PERTURBATIONS OF DYNAMICS OF HOMOGENEOUS TWO-DIMENSIONAL CELLULAR AUTOMATA*†

DANUTA MAKOWIEC

Institute of Theoretical Physics and Astrophysics, Gdańsk University
Wita Stwosza 57, 80-952 Gdańsk, Poland
e-mail fizdm@halina.univ.gda.pl

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The probabilistic approach provides a useful tool for understanding the nature of dynamics of cellular automata. It allows not only clarification of different results of the evolution but also gives explanation to the physical meaning of the rules.

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Cellular automata (CA) are widely known not only as being the micro-dynamical models for complex nonlinear dynamical systems[1, 2], but also as systems providing possibility to describe properties of some equilibrium statistical systems[3, 4]. These are reasons why an improved understanding of the statistical properties of cellular automaton time evolution would be desirable. A classification of cellular automaton rules according to their dynamical properties could be useful not only for the design of CA rules for particular purposes, but also this could result in approximation schemes applicable to CA rules of physical interest, such as lattice gasses.

The probabilistic approach, presented in [3, 4], gives us a useful tool and, moreover, makes possible to improve our theoretical understanding of these very simple extended dynamical systems. Especially, it allowed us to explain the following features of the homogeneous and symmetric CA observed in computer simulations [5]:

— the possibility of a rule to lead to the stable pattern of a lattice;

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— the process of vanishing of some neighbourhoods and creating patterns from one particular configuration only in the process of getting the stabilization;
— the time (number of evolution steps) necessary to get the pattern stabilization;
— the independence of the boundary conditions for almost all rules.

This article will focus in the time evolution of the perturbated CA. Since the probabilistic approach is useful in the description of perturbation we worked with, the introductory section will review its results and explain above enumerated properties. The influence of perturbations to the evolution of CA is discussed in Section 2. The distance between perturbated and purely deterministic dynamics is established in terms of probability distribution of configurations on a lattice. Our considerations are illustrated by results of computer experiments. Moreover, since relation established between d-dimensional probabilistic cellular automata (PCA) which includes deterministic cellular automata (CA) as a special limit, and equilibrium statistical models (ESM) in $(d+1)$-dimensional spin systems [6], we present hints to some $(d+1)$-dimensional spin ESM — Section 3.

1. Probabilistic approach to CA

Let us consider a square periodic lattice, $L \times L$, denoted by $[\sigma]$, where each site is occupied by a two-level subsystem — *spin*. Let us also assume that the initial state of any spin is set randomly with probability $^1/_2$ to get any of two possible states.

By *neighbourhood* $\Theta_i(t) = (E_i(t), N_i(t), W_i(t), S_i(t))$, at time t of any i-th spin on the lattice we mean the sequence of four spin's states corresponding to the states of four nearest neighbours of the i-th spin. The letters in the introduced notation are chosen to follow the geographic directions on a map (in the anticlockwise direction).

A rule on a square lattice is defined by conditional probabilities $P_i(\sigma_i|\Theta_i)$ describing the probability to find at i-th lattice site a spin in σ_i state at next time step, while Θ_i is its present neighbourhood. Generally, as being probabilities they have the property:

$$P_i(-\sigma_i|\Theta_i) = 1 - P_i(\sigma_i|\Theta_i)\,, \tag{1.1}$$

The dynamics is called *deterministic* if a rule takes its values with probability 1 [3, 4]. Hence, in case of deterministic CA, the initial state of a lattice is the only source of randomness.

If the probabilities defined above do not depend on a lattice site, a rule (and CA) is called *homogeneous*.

Furthermore, let us restrict ourselves to *up-down* symmetric rules. It means that only PCA satisfying the following condition are to be considered:

$$P(\sigma_i|\Theta_i) = P(-\sigma_i| - \Theta_i)\,, \tag{1.2}$$

where $-\Theta_i(t) = (-E_i(t), -N_i(t), -W_i(t), -S_i(t))$.

So, the number of independent conditional probabilities reduces to eight. Let us identify them by the following notation:

$$\begin{aligned}
\theta_0 &= (-1,-1,-1,-1)\,, & \theta_1 &= (1,-1,-1,-1)\,,\\
\theta_2 &= (-1,1,-1,-1)\,, & \theta_3 &= (-1,-1,1,-1)\,,\\
\theta_4 &= (-1,-1,-1,1)\,, & \theta_5 &= (1,1,-1,-1)\,,\\
\theta_6 &= (1,-1,1,-1)\,, & \theta_7 &= (1,-1,-1,1)\,.
\end{aligned} \tag{1.3}$$

Other configurations — *up-down* symmetric to θ_i, $i = 0,\ldots,7$ are numbered as follows: for $i = 8,\ldots,15$, $\theta_i = -\theta_{15-i}$.

Let us identify the eight generated conditional probabilities by :

$$a_i = P(1|\theta_i) \qquad \text{for} \quad i = 0,\ldots,7\,. \tag{1.4}$$

Then any P satisfying both conditions (1.1) and (1.2) can be written in the following way:

$$P(\sigma_i|\Theta_i) = \tfrac{1}{2}\left[1 + \sigma_i h(\Theta_i)\right] \tag{1.5}$$

with

$$\begin{aligned}
h(\Theta_i) =& E_i\omega_E + N_i\omega_N + W_i\omega_W + S_i\omega_S\\
&+ E_i(E_i\omega_{EE} + N_i\omega_{EN} + W_i\omega_{EW} + S_i\omega_{ES})
\end{aligned}$$

for $\Theta_i \in \{\theta_0,\ldots,\theta_7\}$. If $-\Theta_i \in \{\theta_0,\ldots,\theta_7\}$ then $h(\Theta_i) = -h(-\Theta_i)$.

The parameters $\{\omega\}$ are related to $\{a_0,\ldots,a_7\}$ through the relations:

$$\begin{aligned}
\omega_E &= \tfrac{1}{2}\Big(a_0 - \sum_{i=1}^{4} a_i + \sum_{i=5}^{7} a_i\Big)\,,\\
\omega_{\substack{N\\W\\S}} &= \tfrac{1}{2}\Big(- a_0 - a_1 + a_{\substack{2\\3\\4}} + a_{\substack{5\\6\\7}}\Big)\,,\\
\omega_{EE} &= \tfrac{1}{2}\Big(- a_0 - a_1 + \sum_{i=2}^{4} a_i + \sum_{i=5}^{7} a_i - 2\Big)\,,\\
\omega_{\substack{EN\\EW\\ES}} &= \tfrac{1}{2}\Big(a_0 - a_1 - a_{\substack{2\\3\\4}} + a_{\substack{5\\6\\7}}\Big)
\end{aligned}$$

and the indexes in two expressions represent three different coefficients (they ought to be read horizontally).

The eight configurations enumerated in (1.3), can be grouped in 3 separated sets: $A = \{\theta_0\}$, $B = \{\theta_1, \ldots, \theta_4\}$, $C = \{\theta_5, \ldots, \theta_7\}$. Notice, that it is useful to consider any rule separately on configurations belonging to the same subdomain — suitable A- B- or C-action. For example, the rule with property $r(\theta_1) = -1$, $r(\theta_2) = -1$, $r(\theta_3) = -1$, $r(\theta_4) = 1$ can be seen as the shift of the southern neighbour (S_i) of a (i–th) spin , if only B-subdomain happens. If the opposite values are taken, as the next example, the B-action can be seen as an anti shift from South ($-S_i$). (See [7, 8] for details of this approach).

The results of our computer simulations [7–9] prove that actions undertaken over B-subdomain are dominating over the evolution. It means that neighbourhoods, on which a rule contradicts B-action, are eliminated from the pattern during the evolution process. If the elimination process goes successfully, a rule stabilizes on a newly created pattern almost always as a translation. If a strong disagreement between actions occurs, the stabilization is not reached. But a kind of a pattern stabilization is reached anyway. It can be described by macroscopic functions: *magnetization* — the total sum of all spins of a lattice, is almost fixed or *activity* — the number of spins changing theirs states at time steps, varies around a fixed value.

The evolution of automata can be expressed in terms of probability distribution D; it means by the probability of the occurrence of a configuration θ_i on a lattice $[\sigma]$:

$$D\left([\sigma]\right)(i) = \frac{n(\theta_i)}{L^2} \qquad i = 0, \ldots, 15\,, \tag{1.6}$$

with $n(\theta_i)$ the total number of θ_i neighbourhoods in a pattern of a lattice. Notice, that the probability distribution D enables to restore all properties of the magnetization [9].

If $a_0, \ldots, a_7$ (1.4) take only two values 0 or 1 (deterministic case), then h function takes also only two values -1 or 1. The dependance on neighbourhoods of h function for deterministic rules is presented in details in [5]. The probability distributions of neighbourhoods, found analytically in [5] thanks to h function properties, are in a very good agreement with results obtained in simulations.

Remembering that deterministic rules usually stabilize on patterns, where their final dynamics reduces to the simpler one, often a shift, one can rewrite $h(\Theta_i)$ in more appropriate form as follows:

$$h(\Theta_i) = h_0(\Theta_i) + h_\Delta(\Theta_i)\,, \tag{1.7}$$

where $h_0(\Theta_i)$ is a part responsible for the final shift stabilization, and $h_\Delta(\Theta_i)$ points on configurations which have to be eliminated before getting the stabilization.

Hence, one can define the full stabilization by means CA after many time steps with a pattern with the property: $h_\Delta(\Theta_i) = 0$ for all nodes $i = 1, \ldots, L^2$.

We have found the following types of $h_0(\Theta_i)$: S_i, $-S_i$, $-N_iS_i$, $-W_iS_i$, -1, 1 and the following components of $h_\Delta(\Theta_i)$: $\pm\frac{1}{2}(1+E_i)(1+N_i)$ for θ_5, $\pm\frac{1}{2}(1+E_i)(1+W_i)$ for θ_6, $\pm\frac{1}{2}(1+E_i)(1+S_i)$ for θ_7, $\pm\frac{1}{2}(1-E_i)(-E_i+N_i+W_i+S_i)$ for θ_0. The particular role of E_i neighbour comes from the fact that this neighbour has been taken as a starting point for any comparison.

Now it is easy to verify the following conclusions:

The stabilization of any rule is possible:

— if the number of configurations that have to be eliminated is smaller than the number of configurations supporting the main dynamics; example: $S_i+\frac{1}{2}(1+E_i)(1+N_i)$ stabilizes very quickly after the elimination of θ_5-configurations from a pattern, while a rule

$$
\begin{aligned}
&-S_i + \tfrac{1}{2}(1+E_i)(1+N_i) + \tfrac{1}{2}(1+E_i)(1+W_i) \\
&+ \tfrac{1}{2}(1+E_i)(1+S_i) + \tfrac{1}{2}(1-E_i)(-E_i+N_i+W_i+S_i)
\end{aligned}
$$

never reaches stabilization;

— if one favourizable by B-action configuration is not eliminated by h_Δ part; example: $-N_iS_i + \frac{1}{2}(1+E_i)(1+N_i)$ stabilizes in no more than 100 steps when lattice size L=44 is considered, while $-N_iS_i + \frac{1}{2}(1+E_i)(1+W_i)$ never reaches stabilization.

At the end of this introduction we must mention the existence of the strong discrepancy between analytically found solutions for probability distribution of neighbourhoods and their values found in computer experiments for rules of the following class: $(\bullet\,, s)$ — $\bullet$ denotes any of the rule from the set $\{-1, -N_iS_i, -W_iS_i\}$; s means shift from South if C-domain happens. The source of it lies in the fact that the probability distributions for these rules obtained according to some iteration procedure, briefly described in [5], varying in time, pass a local extremum, Fig. 1, while in the case of other rules, it does not happen. The periodic or helical boundary conditions, added to a rule simulated in a computer, fix the stabilization at this local extremum.

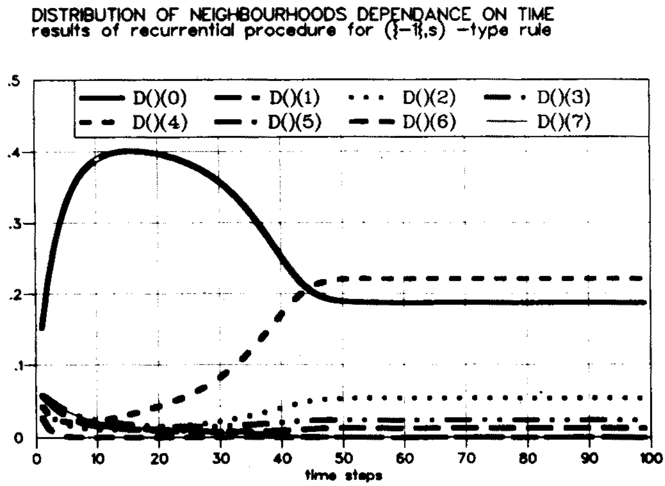

Fig. 1. The dependence on time of probability distribution of neighbourhoods found analytically according to the iteration procedure for the rule of a $\{-1\}$-type. This procedure concerns an unbounded lattice — a lattice with evolution spreading without any constrains. The computer simulated CA, where to a rule some boundary conditions are added, reach the full stabilization at patterns propertied with values at the local maximum of D()(0).

2. Perturbations of deterministic dynamics

a) uniformly perturbated dynamics

The simplest and very often used perturbation of deterministic dynamics has been considered in the following form:

$$h(\Theta_i) = (1 - 2\epsilon) h_{\mathrm{D}}(\Theta_i) , \tag{2.1}$$

where $h_{\mathrm{D}}(\Theta_i)$ denotes any deterministic function: $|h_{\mathrm{D}}(\Theta_i)| = 1$. (2.1) means that the probability that any chosen symmetric and homogeneous rule acts is $1 - \epsilon$, while the probability that any other rule is active is ϵ.

It is interesting to ask whether, and if yes how, the value of perturbation ϵ influence getting the stationarity for given CA. One can evaluate the efficiency of a rule by time needed by a ϵ-perturbated rule to recover its deterministic properties such as probability distribution of configurations of a lattice. According to this criterion, stabilizing rules can be divided in two groups. Again, the division follows B-action properties. One can approximately describe the mean time dependence as:

(i) case $\{-1, -N_iS_i, -W_iS_i\}$ classes of stabilizing rules

$$\langle T \rangle = \frac{1}{\alpha_1 \epsilon + \beta_1} ,$$

(ii) case $\{S_i, -S_i\}$ classes of stabilizing rules

$$\langle T \rangle = \exp\{-(\alpha_2 \epsilon + \beta_2)\}\,.$$

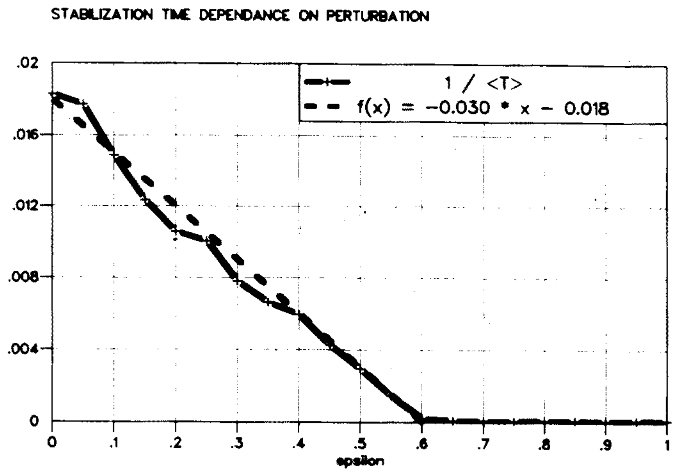

Fig. 2.a. Efficiency of the rule of a $\{-1\}$-type (cases $-NS$, $-WS$ are similar). At $\epsilon > 0.6$ the perturbation was too strong to reconstruct deterministic rule's properties in time shorter than 12,000 steps. (L=44)

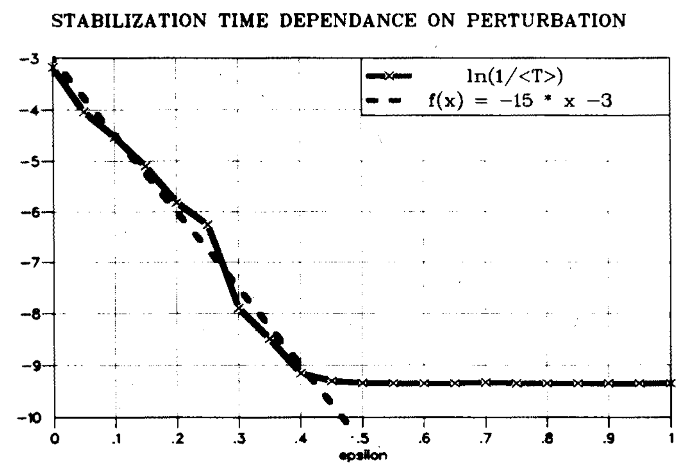

Fig. 2.b. Efficiency of the rule of S-type.(case $-S$ is similar). At $\epsilon > 0.4$ the perturbation was too strong to reconstruct deterministic rule's properties in time shorter than 12,000 steps. (L=44)

When the lattice size $L = 44$ is considered the values of parameters are as in Fig. 2.a and Fig. 2.b, suitable. In case of $\{-1\}$ rules, the STD-errors of obtained results are about 50%, but in all other cases the STD-errors are much smaller and do not exceed 10%. In our computer observation we restricted ourselves to 12 000 steps. So, the further dependance has not been found.

b) kicked dynamics

The following evolution can be considered as another type of perturbated dynamics: a determininistic rule is perturbated only on one configuration. So

$$a_i = 0 \quad \text{or} \quad 1 \quad \text{if} \quad i \neq j \quad \text{and} \quad a_j = \epsilon . \tag{2.2}$$

Let us introduce a Bernoulli stochastic variable ξ_j^K acting on j-th particular configuration, with distribution determined by the following condition:

$$\xi_j^K = \begin{cases} 1 & \text{with probability } \epsilon_K \\ 0 & \text{with probability } 1 - \epsilon_K \end{cases}$$

if only θ_j happens.

Moreover, let us consider the following products of zeros and ones for identification of any neighbourhood on a square lattice [10]: to any $K = 0, \dots, 4$ and any set $\{i_1, \dots, i_K\}$ of $i_k = 1, \dots, 4$ we consider

$$Q^K(\Theta_i; i_1, \dots, i_K) = \prod_{l=1}^{K} \tfrac{1}{2}(1 - \sigma^i_{i_l}) \quad \prod'_{k \neq 1,\dots,K} \tfrac{1}{2}(1 + \sigma^i_{i_k}),$$

where $\Theta_i = (\sigma^i_1, \sigma^i_2, \sigma^i_3, \sigma^i_4)$ and prim on the product indicates that only distinct (distinct from each other and the $i_1, \dots, i_K$) i_k values are considered.

One can see that for given Θ_i: $\sum_K \sum_{i_1,\dots,i_K} Q^K(\Theta_i; i_1, \dots, i_K) = 1$ only for one set of i_k's: each i_k points on the neighbour in Θ_i that is in down state: -1, and there are not any other neighbours being in this state.

Let us consider the shift dynamics S_i perturbated on θ_4 configuration by random shift of the North neighbour instead of southern one. So,the dynamics (2.2) can be rewritten in a microdynamical form:

$$\sigma_i = \sum_K \sum_{i_1,\dots,i_K} Q^K(\Theta_i; i_1, \dots, i_K)(1 - \xi_j^K) S_i + \xi_j^K N_i Q^j(\Theta_i; i_1, \dots, i_j), \tag{2.3}$$

where the set $\{i_1, \dots, i_j\}$ points on θ_j configuration. Varying j one can observe the influence of the particular configuration "kicked" by North neighbour instead of simple shift from South dynamics. Moreover, varying the neighbour kicking the j-th configuration, one can find out the role played by particular neighbours. Additionally, one can consider kicking process on more than one configuration. Since we are interested in symmetric rules only, we restricted our considerations to the case when both configurations of a pair of symmetric configurations are kicked with the same probability. Results of our observation are collected in Table I. It is easy to see that kicked dynamics recovers patterns matching with some deterministic CA — particularly, the possibility of reaching the full stabilization.

TABLE I

The characteristic features of the probability distribution of neighbourhoods $D([\sigma])(i) = D_i$ obtained after 500 steps of time according to the shift evolution (2.3) symmetrically kicked with $\epsilon_K = \frac{1}{2}$ from E and N directions.(W-case, due to the mirror symmetry, can be easily found).

Kicked conf Kicking nghb	θ_0	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7
E		$D_7 = \frac{1}{2}$			$D_0 = \frac{1}{2}$	$D_5 = 0$	$D_6 = 0$	
N			$D_6 = \frac{1}{2}$		$D_0 = \frac{1}{2}$	$D_5 = 0$		$D_7 = 0$
		B-domain is kicked (θ_1–θ_4)						
E		$D_0 = D_7 = \frac{1}{8}$		$D_1 = \ldots = D_4 = \frac{1}{16}$				
						C-domain is kicked (θ_5–θ_7)		
N						$D_0 = D_1 = D_3 = D_6 = \frac{1}{8}$		

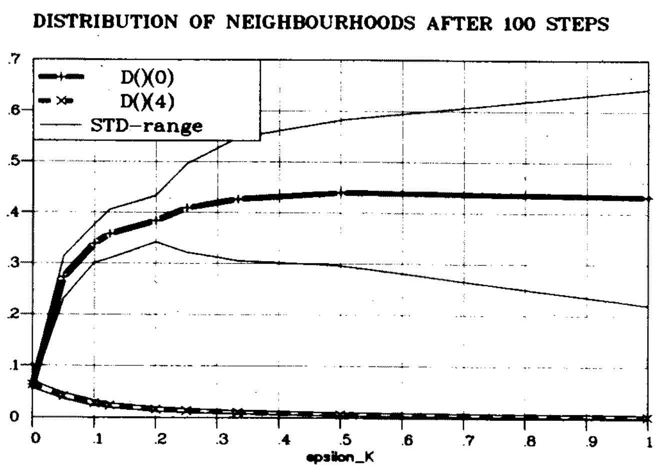

Fig. 3. Efficiency of the kicked shift type rule. The results of a single experiment are scattered in a large interval (STD — error). This basic feature of $\{-1\}$-rule is noticeable since $\epsilon_K \simeq 0.05$.

Since the pure deterministic rule of a $\{-1\}$—type stabilizes in about 100 steps when the size of lattice is L=44, one can compare the probability distribution of the crucial configuration θ_0 after 100 steps against the "kicking" perturbation (Fig. 3). It is easy to see that even at very small ϵ_K, $\epsilon_K \simeq 0.05$, the process of the formation of a suitable pattern is to be started.

3. ESM arising from perturbated dynamics

It has been proved [6] that stationary (or cyclic) d-dimensional PCA provide some $(d+1)$-dimensional Equilibrium Statistical Model (ESM). Corresponding to (2.1) PCA, single-site Hamiltonian of 3-dimensional system can be formally expressed in the form:

$$H(\sigma_i,\Theta_i) = -\beta\sigma_i h_{\mathrm{D}}(\Theta_i) + \ln\cosh\beta\,,\qquad \beta = \frac{1}{2}\ln\frac{1-\epsilon}{\epsilon}\,. \tag{3.1}$$

If h_{D} is in the form of (1.5) then one can divide above Hamiltonian into three parts of interactions:

i) energy in any external field energy $H_0 = -\beta\sigma_i\omega_{EE}$

ii) two-body interactions: $H_2 = \beta\sigma_i \sum_{j=E_i,N_i,W_i,S_i} \omega_j\sigma_j$

iii) three- body interactions: $H_3 = \beta\sigma_i E_i \sum_{j=N_i,W_i,S_i} \omega_{Ej}\sigma_j$

Stationarity in case PCA means that there exists a probabilistic measure μ defined on a space of all possible lattice states, such that the dynamics of PCA does not influence it. For any rule, let us consider a measure concentrated on such states of a lattice that the distribution of neighbours D (1.6) reflects the properties of patterns obtained in fully deterministic dynamics. According to (1.7) for any stationary PCA considered we can formally associate a 3-dimensional ESM with interactions described by following Hamiltonian:

$$H(\sigma_i,\Theta_i) = \ln\cosh\beta - \beta\sigma_i\left(h_0(\Theta_i) + h_\Delta(\Theta_i)\right)$$

Hence, any fully stable state of PCA being of a shift type (case $h_0 = \pm S_i$) corresponds to ESM with only two-body stable interactions: $\pm\sigma_i S_i$. Other types of interactions appear as the influence of noise — somewhat analogous to temperature.

The special interest of many physicists is put to the case $h_0 = -1$, because it provides non-ergodic systems of ESM. In the probability distribution of neighbourhoods description non- ergodicity denotes that $D[\sigma]$ for any stationary lattice $[\sigma]$ can be non—symmetric with respect to up-down symmetry: $D[\sigma](0) \neq D[\sigma](15)$. In the case described the crucial interactions of ESM Hamiltonian reduce to single one: $\pm\sigma_i$. Let us, as example, study carefully the following PCA: $h_D(\Theta_i) = -1 + \frac{1}{2}(1 + E_i)(1 + N_i)$. This property can be rewritten equivalently to the more totalistic form as:

$$h_{\rm D}(\Theta_i) = \tfrac{1}{2}(E_i + N_i + W_i + S_i) + \tfrac{1}{2}(E_i N_i + S_i W_i)\,. \qquad (3.2)$$

Notice following properties of (3.2):

— $E_i + N_i + W_i + S_i = 0$ if $\Theta_i \in \boldsymbol{C}$

— $E_i N_i + S_i W_i = 0$ if $\Theta_i \in \boldsymbol{B}$

— $h_D = -1$ if $\Theta_i \in \boldsymbol{A} \cup \boldsymbol{B}$

The invariant states of deterministic CA dynamics: $\{[\sigma] : \Theta_i \in \boldsymbol{A},\ i = 1, \ldots, L^2\}$, provide the ESM with couplings between spins of the following type:

$$H(\sigma_i, \Theta_i) = \ln\cosh\beta - \frac{\beta\sigma_i}{2}(E_i + N_i + W_i + S_i)$$

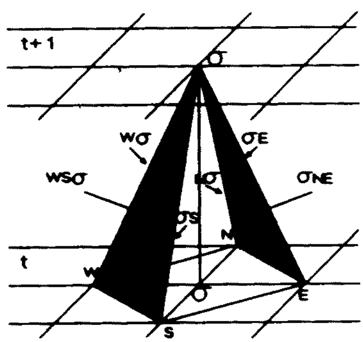

Fig. 4. A 3-dimensional lattice corresponding to PCA considered and the interactions between layers t and $t+1$ in ESM corresponding to -1-type of rule.
The lattice is formed by stacking square plane lattices. The four corners of each neighbouring square E, N, W, S determine, up to noise, the spin at the apex of the pyramid whose base is that square. Both sites at the vertices of each shaded triangle influence additionally the site at the apex. (three-body interactions). We omit the index i of the node, but we hope that does not influence legibility of the picture.

Interactions (3.2) between layers of 3-dimensional system are shown in Fig. 4. Notice, that the nonlinear part in the second term of (3.2) will become nonzero only if the first part is zero.

Kicked dynamics uniformly perturbated also provides ESM: $(1-2\epsilon)h_K$, with h_K obtained according to (1.5) with (2.2) properties. In case of the same rule as considered in (3.2), it means

$$h_K(\Theta_i) = (1-\epsilon_K)S_i + \epsilon_K h_D(\Theta_i). \tag{3.3}$$

Since (3.3) provides a competition between two deterministic rules: shift and $\{-1\}$-interactions, the totalistic property of a linear part has been broken in the corresponding ESM Hamiltonian.

4. Concluding remarks

1. Starting at the random state of a lattice $[\sigma]$, one can find that there are deterministic rules forcing the lattice to follow theirs properties if only the noise parameter is small enough. Kicked dynamics analysis underlines the influence of the action taken over B-domain to the final result. Within B-domain one can point on the neighbour of a configuration which is crucial to the stabilization of a pattern. The classification problem of homogeneous CA, partially solved by looking at the final pattern properties in [11] , ought to be revised from the above mention dynamical point of view.
2. Since (3.2), the resulting ESM can be seen as the Toom model [4] on the 3-dimensional lattice but with square plane lattices in the place of triangular ones. But the Tooms proof of the existence of a phase transition, adopted to the model considered, does not work. Our uniformly perturbated dynamics is still homogeneous and symmetric. Hence, following Tooms's arguments, starting at the fixed stable configuration $[\sigma] \equiv 1$, the rule of PCA can only flip all spins of a lattice. It means:

$$[\sigma(0)] \equiv 1 \longrightarrow [\sigma(1)] \equiv \begin{cases} 1 & \text{with probability } 1-\frac{1}{2}\epsilon \\ -1 & \text{with probability } \frac{1}{2}\epsilon \end{cases},$$

because the result depends only on $P(\sigma_i, \theta_0)$ The deterministic dynamics (3.3) clearly also has $[\sigma] \equiv 1$ and $[\sigma] \equiv -1$ as invariant states as well as all combinations of these states. Moreover, homogeneity of the dynamics is broken by kicks. But kicks do not affect θ_0 or θ_{15} and, again, the Toom's arguments cannot be applied here.

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