ON Q-PERTURBATIONS OF COMMUTATION RELATIONS AND Q-INDEPENDENCE*†

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Real and complex perturbations of commutations relations are discussed. Using this approach we present some new ideas in the quantum probability inspired by C^*-algebraic description of quantum statistical mechanics as well as from the ideas of Voiculescu.

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1. Introduction

This paper is divided on four sections. We start with a description of a one parameter family of C^* algebras $\boldsymbol{A}_q(H)$ generated by q-perturbed commutation relations. Our discussion will show that there are , at least, two different possibilities for the choice of parametrization of $\boldsymbol{A}_q(H)$. The first one is to consider the real interval; $q \in [-1, +1]$. Then $q = -1$ corresponds to fermions while $q = +1$ corresponds to bosons. The second choice

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of parametrization leads to complex numbers q; $q = e^{i\varphi}$. Then $\varphi = 0$ corresponds to bosons while $\varphi = \pi$ to fermions. It will be shown that the complex interpolation corresponds to the "perturbation" of the full set of commutation relations whereas the real "perturbation" corresponds to the interpolation of a part of commutation relations.

Let us recall that the fractional statistics, parameterized by complex numbers that interpolate continuously between bosons and fermions, are encountered in models based on lower dimensional configuration space, see *e.g.* Wilczek [1] and the references given there. Our conclusions are that the above results are strongly related to various generalizations of the Pauli principle.

In the last section we consider a probability calculus over C^* algebra $\boldsymbol{A}_q(H)$ for the real parametrization. In particular a definition of q-independence is given. Let us remark that our definition includes the "free" probability, which was founded by D. Voiculescu [2]. One of the advantages of this approach is that a unification of some ideas in probability calculus over bose and fermi systems is reached. Also, it gives us a possibility to study a new type of central limit theorems. An example of such theorem is offered. Illustrative physical models are given.

2. q-perturbations of commutation relations — real parametrization

Let H be a Hilbert space with the inner product $< .,. >$ and let q be a number with $-1 \le q < +1$. We will study C-antilinear maps $a : H \to \boldsymbol{A}$ into a C^*-algebra with identity such that

$$a(f)a^*(g) - qa^*(g)a(f) = < f, g > \mathbf{1}. \tag{1}$$

Let us recall

Proposition 1 ([3])

Let $\boldsymbol{A}$ be a C^*- algebra with unit, $q \in (-1, +1)$, $c \in \boldsymbol{R}$ and $a \in \boldsymbol{A}$ a nonzero element satisfying:

$$aa^* - qa^*a = c\mathbf{1}. \tag{2}$$

Then $c > 0$, and either $aa^* = a^*a = (c/1-q)\mathbf{1}$, or the spectra of aa^* and a^*a are equal to the closure of the sequence

$$c\frac{1-q^n}{1-q}, \tag{3}$$

where $n \in \boldsymbol{N}$, and $n \ge 0$ for a^*a and $n \ge 1$ for aa^* □

Corollary 2 ([3])

Suppose $a : H \to \boldsymbol{A}$ satisfies (1). Then

$$\|a(f)\| = c(q)\|f\|, \tag{4}$$

where:

$$c(q) = \begin{cases} (\sqrt{1-q})^{-1} & \text{for} \quad 0 \le q < 1 \\ 1 & \text{for} \quad -1 < q \le 0 \end{cases} \tag{5}$$

□

Now, we are in position to formulate the first problem: The above facts clearly say that (1) interpolates between CCR — canonical commutation relations — (for $q = 1$; bosons which do not have any bounded realizations), and CAR — canonical anticommutation relations — (for $q = -1$; fermions). Both, for fermions and bosons, the relations (1) are supplemented by the relations $a^\dagger(f)a^\dagger(g) - qa^\dagger(g)a^\dagger(f) = 0$ where $a^\dagger$ stands for either creation or annihilation operator. So we should ask for a form of such supplementary relations for $|q| < 1$. The difficulty of this problem lies in the fact the direct generalization leads to an undefined system. On the other hand, from the physical point of view, it is of fundamental importance to know the form of "generalized" Pauli principle. Thus we should look for a form of relations between $a^\dagger(f)$ and $a^\dagger(g)$ for $|q| < 1$. Let us start with

Proposition 3

Suppose that $a : H \to \boldsymbol{A}$ satisfies (1). Then:

$$\|a(f)a(f)\| = \|a^*(f)a^*(f)\| = \varphi(q)\|f\|^2, \tag{6}$$

where

$$\varphi^2(q) = \begin{cases} c^2(q)[qc^2(q) + 1] & q \in [-1/2, 1) \\ \psi(\lambda_0) & q \in (-1, -1/2) \end{cases} \tag{7}$$

and $\psi(\lambda_0) = \sup_{\lambda \in \sigma(a(f)a^*(f))} \{q\lambda^2 + \|f\|^2\lambda\} \neq 0$, $\sigma(b)$ stands for the spectrum of b.

□

Remark

The proof of this theorem as well as other mathematical details will be published elsewhere [4].

Corollary 4

For $q \neq -1$

$$\|a^*(f)a^*(f)\| = \|a(f)a(f)\| \neq 0,$$

i.e. there is a violation of the Pauli principle.

□

Corollary 5

Since CAR algebra is uniquely defined (up to isomorphism) we infer that even a small perturbation of Fermi statistics leads to a nonisomorphic algebraic structure. In other words, CAR algebra is not stable with respect to the above perturbations (*cf.* [5] and [6]).

□

Corollary 6

Relations

$$a^{\dagger}(f)a^{\dagger}(g) - c(q; f, g)a^{\dagger}(g)a^{\dagger}(f) = 0 \tag{8}$$

are in contradiction with relations (1) where $(q; f, g) \to c(q; f, g)$ is a $\boldsymbol{C}$-valued function with the following property

$$c(q; f, g) \neq 1\,, \tag{9}$$

for $q \in (-1, +1)$, $f, g \in H$.

□

In spite of this instability one can prove (cf. [3])

Proposition 7

Let H be a Hilbert space, $q \in (-1, +1)$ and let $\boldsymbol{A}_i^q(H)$ $i = 1, 2$ be two C^*-algebras generated by the identity and elements $a_i(f)$, $f \in H$, satisfying q-relations (1). It follows that $\boldsymbol{A}_i^q(H)$ is determined up to *-homomorphism, *i.e.* there exists a unique *-homomorphism π such that

$$\pi(a_1(f)) = a_2(f)$$

□

Having the description of the universal C^*-algebra associated with the q-relations (1), let us consider as an example the interesting, from the physical point of view, representation of $\boldsymbol{A}^q(H)$.

Let $\mathcal{F}(H)$ denote the full Fock space $\bigoplus_{r=0}^{\infty}(\otimes^r H)$ where $\otimes^0 H$ is a one dimensional Hilbert space, and H is an infinite dimensional separable Hilbert space.

Define linear maps

$$a^*(f) f_1 \otimes f_2 \cdots \otimes f_n = f \otimes f_1 \otimes f_2 \cdots \otimes f_n \tag{10}$$

and

$$P_q^{(n)} f_1 \otimes \cdots \otimes f_n = \sum_{\rho \in S_n} q^{|\rho|} f_{\rho(1)} \otimes \cdots \otimes f_{\rho(n)}\,, \tag{11}$$

where $q \in [-1, +1]$, S_n is the symmetric group, and $|\rho|$ is the number of inversions of ρ.

Let $P_q = \bigoplus_n P_q^{(n)}$. Then (*cf.* [7]) $P_q > 0$ for $q \in (-1, +1)$. Consequently, we can define

$$a_q(f) = P_q^{-1/2} a(f) P_q^{1/2}, \tag{12}$$

$$a_q^*(f) = P_q^{1/2} a^*(f) P_q^{-1/2}. \tag{13}$$

Proposition 8

i) $a_q^*(f), a_q(g)$ are bounded operators on $\mathcal{F}(H)$ for $q \in (-1, +1)$ and $f, g \in H$.

ii)

$$a_q(f)a_q^*(g) - q a_q^*(g) a_q(f) = \langle f, g \rangle 1, \tag{14}$$

where $< ., . >$ denotes the inner product of H.

Proof: Direct calculations.

□

Remark

Let $O_F^q(H)$ denote the C^*-algebra generated by $\{a_q(f), a_q^*(g) : f, g \in H\}$ for $q \in (-1, +1)$. Thus $O_F^q(H)$ is a representation of $A^q(H)$. One can consider $O_F^q(H)$ as a q-perturbation of the C^*-algebra $O_F(H)$ considered by Cuntz (see [8] and [9]).

3. q-perturbation of commutation relations — complex parametrization

In this section we will consider the perturbations of the full set of commutation relations.

Let H be a real Hilbert space with an even or infinite dimension and a nondegenerate inner product $s(.,.)$. By $\{e_i\}$ we denote an orthonormal basis in H. Let us define the operator $J : H \to H$ in the following way:

$$Je_i = e_{i+1}, \tag{15a}$$

$$Je_{i+1} = -e_i, \tag{15b}$$

for $i = 2k - 1$, $k = 1, 2, \ldots$

If one defines

$$(\lambda_1 + i\lambda_2) f = \lambda_1 f + \lambda_2 J f, \tag{16}$$

for all $\lambda_i \in \boldsymbol{R}$ and $f \in H$, and

$$(f, g) = s(f, g) + i s(f, Jg), \tag{17}$$

then $(H,(.,.))$ can be considered as a complex Hilbert space. We will study $\mathbf{R}$-linear maps $B : H \to \mathcal{A}$ into a C^*-algebra with identity such that

$$B(e_n)B(e_m) - e^{i\varphi}B(e_m)B(e_n) = 0\,, \tag{18a}$$

for $n > m$; $\varphi \in (0,\pi]$,

$$(1-\Phi(\varphi))\big(B(e_n)B(e_n) + B(Je_n)B(Je_n)\big) \\ -i(1+\Phi(\varphi))\big(B(Je_n)B(e_n) - B(e_n)B(Je_n)\big) = 2 \tag{18b}$$

and

$$B(e_n)^* = B(e_n)\,. \tag{18c}$$

where $\Phi(\varphi)$ is a real-valued function of φ such that $\Phi(0) = 1$, $\Phi(\pi) = -1$. One can introduce annihilation and creations operators $a_\varphi(f)$ and $a^*_\varphi(f)$ by

$$a_\varphi(f) = \frac{1}{\sqrt{2}}\big(B(f) + iB(Jf)\big)\,, \tag{19a}$$

$$a^*_\varphi(f) = \frac{1}{\sqrt{2}}\big(B(f) - iB(Jf)\big)\,, \tag{19b}$$

then one has

Proposition 9
The family of operators $a_\varphi(e_n)$, $a^*_\varphi(e_m)$, $m, n = 1, 2\ldots$, satisfy

$$a_\varphi(e_n)a^*_\varphi(e_m) - e^{i\varphi}a^*_\varphi(e_m)a_\varphi(e_n) = 0\,, \tag{20a}$$

$$a_\varphi(e_n)a_\varphi(e_m) - e^{i\varphi)}a_\varphi(e_m)a_\varphi(e_n) = 0\,, \tag{20b}$$

for $n > m$, and

$$a_\varphi(e_n)a^*_\varphi(e_n) - \Phi(\varphi)a^*_\varphi(e_n)a_\varphi(e_n) = 1\,, \tag{20c}$$

for $\varphi \in (0,\pi]$.
Proof: Direct calculations. □

Let us make a few remarks concerning the complex unification of commutation relations.

Remarks

i) The basic idea of the above construction is to replace $f \to a(f)$ and $g \to a^*(f)$ by the family of maps $f \to B(f)$. Then, for example, the full canonical anticommutation relations can be described by one formula $\{B(f), B(g)\} = s(f,g)$. Therefore, a perturbation of this formula gives a perturbation of the full CAR relations.

ii) Let us remark that this approach has a "noncanonical" feature — namely formulas (20) contain the unpleasant condition: $n > m$. It seems that this is a property of the perturbation of the full set of commutation relations (*cf.* [10, 11]).

iii) Let us repeat that the complex parametrization of q-perturbations is encountered in models based on lower dimensional configuration space.

4. q-independence

In the stochastic calculus one of the basic notions is that of independence. Whereas in classical (commutative) probability theory there is only one possible definition of independence of random variables (*cf.* Feller [12], vol.II), the situation in quantum (noncommutative) probability theory is different. Namely, one can define various notions of independence. In particular, here, we propose a new one. For this purpose, let us observe that operators $\{a(f), a^*(g); f, g \in H\}$, considered in Sections 2 and 3, give a model for a generalization of the Voiculescu independence (see [Voi]). Namely, one can define q-independence. In order to present the generalization of Voiculescu definition of independence we must introduce the notion of probability system.

Let $\mathcal{A}$ be a unital C^*-algebra and ϕ be a state on $\mathcal{A}$, *i.e.* a positive normalized functional on $\mathcal{A}$. A pair $(\mathcal{A}, \phi)$ will be called a probability system.

Definition 10

q-independence of $(\mathcal{A}, \phi)$ is understood as the following property of a family of subalgebras $(\mathcal{A}_i)_{i \in I}$ with respect to $(\mathcal{A}, \phi)$ ($\mathcal{A}_i \subset \mathcal{A}$, $\mathcal{A}$ is the fixed C^*-algebra with the fixed state ϕ): for an arbitrary n, and a n-tuple $(a_1, \ldots, a_n)$ such that $a_i \in \mathcal{A}_i$ where $i_1 \neq i_2 \neq i_3 \ldots \neq i_n$ and $\phi(a_j) = 0$, $j \in I$, one has

$$\phi(a_1 \ldots a_n) = \alpha(q; V)\phi(V_1) \ldots \phi(V_p), \tag{21}$$

where $V = \{V_1, \ldots, V_p\}$ is the partition of $\{1, \ldots, n\}$ associated with $\{a_1, \ldots, a_n\}$, $\phi(V_j) = \phi(a_{j_1} \ldots a_{j_n})$ and $\alpha(q, V)$ is a rotational function with respect to q such that $\alpha(0, V) = 0$. □

Let us remark that the case $\alpha(q, V) = 0$ corresponds to the Voiculescu definition. The q-independence with respect to $(\mathcal{A}, \phi)$ can be partially characterized as follows.

Theorem 11

Let $(\mathcal{A}_i, \phi_i)$, $i \in J$ be a family of probability systems and $q \in (-1, +1)$. Then there exists a probability system $(\mathcal{A}^q, \phi)$ and embeddings

$$h_i : \mathcal{A}_i \to \mathcal{A}^q$$

such that

1) $\mathcal{A}^q$ is generated by $\bigcup_{i\in J} h_i(\mathcal{A}_i)$
2) $\phi \circ h_i = \phi_i$
3) The family $\left(h_i(\mathcal{A}_i)\right)_{i\in J}$ considered as family of subalgebras of $\mathcal{A}^q$ is q-independent with respect to $(\mathcal{A}^q, \phi)$. □

Remark

The proof of this theorem as well as other mathematical details will be published elsewhere [4].

To get a physical model for the considered type of independence it is enough to consider as $\mathcal{A}_i$ the subalgebras of $O^q_F(H)$ generated by $\{a_q(f^i);\ f^{(i)} \in \mathcal{S}(\boldsymbol{R}^3) \cap O_i\}$ where $O_1 = \{f \in L(\boldsymbol{R}^3);\ \mathrm{supp} f \subseteq \mathrm{ball}(0, \frac{1}{4})\}$. $O_i = \{f(x-i); f \in O_1\}$. $\phi(.)$ is taken as $(\Omega, .\Omega)$, Ω is a vacuum and $\mathcal{S}(\boldsymbol{R}^3)$ denotes the Schwartz space. We put $H = L^2(\boldsymbol{R}^3)$.

Having this type of independence one can repeat the way of Bożejko and Speicher giving versions of q-central limits theorems. Namely, for example one can prove

Proposition 12

Let $\{a_i\}_{i\in N}$ be a sequence in $\mathcal{A}$ such that:

1)
$$\phi(a_i) = 0$$

2) $\{a_i\}$ generates q-independent family of subalgebras in $\mathcal{A}$,
3) $\phi(a_i^2) = \beta^2$,

Define $S_N = N^{-\frac{1}{2}}(a_1 + \cdots + a_N)$. Then for $r \in \mathbf{N}$ one has

$$\lim_{N\to\infty} \phi(S_N^r) = \begin{cases} 0 & \text{if } r \text{ is odd} \\ \sum_V \alpha(q;V)\beta^2 & \text{if r is even} \end{cases},$$

where the summation is over all partitions $V = \{V_1, \ldots, V_{\frac{r}{2}}\}$ such that $\mathit{card} V_i = 2$.

Proof: Direct calculations.□

Remark

To get a model for this theorem it is enough to take as $a_i = a_q(f) + a_q^*(f)$ with $f \in O_i$.

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