) = 1 - z\left(|\Lambda_0|^{-1}\Lambda_0^{-j} + |\Lambda_1|^{-1}\Lambda_1^{-j} + |\tilde{\Lambda}_2|^{-1}\tilde{\Lambda}_2^{-j}\right), \tag{42}$$

or, with the data as given above,

$$Z_+(z) = \frac{\prod_{j=0}^{\infty}\left[1 - z\left((1+(-1)^j)\left(\frac{1-s}{2}\right)^{j+1} + s^{j+1}(-1)^j\right)\right]}{\prod_{j=0}^{\infty}\left[1 + zs^{2j+2}\right]}. \tag{43}$$

Note that for odd j the numerator cancels the denominator exactly. Thus the zeta function in the symmetric subspace becomes entire again:

$$Z_+(z) = \prod_{j=0}^{\infty}\left[1 - z\left(2\left(\frac{1-s}{2}\right)^{2j+1} + s^{2j+1}\right)\right]. \tag{44}$$

The odd subspace is slightly more complicated, since the terms with an odd number of 2's in the symbol string for the orbit pick up an overall sign change in the zeta function. Nevertheless, an expansion like (33) shows that the grouping of terms survives and that the additional sign only affects the sign between groups, not within groups, so that the cancellations still survive. A calculation as in the symmetric subspace then yields

$$Z_-(z) = 1 - z\left(|\Lambda_0|^{-1}\Lambda_0^{-j} + |\Lambda_1|^{-1}\Lambda_1^{-j} - |\tilde{\Lambda}_2|^{-1}\tilde{\Lambda}_2^{-j}\right) \tag{45}$$

$$= \prod_{j=0}^{\infty}\left[1 - z\left(2\left(\frac{1-s}{2}\right)^{2j+2} - s^{2j+2}\right)\right]. \tag{46}$$

Obviously, the full zeta function is the product of the two subspace zeta functions, $Z(z) = Z_+(z)Z_-(z)$.

6. Conclusions

The above analysis is admittedly rather tedious and, as far as simple piecewise linear maps is concerned, much too heavy a machinery. Nevertheless, this sort of calculation is typical for symmetry reductions of zeta functions as they appear *e.g.*, in semiclassical calculations. The above example shows that in the end the complicated looking expressions (27), and (28) actually give the correct results.

Because of the way the missing factors from boundary orbits are included in Eq. (41) one might worry about the analyticity properties of the symmetry reduced zeta functions. Presumably the full zeta function as well as the symmetry reduced ones are entire [8,15]. Then the poles due to zeros in the denominator have to be cancelled by zeros in the numerator, as observed in the example (Eq. (44)). In more complicated examples the positions of these zeros will also be known and can be used to improve the convergence of cycle expanded zeta functions as in the calculations in [16].

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