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SOLVING QCD CASCADE TWENTY YEARS LATER*

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Multiparton correlations in a QCD jet are calculated analytically in the double logarithmic approximation for constant α_s. For a well developed cascade, *i.e.* far from the low energy cut-off, correlations show characteristic scale invariant power behaviour. Cumulant moments in restricted angular cells also follow the power dependence on the volume of a cell with known exponents.

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1. Introduction

Perturbative QCD and its analytic solution has a long history [1–6]. The original success of the theory came from the description of the deep inelastic scattering. In later years the time like region was also understood and many predictions, *e.g.* for total multiplicity and single parton inclusive densities, have been formulated and confronted with experiment. Theory and experiment agree to such an extent that many physicists consider QCD as a closed chapter. On the other hand the low energy nonperturbative sector of QCD remains an open problem, and questions of the confinement and particle spectrum can be attacked only numerically [7]. There exists however an intriguing possibility that perturbative QCD "knows" more about its low energy sector than could be a priori expected. In particular the energy spectra of particles produced in e^+e^- collisions agree very well with those of partons if the cascade is evolved to sufficiently low cutoffs. This led to the concept of local parton hadron duality [8], which still puzzles many people. It will be interesting to extend these studies to multiparton

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** On leave from the Jagellonian University, Cracow, Poland distributions. So far only correlations between parton energies have been derived in Double Log Approximation (DLA) [9] and recently in modified LLA [10].

Another recent interest in multiparticle correlations comes from the studies of multiplicity fluctuations in variable phase space intervals [11,12] and their possible connection to the underlying fractal structure of the QCD cascade [13]. One can infer from these studies that the multiparticle correlation functions increase for decreasing distance in a two- or three-dimensional momentum space. This rises the question about the singularity structure of multiparton densities, not fully addressed so far.

In this article we present a scheme to solve the fully differential multiparton densities. Then the explicit expressions for the two-gluon density in the relative angle, and for the fully differential angular correlations are given. We also prove the selfsimilarity of the QCD cascade at asymptotically high energies and derive intermittency exponents by direct calculation of the multiplicity moments.

All results were obtained in the collaboration with Wolfgang Ochs. We discuss here only the fixed α_s case in DLA [14]. Generalization for the running α_s was also done recently [15,16].

2. A scheme to derive multiparton densities

The master equation for the generating functional of multiparton densities [4,17] reads in the DLA [9]

$$Z_P\{u\} = \exp\left(\int\limits_{\Gamma_P(K)} \mathcal{M}_P(K)[u(K)Z_K\{u\} - 1]d^3K \right) , \tag{1}$$

where the subscript P stands for the momentum vector of the parent parton and the half opening angle ($P = \{\vec{P}, \Theta\}$); $\Gamma_P(K)$ denotes the phase space for the d^3K integration ($\Gamma_P(K) = \{K : K < P, \Theta_{KP} < \Theta, K\Theta_{KP} > Q_0\}$ where Q_0 is a cutoff parameter). Knowledge of the functional dependence of $Z\{u\}$ on $u(\vec{k})$ is equivalent to knowing all normalized cross sections. For example $\frac{1}{n!}\delta^n/\delta u(k_1)\ldots\delta u(k_n)Z\{u\}|_{u=0} = p(k_1,\ldots,k_n)$ gives probability densities for exclusive production, while Taylor expansion around $u(\vec{k}) = 1$ generates the inclusive densities

$$\rho_P^{(n)}(k_1,\ldots,k_n) = \delta^n Z_P\{u\}/\delta u(k_1)\ldots\delta u(k_n)\,|_{u=1} . \tag{2}$$

In general $u(\vec{k})$ can be thought of as the profile or the acceptance function. In particular setting $u(\vec{k}) = v$ for all $\vec{k}$ gives the generating function of the distribution of the total multiplicity.

Differentiating functionally Eq.(1) we get integral equations for inclusive multiparton densities

$$\rho_P^{(1)}(k) = \mathcal{M}_P(k) + \int_\Gamma \mathcal{M}_P(K)\rho_K^{(1)}(k)d^3K\,,$$
$$\rho_P^{(2)}(k_1,k_2) = \rho_P^{(1)}(k_1)\rho_P^{(1)}(k_2) + \mathcal{M}_P(k_1)\rho_{k_1}^{(1)}(k_2) + \mathcal{M}_P(k_2)\rho_{k_2}^{(1)}(k_1)$$
$$+ \int_\Gamma \mathcal{M}_P(K)\rho_K^{(2)}(k_1,k_2)d^3K\,, \tag{4}$$

and for arbitrary order

$$\rho_P^{(n)}(k_1,\ldots,k_n) = d_P^{(n)}(k_1,\ldots,k_n) + \int_\Gamma \mathcal{M}_P(K)\rho_K^{(n)}(k_1,\ldots,k_n)d^3K\,. \tag{5}$$

Here, and in the following, the symbol Γ denotes generically all boundaries of the phase space integration, and $\mathcal{M}_P(K)$ is the probability for bremsstrahlung of a single gluon

$$\mathcal{M}_P(K)d^3K = a^2\frac{dK}{K}\frac{d\Theta_{PK}}{\Theta_{PK}}\frac{d\Phi_{PK}}{2\pi}\,, \tag{6}$$

where $a^2 = (2/\pi)C_V\alpha_s$.

In practical applications three dimensional integrals separate into usually simple momentum integrals, and into 2-dimensional angular integrals with often complicated boundaries. However, in the DLA one can simplify boundary restrictions and reduce the number of relevant variables, by identifying the regions of the phase space which give the dominant logarithms. In these regions only simple poles from inner bremsstrahlung dominate the integrand. This "pole dominance" approximation is an inherent part of the DLA and was used successfully for calculations of the total multiplicities and momentum distributions [1–5]. We illustrate this approximation technique by solving Eq. (3) for the single parton inclusive distribution in $\vec{k}$. To this end we rewrite the d^3K integration explicitly, see Fig. 1,

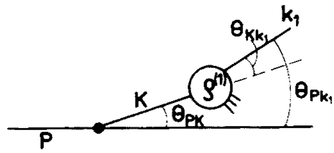

Fig. 1. Kinematics and variables of the integral equation, Eq. (8), for the single parton inclusive density.

$$\rho(\vartheta, k, P) = \mathcal{M}_P(k) + \frac{a^2}{2\pi} \int\limits_{\Gamma} \frac{dK}{K} \frac{d\Omega_K}{\Theta_{PK}^2} \rho(\Theta_{Kk}, k, K) \,. \tag{7}$$

Because of the angular ordering $\Theta_{PK} > \Theta_{Kk}$. Consequently the dominating singularities are those in Θ_{Kk}, $\Theta_{Kk} \sim 0$. At this point $\Theta_{PK} \sim \Theta_{Pk} \equiv \vartheta$. Choosing the polar angle of the $d\Omega_K$ integration as Θ_{Kk} we get

$$\rho(\vartheta, k, P) = \mathcal{M}_P(k) + \frac{a^2}{\vartheta^2} \int\limits_{k}^{P} \frac{dK}{K} \int\limits_{\frac{Q_0}{k}}^{\vartheta} \frac{d\Theta_{Kk}}{\Theta_{Kk}} \left[\Theta_{Kk}^2 \rho(\Theta_{Kk}, k, K)\right] , \tag{8}$$

where the factor in the square brackets does not have Born pole Θ_{Kk}^{-2}. This equation can be further simplified by introducing, natural for this problem, variables $x = \ln(P/k)$, $z = \ln(K/k)$, $\zeta = \ln(\vartheta Q_0/k)$, $\xi = \ln(\Theta_{Kk} Q_0/k)$ and corresponding density $\rho(x, \zeta) \equiv d^2 n / dx d\zeta = 2\pi k^3 \vartheta^2 \rho(\vartheta, k, P)$. Then

$$\rho(x, \zeta) = a^2 + a^2 \int\limits_0^x dz \int\limits_0^\zeta d\xi \rho(z, \xi) \,. \tag{9}$$

This can be easily solved (*e.g.* by iteration) and gives

$$\rho_P^{(1)}(\vec{k}) = \mathcal{M}_P(\vec{k}) I_0(2a\sqrt{x\zeta}) \,. \tag{10}$$

We now present our general solution for arbitrary density, Eq. (5). First, it is easy to prove that the inhomogenous term $d_P^{(n)}$, which corresponds to the direct emission from the parent parton, is built from various products of the correlation functions of *lower* order. Only if all n derivatives act on the Z function under the d^3K integral in the exponent of Eq. (1) can we recover the n-th order density. This gives the last term in Eq. (5). In all other cases the n derivatives are split among various factors resulting from the differentiation of the exponent giving various products of the densities of lower order. Integral of any particular densiy, of lower than n order, can be always replaced by the density itself and corresponding inhomogenous term of yet lower order, by using apropriate integral equation for the lower order.

Second, the unknown function $\rho^{(n)}$ appears only in the last term in Eq. (5). Simplicity of this structure allows us immediately to solve equation (5) by iterations

$$\rho_P^{(n)}(k_1,\ldots,k_n) = \sum_{r=0}^{\infty}\int_\Gamma d^3K_1\ldots d^3K_r\mathcal{M}_P(K_1)\mathcal{M}_{K_1}(K_2)$$

$$\ldots\mathcal{M}_{K_{r-1}}(K_r)d_{K_r}^{(n)}(k_1,\ldots,k_n)\,, \tag{11}$$

or symbolically

$$\rho^{(n)} = \left(\frac{1}{1-\hat{\mathcal{M}}}\right)\circ d^{(n)}\,. \tag{12}$$

Further, we note that *all but one* K_m integrations ($m=1,\ldots,r-1$) can actually be done, since the complete K_m dependences are given by the Born cross sections and variable boundaries of the inner integrations. Hence changing the orders of integrations we get the following integral representation of the solution

$$\rho_P^{(n)}(k_1,\ldots,k_n) = d_P^{(n)}(k_1,\ldots,k_n) + \int_\Gamma R_P(K,\sigma)d_K^{(n)}(k_1,\ldots,k_n)d^3K\,, \tag{13}$$

where the resolvent $R_P(K,\sigma)$ is given by

$$R_P(K,\sigma) = \sum_{r=1}^{\infty}\int_{\Gamma,\Theta_{K_{r-1}K}>\sigma} d^3K_{r-1}\ldots d^3K_1\mathcal{M}_P(K_1)\ldots\mathcal{M}_{K_{r-1}}(K)\,. \tag{14}$$

For constant α_s, $R_P(K,\sigma)$ can be calculated explicitly

$$R_P(K,\sigma) = \mathcal{M}_P(K)I_0\left(2a\sqrt{\ln(P/K)\ln(\Theta_{KP}/\sigma)}\right)\,, \tag{15}$$

but the results [13, 14] are valid for running α_s as well. In these formulae σ is the minimal opening angle of a jet K and has to be determined for each case separately; in general it depends on all momenta $\sigma=\sigma(k_1,\ldots,k_n,K)$, for example, $\sigma=\Theta_{k_1K}$ for $n=1$.

A role of σ can be better understood if we consider in detail contribution from the second iteration, see also Fig. 2,

$$\rho_P^{(n,2)}(\mathcal{K}) = \int_{\Gamma_1} d^3K_1\mathcal{M}_P(K_1)\int_{\Gamma_2} d^3K_2\mathcal{M}_{K_1}(K_2)d_{K_2}^{(n)}(\mathcal{K})\,. \tag{16}$$

Γ_1 and Γ_2 denote appropriate boundaries resulting from all three constraints $\Gamma_P(K_1)$, $\Gamma_{K_1}(K_2)$ and $\Gamma_{K_2}(\mathcal{K})$, where $\mathcal{K}$ denotes collectively all momenta

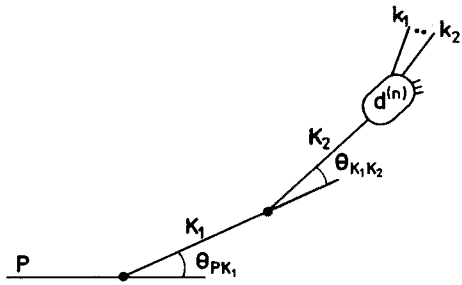

Fig. 2. Second iteration of the integral equation, Eq. (5).

of the final partons $\mathcal{K} = \{k_1, \ldots, k_n\}$. After changing the orders of the K_1 and K_2 integration we get

$$\rho_P^{(n,2)}(\mathcal{K}) = \int\limits_{\Gamma_2} d^3K_2 \left[\int\limits_{\Gamma_1, \Theta_{K_1K_2} > \sigma} d^3K_1 \mathcal{M}_P(K_1)\mathcal{M}_{K_1}(K_2) \right] d_{K_2}^{(n)}(\mathcal{K}), \tag{17}$$

which allows to identify the term in the square brackets with the second order contribution to the resolvent, Eq. (14). However, because the K_1 integration is now done *at fixed* K_2, its boundaries are influenced by the constraints implied on K_2 by the configuration of the final partons $\mathcal{K}$. In particular, angular ordering requires that $\Theta_{K_1K_2} > \sigma(\mathcal{K}, K_2)$ with σ defined as above. Similar considerations lead to the general formula, Eqs. (14), (15) with σ being the lower cutoff of the emission angle Θ_{PK} and, at the same time, the opening angle or a measure of the "virtuality" of a jet K.

Eq. (13) has a simple interpretation in terms of the cascading process (see Fig. 3). In fact, the resolvent $R_P(K, \sigma)$ is nothing but the inclusive distribution of a jet (K, σ) in a jet (P, Θ). Indeed the inclusive density of "elementary" partons $\rho_P^{(1)}(k)$, Eq. (10), coincides with the resolvent, Eq. (15), if the angular cutoff for emission of an elementary parton Q_0/k is replaced by σ for a virtual jet.

Eqs. (13), (15) give the recursive prescription for explicit calculation of general (fully or partly differential) multiparton densities of arbitrary order.

3. Correlation functions from the resolvent representation

3.1. Density in the relative angle

As a first application we consider the distribution of the relative angle between two partons in a jet with primary parton momentum P and jet

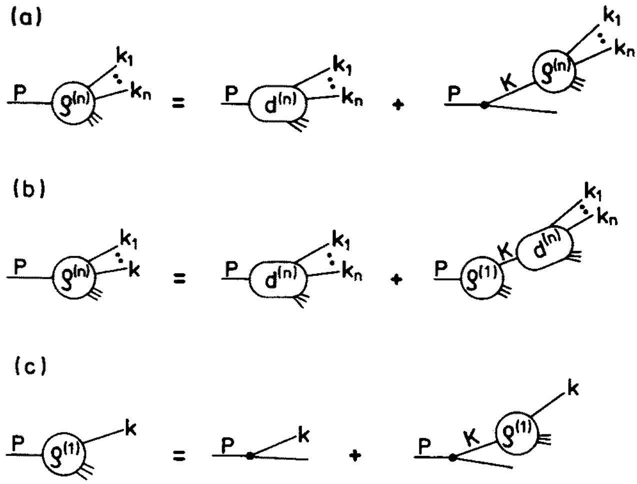

Fig. 3. Graphic representation of the integral equation, Eq. (5) (a), its solution, Eq. (13) (b), and of the equation for the single parton density or for the resolvent, Eq. (15) (c).

half opening angle Θ for fixed α_s,

$$\rho^{(2)}(\vartheta_{12}, P, \Theta) = \int \rho_P^{(2)}(k_1, k_2)\delta(\Theta_{k_1 k_2} - \vartheta_{12}) d^3k_1 d^3k_2 \,. \qquad (18)$$

Integrating Eq. (4) over the phase space of final partons, at fixed ϑ_{12}, gives the integral equation

$$\rho^{(2)}(\vartheta_{12}, P, \Theta) = g(\vartheta_{12}, P, \Theta) + a^2 \int_{Q_0/\vartheta_{12}}^{P} \frac{dK}{K} \int_{\vartheta_{12}}^{\Theta} \frac{d\Psi}{\Psi} \rho(\vartheta_{12}, K, \Psi) \,. \qquad (19)$$

The direct term is defined as

$$g(\vartheta_{12}, P, \Theta) = \int d_P^{(2)}(k_1, k_2)\delta(\Theta_{k_1 k_2} - \vartheta_{12}) d^3k_1 d^3k_2 \,. \qquad (20)$$

Here $d_P^{(2)}(k_1, k_2)$ is given by the inhomogenous part of Eq. (4) and can be written as $d^{(2)} = d_{\rm prod} + d_{\rm nest}$ where the "product" term corresponds to two independent emissions from the primary parton ($P \to 1, P \to 2$) and the "nested" term to chain emissions ($P \to 1 \to 2$ and $P \to 2 \to 1$).

In deriving Eq. (19) we have followed similar procedure which led to Eq. (8). However now the polar angle $\Psi = \Theta_{PK}$, and as a consequence the relative angle ϑ_{12} sets the lower bounds for Ψ and K integrations in accordance with our arguments following Eq. (7). Namely, the leading logarithms always emerge from $\vec{K} \parallel \vec{k}_1(\vec{k}_2)$. In this configuration minimal virtuality of a jet K, which subsequently emits a pair (k_1, k_2) with given angular separation, is controlled by the relative angle ϑ_{12}. Note that the individual directions $\hat{n}_{k_1}, \hat{n}_{k_2}$ are already integrated over, therefore the "most narrow" singularities $\Theta_{Kk_1}, \Theta_{Kk_2} \sim Q_0/K$ are included in $\rho^{(2)}(\vartheta_{12}, \Psi, K)$. Therefore the relative angle ϑ_{12} is the only remaining scale controlling the singularities of the integrand in Eq. (19). This is an important difference between the partially and fully differential correlations which will be discussed later.

Equation (19) has the solution

$$\rho^{(2)}(\vartheta_{12}, P, \Theta) = g(\vartheta_{12}, P, \Theta) + a^2 \int_{Q_0/\vartheta_{12}}^{P} \frac{dK}{K} \int_{\vartheta_{12}}^{\Theta} \frac{d\Psi}{\Psi} I(\frac{P}{K}, \frac{\Theta}{\Psi}) g(\vartheta_{12}, K, \Psi), \tag{21}$$

with the resolvent

$$I(x,y) = I_0(2a\sqrt{ln(x)ln(y)}). \tag{22}$$

After carrying out the momentum integrals in Eq. (20) the direct term can be represented by products of single particle angular distributions

$$\rho_P^{(1)}(\vartheta_i) = \frac{a}{2\pi\vartheta_i^2} \sinh(a\ln(\vartheta_i/\kappa))(\kappa = Q_0/P, \vartheta_i \equiv \Theta_{Pk_i}). \tag{23}$$

In order to integrate the product term over the angles, Eq. (20) we use again the pole dominance, this time however the kinematics is different. We are looking now for the regions in the phase space of children partons $\vec{k_1}$, $\vec{k_2}$ which give dominant logarithms at *fixed* parent momentum P. Secondly, the integrand in Eq. (20) contains products (in case of $d_{\rm prod}$) of singularities in Θ_{Pk_1} and Θ_{Pk_2}. In this situation our receipe reads

$$g_{\rm prod}(\vartheta_{12}, P, \Theta) = \rho_P^{(1)}(\vartheta_{12}) \int_{\kappa}^{\vartheta_{12}} \frac{d\Theta_{Pk_1}}{\Theta_{Pk_1}} \rho_P^{(1)}(\Theta_{Pk_1}) + (1 \leftrightarrow 2), \tag{24}$$

i.e., the singular part around each pole ($\Theta_{Pk_1} \sim 0$, say) is integrated down to the elementary cutoff Q_0/P while the remaining slowly varying function $\rho_P^{(1)}(\Theta_{Pk_2}(\Theta_{Pk_1}, \vartheta_{12}, \phi))$ is approximated by its central value $\rho_P^{(1)}(\vartheta_{12})$. The upper bound for Θ_{Pk_1} is chosen as ϑ_{12} since beyond this scale the integrand does not have the logarithmic divergence. Final result is simple

$$g_{\rm prod}(\vartheta_{12}, P, \Theta) = \frac{2a}{\vartheta_{12}} \sinh(a\ln\frac{\vartheta_{12}}{\kappa})\{\cosh(a\ln\frac{\vartheta_{12}}{\kappa}) - 1\}, \tag{25}$$

while the nested term can be integrated exactly due to its hierarchical structure

$$g_{\rm nest}(\vartheta_{12}, P, \Theta) = \frac{2a^2}{\vartheta_{12}}\{\cosh(a\ln\frac{\vartheta_{12}}{\kappa}) - 1\}\ln\frac{\Theta}{\vartheta_{12}}\,. \qquad (26)$$

Note that integrating the direct term $g_{\rm prod}$ over the whole range of the relative angles gives a square of the total multiplicity in a cone

$$\int_{\kappa}^{\Theta} d\vartheta_{12} g_{\rm prod}(\vartheta_{12}, P, \Theta) = \left(\cosh\left(a\ln\frac{P\Theta}{Q_0}\right) - 1\right)^2 = \overline{n}(P\Theta)^2\,, \qquad (27)$$

as it should. This shows that our approximation, Eq. (24) correctly identifies all leading logarithms.

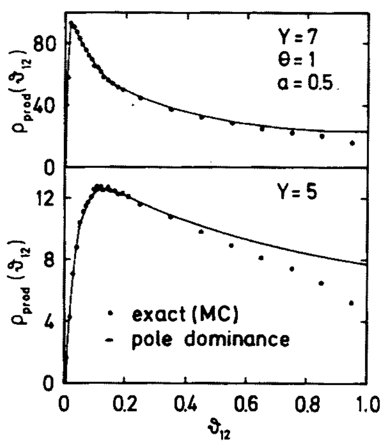

Fig. 4. Test of the pole dominance in the angular correlations for two energies ($Y = \ln(P/Q_0)$). Comparison of the $O(a^4)$ double log formula (solid line) with the MC integration over the final phase space.

To illustrate the quality of the pole dominance approximation we compare results for $g_{\rm prod}$ at the Born-level. In this case $\rho_P^{(1)}(\vartheta_i, k_i) \sim a^2/(\vartheta_i^2 k_i^3)$ and the 6-fold integral in Eq. (20) can be calculated exactly by numerical integration. The comparison between the exact result (MC method) and the approximation is shown in Fig.4. Whereas the result of the exact calculation vanishes for $\vartheta_{12} \to 0$, in our approximation it vanishes already for $\vartheta_{12} \to \kappa$; for $\vartheta_{12} \to \Theta$ on the other hand our approximation yields too large values. In between these limits there is a range of reasonable agreement and it improves with the energy. This agrees with the known applicability range of the DLA.

With the direct term given by Eqs. (25) and (26) the integral in Eq. (21) can be done analytically yielding our final result

$$\rho^{(2)}(\vartheta_{12}, P, \Theta) = \frac{a}{\vartheta_{12}} \sum_{m=0}^{\infty} y^{2m+1} I_{2m+1}(z) - \frac{2a}{\vartheta_{12}} \sinh\left(\frac{1}{4} yz\right), \qquad (28)$$

where $y = 2\sqrt{\ln(\vartheta_{12}/\kappa)/\ln(\Theta/\vartheta_{12})}$, $z = 2a\sqrt{\ln(\vartheta_{12}/\kappa)\ln(\Theta/\vartheta_{12})}$, and $I_l(z)$ is the modified Bessel function of the order l. It turns out that this sum is rapidly convergent and only few terms are sufficient for numerical evaluation. The same result can be obtained by solving the corresponding differential equation.

A particularly simple expression is obtained for the special point $y = 1$, *i.e.* $\vartheta_{12} = \kappa(\Theta/\kappa)^{(1/5)} = \tilde{\vartheta}$, where one finds

$$\rho^{(2)}(\tilde{\vartheta}, P, \Theta) = \frac{a}{2\tilde{\vartheta}} \left(\sinh\left(\frac{4a}{5} \ln \frac{\Theta}{\kappa}\right) - 4 \sinh\left(\frac{a}{5} \ln \frac{\Theta}{\kappa}\right)\right). \qquad (29)$$

This can be generalized by introducing a concept of the dual parameters in resemblance to the statistical physics. Define the dual variables $\vartheta^*_{12}, P^*, \Theta^*$ such that $y^* \equiv y(\vartheta^*_{12}, P^*, \Theta^*) = 1/y$ and $z^* = z$. Then, there exists a simple expression for the *sum* of the two correlations ($\overline{\rho}(\vartheta, P, \Theta) \equiv (\vartheta/2a)\rho^{(2)}(\vartheta, P, \Theta) + \sinh(a \ln(\vartheta/\kappa))$)

$$\overline{\rho}(\vartheta, P, \Theta) + \overline{\rho}(\vartheta^*, P^*, \Theta^*) = \sinh\left(\frac{z}{2}\left(y + \frac{1}{y}\right)\right). \qquad (30)$$

Dual relations of this type may have broader applications for fully differential correlations since then one has more flexibility to satisfy both relations $y^* = 1/y$ and $z^* = z$.

In the high energy limit, $\kappa = Q_0/P \to 0$, our result, Eq. (28) assumes a simple form (see Appendix A for the details)

$$\rho^{(2)}(\vartheta_{12}, P, \Theta) \simeq \frac{a}{2\vartheta_{12}} \left(\frac{\vartheta_{12}}{\kappa}\right)^{2a} \left(\frac{\Theta}{\vartheta_{12}}\right)^{a/2}, \qquad (31)$$

which proves a selfsimilar nature of the QCD cascade for fixed α_s. Note that this simple power law emerges only for the well developed cascade, *i.e.* for ϑ_{12} much bigger than the elementary cut-off. Eq. (31) represents the two components of Eq. (21): the first two factors correspond to the direct term ($g \simeq g_{\text{prod}}$ in this limit), the last factor to the enhancement from the emissions of the intermediate parent jets. The exponent in the first term $2a = 4\sqrt{3\alpha_s/2\pi}$ agrees with the asymptotic result of Gustafson and Nilsson derived with a different method and for different observables [18]. However, the last factor changes this result by 25%. In addition we find that the dual sum, Eq. (30), is given exactly by Eq. (31). Hence in the combinations analogous to (30) the terms nonleading in ϑ_{12}/κ cancel. Finally, we observe that, for $\vartheta_{12} \gg \kappa$, the leading powers of ϑ_{12}/κ cancel in the normalized density,

$$r_{12}(\vartheta_{12}) = \frac{\rho^{(2)}(\vartheta_{12}, P, \Theta)}{\rho^{(2)}_{prod}(\vartheta_{12}, P, \Theta)} \tag{32}$$

and we get the infrared safe and free of the narrow divergences result

$$r_{12} \simeq \left(\frac{\Theta}{\vartheta_{12}}\right)^{a/2}, \quad \frac{\vartheta_{12}}{\kappa} \gg 1, \tag{33}$$

where the deviation from unity is due to the emissions of the intermediate parent jets.

3.2. Fully differential angular correlations

This calculation shares common features of both cases considered earlier, namely the angular ordering in the single parton density, Eq. (8), and integration of a product of the singularities as in Eq. (24). We begin by integrating our resolvent representation, Eq. (13) over parton momenta. One obtains

$$\rho^{(2)}_P(\Omega_1, \Omega_2) = g^{(2)}_P(\vartheta_1, \vartheta_2) + \frac{a^2}{2\pi}\int \frac{dK}{K}\frac{d\Omega_K}{\Theta^2_{PK}} I\left(\frac{P}{K}, \frac{\Theta_{PK}}{\sigma}\right) g^{(2)}_K(\Theta_{Kk_1}, \Theta_{Kk_2}), \tag{34}$$

with the resolvent $I(x, y)$ defined in Eq. (22) and the inhomogenous term given by the product

$$g^{(2)}_P(\vartheta_1, \vartheta_2) = \rho^{(1)}_P(\vartheta_1)\rho^{(1)}_P(\vartheta_2), \tag{35}$$

with the angular densities given by Eq. (23). Similarly to the previous cases, nested terms do not contribute to the leading behaviour in the high energy limit and will be neglected. If needed, they can be included without any difficulty. Saturating the angular integration by two poles $\Theta_{Kk_1}(\Theta_{Kk_2}) \sim 0$ we get for the connected correlation function, $\Gamma^{(2)} \equiv \rho^{(2)} - \rho^{(1)}\rho^{(1)}$,

$$\Gamma_P^{(2)}(\vartheta_1,\vartheta_2,\vartheta_{12}) = \frac{a^3}{2\pi}\int\limits_{Q_0/\vartheta_{12}}^{P}\frac{dK}{K}I\left(\frac{P}{K},\frac{\vartheta_1}{\vartheta_{12}}\right)$$

$$\left(\frac{1}{\vartheta_1^2}\rho_K^{(1)}(\vartheta_{12})\int\limits_{\kappa_K}^{\vartheta_{12}}\frac{d\Theta_{Kk_1}}{\Theta_{Kk_1}}\sinh\left(a\ln\left(\frac{\Theta_{Kk_1}}{\kappa_K}\right)\right)+(1\to 2)\right). \quad (36)$$

As before $\hat{n}_K$ was replaced by $\hat{n}_{k_1}$ ($\hat{n}_{k_2}$ in the $1\leftrightarrow 2$ term) in all nonsingular expressions. In particular, $\Theta_{PK}\to\vartheta_1$ and $\Theta_{Kk_2}\to\vartheta_{12}$. Note also that $\sigma(k_1,k_2,K)=\vartheta_{12}$ in this case since the minimal virtuality (emission angle Θ_{PK}) required for the parent K to emit k_2 is indeed controlled by ϑ_{12} if $\vec{K}\|\vec{k}_1$. The upper limit of the Θ_{Kk_1} integration is given by ϑ_{12} and not by Θ_{Pk_1} as one might have guessed from the angular ordering. This is a consequence of the arguments following Eq. (24) [1]. For the same reason the lower bound of the K integration is controlled by ϑ_{12} and not by Θ_{Pk_1}. Integrating Eq. (36) we obtain

$$\Gamma_P^{(2)}(\vartheta_1,\vartheta_2,\vartheta_{12}) = \frac{a^2}{(2\pi)^2\vartheta_1^2\vartheta_{12}^2}\sum_{m=0}^{\infty}(2^{2m}-1)\left(\frac{l}{L_1}\right)^{m+1}I_{2m+2}(2a\sqrt{lL_1})+(1\to 2), \quad (37)$$

where $l=\ln(\vartheta_{12}/\kappa), L_1=\ln(\vartheta_1/\vartheta_{12})$ and $\vartheta_1(\vartheta_2)>\vartheta_{12}$ respectively. Eq. (37) is the new result for the full angular dependence of the two parton correlation function in DLA. Similarly to the density in the relative angle, Eq. (31), we find a characteristic power behaviour at high energies

$$\Gamma^{(2)}\simeq\frac{a^2}{2(4\pi)^2}\frac{1}{\vartheta_{12}^2}\left(\frac{\vartheta_{12}}{\kappa}\right)^{2a}\left(\frac{1}{\vartheta_1^2}\left(\frac{\vartheta_1}{\vartheta_{12}}\right)^{a/2}+\frac{1}{\vartheta_2^2}\left(\frac{\vartheta_2}{\vartheta_{12}}\right)^{a/2}\right), \quad (38)$$

which shows again the fractal (or intermittent) nature of the cascade.

With the aid of Eq. (13) our scheme can now be applied to higher order correlation functions. In order to derive analytic expressions it is essential to use the pole dominance at each stage of the recursive calculations. In particular, whenever the product of two or more singularities in Θ_{Kk_i} appears, it is sufficient to restrict the $\vec{K}$ integration to the vicinity of each singularity separately and approximate the remaining, slowly varying, function by its

[1] However, the connected correlation Γ in Eq. (36) does not vanish only if $\Theta_{Pk_1}(\Theta_{Pk_2})>\vartheta_{12}$ hence the angular ordering along the cascade is preserved.

central value. For example the leading (in P) part of the direct term for $n = 3$ reads

$$d^{(3)} = \prod_{i=1}^{3} \rho^{(1)}(i) + \rho^{(1)}(1)\Gamma^{(2)}(2,3) + \text{cycl}\,. \tag{39}$$

It is clear from Eqs (23) and (38) that now we will encounter two and three pole terms which should be treated as discussed above [15]. This procedure indeed reproduces all dominant logarithms. In particular, integrating Eq. (37) at fixed relative angle one recovers known result for the angular correlation [14], and integration over the full phase space gives the well known DLA expression for the second cumulant.

4. Multiplicity moments in restricted phase space

Fractal structure of the final states can be conveniently analysed in terms of the factorial moments counting the average number of n-tuples of particles produced in the restricted regions of the phase space (cells) γ [11]

$$f^{(n)} = \int_{\gamma} \rho^{(n)}\,. \tag{40}$$

Power dependence on the volume of a cell signals the selfsimilarity (intermittency) of the production mechanism, with the exponents being simply related to the fractal dimension of the produced system. In this chapter we shall calculate the intermittency exponents of the QCD jet choosing as a cell the sideway cone $\gamma(\vartheta, \delta)$ with the half opening angle δ and the polar angle with respect to the jet axis ϑ. Factorial moments, Eq. (40), can be calculated directly from the correlation functions derived recursively in previous chapter [15]. However, one can avoid tedious analysis of the many body correlations by deriving and solving integral equations satisfied by moments themselves. This will be done in this Section. First we simplify integral equations satisfied by the correlation functions and next, by suitable integration, derive equations for moments.

4.1. Integral equation for angular correlation functions

Momentum integrated correlation functions satisfy the following integral equation

$$\rho_P^{(n)}(\Omega) = d_P^{(n)}(\Omega) + \int_{\Gamma} d^3K \mathcal{M}_P(K) \rho_K^{(n)}(\Omega')\,, \tag{41}$$

where Ω denotes collectively all angular variables, and Γ stays for the boundary of the parent phase space implied by P and Ω. It is convenient to rewrite this equation in terms of the connected correlation function [15]. One obtains for $n = 2$

$$\Gamma_P^{(2)}(\vartheta_1,\vartheta_2,\vartheta_{12}) = \Delta_P^{(2)}(\vartheta_1,\vartheta_2,\vartheta_{12}) + \int d^3K \mathcal{M}_P(K)\Gamma_K^{(2)}(\Theta_{Kk_1},\Theta_{Kk_2},\vartheta_{12}), \tag{42}$$

where, the inhomogenous term Δ is given by (we again neglect the nested contribution)

$$\Delta_P^{(2)}(\vartheta_1,\vartheta_2,\vartheta_{12}) = \frac{a^2}{\vartheta_1^2}\int\limits_{Q_0/\vartheta_{12}}^{P}\frac{dK}{K}\rho_K^{(1)}(\vartheta_{12})\int\limits_{\kappa_K}^{\vartheta_{12}}\frac{d\Theta_{Kk_1}}{\Theta_{Kk_1}}\left[\Theta_{Kk_1}^2\rho_K^{(1)}(\Theta_{Kk_1})\right] + (1\to 2)\,. \tag{43}$$

Where, as usual, we have used the pole dominance to simplify the parent integration. Similar equations hold for arbitrary order n, the only difference consists of the more complicated expression for the inhomogenous term $\Delta^{(n)}$. Note however, that although $d^{(n)}$ contains product of many poles (or more general, power singularities), which result from varius products of lower correlation functions, the structure of $\Delta^{(n)}$ is simpler. The pole dominance in d^3K integration splits $\Delta^{(n)}$ into a sum of terms where the parent is almost parallel to *each one* of the final partons. Consequently $\Delta^{(n)}$ has a structure

$$\Delta^{(n)} = \int d^3K \mathcal{M}_P(K) d_K^{(n)} = \sum_{i=1}^{n}\Delta_i^{(n)}(\Theta_{Pk_i},\chi)\,, \tag{44}$$

where χ denotes all remaining variables. It follows that χ contains only relative angles ϑ_{jl} between final partons. In particular $\Delta^{(2)}$ as given by Eq. (43) has the form (44).

The structure (44) of the inhomogenous term implies that the parent integration in the integral equation for $\Gamma^{(n)}$ itself , Eq. (42), can be also simplified. It turns out that $\Gamma^{(n)}$ also decouples into a sum of the type (44) and each of individual terms $\Gamma_i^{(n)}$ satisfies the following equation (n=2)

$$\Gamma_i^{(2)}(\vartheta_i,\vartheta_{12},P) = \Delta_i^{(2)}(\vartheta_i,\vartheta_{12},P) + \frac{a^2}{\vartheta_i^2}\int\limits_{Q_0/\vartheta_{12}}^{P}\frac{dK}{K}\int\limits_{\vartheta_{12}}^{\vartheta_i}\frac{d\Theta_{Kk_i}}{\Theta_{Kk_i}}\left[\Theta_{Kk_i}^2\Gamma^{(2)}(\Theta_{Kk_i},\vartheta_{12},K)\right], \vartheta_i > \vartheta_{12}\,. \tag{45}$$

Again the pole dominance was used to approximate the angular part of the parent integration. Note however the difference in the phase space boundaries as compared to Eq. (43). Similar equations hold for higher order correlation functions. They all imply a remarkably simple structure for the connected correlation functions.

4.2. Integral equation for cumulant moments

Since the results of the previous Section show that connected correlation functions are more natural observables for the description of the QCD cascade [15], we discuss here the cumulant moments

$$C_P^{(n)}(\vartheta,\delta) = \int\limits_{\gamma(\vartheta,\delta)} d\Omega_1 \dots d\Omega_n \Gamma_P^{(n)}(\Omega_1,\dots,\Omega_n). \tag{46}$$

For small opening angle $\delta \ll \vartheta$ the difference between the cumulant and factorial moments $F - C = \int_\gamma d$ is nonleading in DLA since the cone γ does not contain narrow singularities of the direct term $d_P^{(n)}$. For $\delta \sim \vartheta$ the difference may be significant, however it vanishes again in the high energy limit (the angular cut-off $\kappa \to 0$). For small δ the integral in Eq. (46) can be approximated, giving for $n = 2$

$$C^{(2)} = \sum_{i=1}^{2} \int d\Omega_i \int d\Omega_{12} \Gamma_i^{(2)}(\vartheta_i,\vartheta_{12}) \simeq 4\pi\delta^2 \int\limits_{\kappa}^{\delta} \Gamma_1^{(2)}(\vartheta,\vartheta_{12})\vartheta_{12} d\vartheta_{12}, \tag{47}$$

since both terms in the sum are equal. For larger δ the integral over Ω_i should be done more carefully. However, one can define new moments which are simply related to Γ for all δ. Because the natural variables for this problem consist always of *one* polar angle $\Theta_{Pk_i} \equiv \vartheta_i$ and all others relative angles, it is useful to define moments of the *associated multiplicity*

$$A^{(n)}(\vartheta,\delta) = \sum_{i=1}^{n} \int \delta(\Omega_i - \Omega)\Gamma_i^{(n)} d\Omega_1 \dots d\Omega_n, \qquad \Omega = (\vartheta,\phi). \tag{48}$$

For small opening angle δ cumulant moments and associated moments are simply related $C \simeq \delta^2 A$. For larger δ associated moments are still readily derivable from the connected correlations and contain complete information about the scaling behaviour.

Finally, it is convenient to introduce $\overline{C} = \vartheta^2 A \simeq \frac{\vartheta^2}{\delta^2} C$ which corresponds to the density in $\zeta = \ln\vartheta$.

Now we are in the position to derive and solve the integral equation for factorial moments. It was shown in the previous section that connected correlation functions depend on rather simple subsets of all angles characterizing the final n-parton configuration. This allows for the straightforward integration of the integral equation Eq. (45) (also for higher n) over the cone, Eq. (47). One obtains (see Appendix B)

$$\overline{C}_P^{(n)}(\vartheta,\delta) = \overline{D}_P^{(n)}(\vartheta,\delta) + a^2 \int_{Q_0/\delta}^{P} \frac{dK}{K} \int_{\kappa_K}^{\delta} \frac{d\psi}{\psi} \overline{C}_K^{(n)}(\psi,\psi) + a^2 \int_{Q_0/\delta}^{P} \frac{dK}{K} \int_{\delta}^{\vartheta} \frac{d\psi}{\psi} \overline{C}_K^{(n)}(\psi,\delta) . \quad (49)$$

Inhomogeneous parts $D^{(n)}$ are given by the corresponding integrals of, once iterated, disconnected contributions $\Delta_P^{(n)}$

$$D_P^{(n)} = \int_{\gamma} \Delta_P^{(n)} . \quad (50)$$

In particular

$$\overline{D}_P^{(2)}(\delta) = 2\pi \sum_{i=1}^{2} \int_{\kappa}^{\delta} \left(\vartheta^2 \Delta_i^{(2)}(\vartheta,\vartheta_{12},P) \right) \vartheta_{12} d\vartheta_{12} , \quad (51)$$

with $\Delta_i^{(2)}$ given by the $i-th$ term of the Eq. (43). It is easy to see that $\overline{D}^{(n)}$ is independent of ϑ. Equation (49) has slightly more complicated structure due to the term $\overline{C}(\psi,\psi)$. Nevertheless, it can be solved in two steps — each step involving the type of equation encountered earlier. Observe that at $\delta = \vartheta$ the last term vanishes and we get simpler equation for the boundary value $\overline{C}_P^{(n)}(\delta,\delta) \equiv \overline{C}_P^{(n)}(\delta)$,

$$\overline{C}_P^{(n)}(\delta) = \overline{D}_P^{(n)}(\delta) + a^2 \int_{Q_0/\delta}^{P} \frac{dK}{K} \int_{\kappa_K}^{\delta} \frac{d\psi}{\psi} C_K^{(n)}(\psi) . \quad (52)$$

Equations of this type are satisfied by the global quantities like the total average multiplicity $\overline{n}(P\Theta)$ or factorial moments in the whole cone (P,Θ). They can be easily solved by iteration

$$\overline{C}^{(n)}(x) = \overline{D}^{(n)}(x) + a \int_0^x \sinh[a(x-t)] \overline{D}^{(n)}(t) dt, \quad x = \ln\left(\frac{P\delta}{Q_0}\right) . \quad (53)$$

Now equation (49) reads in two variables

$$\overline{C}_P^{(n)}(\vartheta,\delta) = \overline{C}_P^{(n)}(\delta,\delta) + a^2 \int\limits_{Q_0/\delta}^{P} \frac{dK}{K} \int\limits_{\delta}^{\vartheta} \frac{d\psi}{\psi} \overline{C}_K^{(n)}(\psi,\delta), \tag{54}$$

with the known boundary term given by Eq. (53). Hence the equation for moments, Eq. (49), was reduced to, standard by now, integral equation with slightly more complicated inhomogenous term.

It is interesting to note that the equation for the two parton density in the relative angle, Eq. (19), is identical with Eq. (54) if we substitute

$$\Theta \to \vartheta, \vartheta_{12} \to \delta. \tag{55}$$

Leading, when $P \to \infty$, behaviours of the inhomogenous terms also coincide for $n = 2$, hence the asymptotic form of the cumulant moment $C^{(2)}$ follows directly from our asymptotic solution, Eq. (31) upon substitution (55) [14]

4.3. Solution for the cumulant moments at high energy

Asymptotic behaviour of the cumulant moments for arbitrary order n can be now readily obtained. We have proven [15] that in the high energy limit

$$\overline{D}^{(n)} \simeq A_n \delta \left(\frac{\delta}{\kappa}\right)^{na}, \qquad \kappa \to 0. \tag{56}$$

Eq. (53) implies the same behaviour for $\overline{C}_P^{(n)}(\delta)$. Substituting this into the resolvent solution of Eq. (54),

$$\overline{C}_P^{(n)}(\vartheta,\delta) = \overline{C}_P^{(n)}(\delta) + a^2 \int\limits_{Q_0/\delta}^{P} \frac{dK}{K} \int\limits_{\delta}^{\vartheta} \frac{d\Psi}{\Psi} I(\frac{P}{K},\frac{\vartheta}{\Psi}) \overline{C}_K^{(n)}(\delta), \tag{57}$$

with $I(x,y)$ given by Eq. (22), we get for the unnormalized cumulants (see Appendix C)

$$C_P^{(n)}(\vartheta,\delta) \sim \left(\frac{\delta}{\kappa}\right)^{na-a/n} \left(\frac{\vartheta}{\kappa}\right)^{a/n} \frac{\delta^2}{\vartheta^2}. \tag{58}$$

This gives for the normalized cumulants,

$$C_{\text{norm}}^{(q)} = \frac{C^{(q)}}{\overline{n}_\gamma^q}, \quad \overline{n}_{\gamma(\vartheta,\delta)} \sim \frac{\delta^2}{\vartheta^2}\left(\frac{\vartheta}{\kappa}\right)^a, \tag{59}$$

the following result which is free of the narrow and infrared divergencies

$$C_{\rm norm}^{(n)} \sim \left(\frac{\delta}{\vartheta}\right)^{a(n-1/n)-2(n-1)} \tag{60}$$

As we have discussed earlier, the asymptotic behaviour of factorial moments is the same for $\delta \ll \vartheta$, since the difference $F - C = \int_\gamma d$ is higher order in the small opening angle δ [2]. More comprehensive discussion of all our results, including running α_s, and with emphasis on the phenomenological applications, will be published elsewhere [19].

5. Conclusions

We have found that multiparton correlations in the QCD cascade have surprisingly simple structure. A recursive scheme exists, which allows for the analytic calculations of the fully differential correlation functions of arbitrary order. We have analysed in detail the solutions for the distribution of the relative angle among two partons in a jet, and for the complete angular dependence of the connected two-parton correlation function. At high energy they show the power behaviour typical for the selfsimilar process. Similarly the cumulant moments in the small angular cells have been calculated. These moments exhibit the power dependence on the volume of a cell with intermittency exponents

$$\lambda_n = \frac{a}{2}(n - \frac{1}{n}) - n + 1 \,. \tag{61}$$

These results were recently generalized to the running α_s case [15, 16]. In addition to the complete theoretical calculation of the exponents (61), results presented here constitute a basis for the higher order calculations of the correlation functions. Such a program would allow for further tests of the local parton hadron duality.

We hope that simplicity of present approach will motivate further study of the subject which in turn could shed more light on the connection between the high and low energy sectors of quantum chromodynamics.

[2] Preliminary result for *factorial* moments have been communicated by R. Peschanski *et al.* at the XXII Int. Symposium on Multiparticle Dynamics in Santiago de Compostela, Spain, in July 1992. Their result agrees with ours for the constant α_s, and for $\delta \ll \vartheta$. For comparison of the running α_s, see Ref. [15].

Appendix A

We derive here the asymptotic form, Eq. (31), of the correlation function (28) at high energies $\kappa = Q_0/P \to 0$. In this limit

$$y = 2\sqrt{\frac{\ln(\vartheta_{12}/\kappa)}{\ln(\Theta/\vartheta_{12})}} \to \infty, \quad z = 2a\sqrt{\ln(\vartheta_{12}/\kappa)\ln(\Theta/\vartheta_{12})} \to \infty, \tag{62}$$

with

$$\frac{z}{y} = a\ln\left(\frac{\Theta}{\vartheta_{12}}\right) \equiv \frac{\sigma^2}{2} = \text{const}. \tag{63}$$

Using the sum rule,

$$\sum_{k=0}^{\infty} y^k I_{\nu+k}(z) = z^{-\nu} e^{yz/2} \int_0^z e^{-y\tau^2/(2z)} I_{\nu-1}(\tau)\tau^{-\nu} d\tau, \quad \nu > 0, \tag{64}$$

one can rewrite the sum in Eq. (28) as

$$\rho^{(2)}(\vartheta_{12}, P, \Theta) = \frac{a}{\sigma^2\vartheta_{12}} \left(e^{\frac{zy}{2}} \int_0^z e^{\frac{-\tau^2}{\sigma^2}} I_0(\tau)\tau d\tau + e^{\frac{-zy}{2}} \int_0^z e^{\frac{\tau^2}{\sigma^2}} I_0(\tau)\tau d\tau \right), \tag{65}$$

with the first $O((\vartheta_{12}/\kappa)^{2a})$ term dominating the high energy behaviour. We have neglected the terms $O((\vartheta_{12}/\kappa)^{a})$. Since the integrand in the leading term is well behaved at large τ, we can extract the large P limit by setting the upper bound to infinity. This gives

$$\rho^{(2)}(\vartheta_{12}, P, \Theta) = \frac{a}{\vartheta_{12}} \left(\frac{\vartheta_{12}}{\kappa}\right)^{2a} \int_0^{\infty} e^{-u^2} I_0(\sigma u) u du, \tag{66}$$

which can be integrated exactly to give finally

$$\rho^{(2)}(\vartheta_{12}, P, \Theta) = \frac{a}{2\vartheta_{12}} \left(\frac{\vartheta_{12}}{\kappa}\right)^{2a} e^{\frac{\sigma^2}{4}}. \tag{67}$$

This is identical with Eq. (31) after taking into account Eq. (63).

Appendix B

We derive here the integral equation for the cumulant moments. Let us begin with $n = 2$. Integrating Eq. (45) over the cone, Eq. (47), gives for each i ($i = 1, 2$),

$$C_i^{(2)}(\vartheta, \delta, P) = D_i^{(2)} + 2\pi\delta^2 \frac{a^2}{\vartheta^2} \int_{\kappa}^{\delta} \vartheta_{12} d\vartheta_{12} \int_{Q_0/\vartheta_{12}}^{P} \frac{dK}{K} \int_{\vartheta_{12}}^{\vartheta} \vartheta_i d\vartheta_i \Gamma_i^{(2)}(\vartheta_i, \vartheta_{12}, K). \tag{68}$$

Changing orders of the ϑ_{12} and (K, ϑ_i) integrations gives two terms

$$\begin{aligned} C_i^{(2)}(\vartheta, \delta, P) &= D_i^{(2)} \\ &+ \delta^2 \frac{a^2}{\vartheta^2} \int_{Q_0/\delta}^{P} \frac{dK}{K} \int_{Q_0/K}^{\delta} \frac{d\vartheta_i}{\vartheta_i} \left(2\pi\vartheta_i^2 \int_{Q_0/K}^{\vartheta_i} \vartheta_{12} d\vartheta_{12} \Gamma_i^{(2)}(\vartheta_i, \vartheta_{12}, K) \right) \\ &+ \frac{a^2}{\vartheta^2} \int_{Q_0/\delta}^{P} \frac{dK}{K} \int_{\delta}^{\vartheta} \vartheta_i d\vartheta_i \left(2\pi\delta^2 \int_{Q_0/K}^{\delta} \vartheta_{12} d\vartheta_{12} \Gamma_i^{(2)}(\vartheta_i, \vartheta_{12}, K) \right). \end{aligned} \tag{69}$$

Recognizing definition (47) in the inner integrals and adding contributions from both i we get

$$\begin{aligned} C_P^{(2)}(\vartheta, \delta) =& D_P^{(2)}(\vartheta, \delta) + a^2 \frac{\delta^2}{\vartheta^2} \int_{Q_0/\delta}^{P} \frac{dK}{K} \int_{\kappa_K}^{\delta} \frac{d\psi}{\psi} C_K^{(2)}(\psi, \psi) \\ &+ \frac{a^2}{\vartheta^2} \int_{Q_0/\delta}^{P} \frac{dK}{K} \int_{\delta}^{\vartheta} \psi d\psi C_K^{(2)}(\psi, \delta), \end{aligned} \tag{70}$$

which gives yet simpler equation (49) for the logarithmic density $\overline{C}$. This result remains valid for general n as well. The essential property, namely emergence of the two terms in the integral Eq. (69), which can be reinterpreted according to the definition (47) is independent of the order of the moment.

Appendix C

We derive in this Appendix the asymptotic form of the cumulant moments Eq. (58) at large energies. In the logarithmic variables

$$x = \ln\frac{P\delta}{Q_0}, \quad t = \ln\frac{\vartheta}{\delta}, \quad z = \ln\frac{P}{K}, \quad \tau = \ln\frac{\vartheta}{\Psi}, \tag{71}$$

equation (57) reads

$$\begin{aligned}\overline{C}^{(n)}(x,t) &= \overline{C}^{(n)}(x) + a^2\int_0^x dz\int_0^t I_0(2a\sqrt{z\tau})\overline{C}^{(n)}(x-z)d\tau \\ &= \overline{C}^{(n)}(x) + a\int_0^x \sqrt{\frac{t}{z}}I_1(2a\sqrt{zt})\overline{C}^{(n)}(x-z)dz\,.\end{aligned} \tag{72}$$

Inserting asymptotic form of the inhomogenous term, Eqs (53), (56) $\overline{C}^{(n)}(x) \simeq C_n^0 \exp(nax)$, and expanding modified Bessel function we get

$$\overline{C}^{(n)}(x,t) = \overline{C}^{(n)}(x) + C_n^0 \exp(nax)\sum_{m=0}^{\infty}\frac{a^{2m+2}t^{m+1}}{m!(m+1)!}\int_0^x z^m \exp(-naz)dz. \tag{73}$$

Leading behaviour is unchanged if we replace upper limit of integration by ∞

$$\begin{aligned}\overline{C}^{(n)}(x,t) =& \overline{C}^{(n)}(x) + C_n^0\exp(nax)\sum_{m=0}^{\infty}\frac{a^{2m+2}}{m!(m+1)!}\left(\frac{t}{na}\right)^{m+1} \\ &\times\int_0^\infty u^m\exp(-u)du\,,\end{aligned} \tag{74}$$

which gives finally

$$\overline{C}^{(n)}(x,t) = C_n^0 e^{(nax+\frac{a}{n}t)}\,. \tag{75}$$

This is equivalent to Eq. (58) for $C^{(n)}$.

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