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GEOMETRY OF TRANSITION TO QUANTUM GRAVITY REGIME*

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It seems highly plausible that in changing from classical general relativity to the quantum gravity regime there is a stage at which in some regions classical regime still remains in power (and the size of these regions gradually shrinks to zero), whereas at some other places topological anomalies, changes of the metric signature, metric degeneracies, and so on, take over. The Lorentz metric behaviour in such situations is investigated in terms of the theory of differential spaces. It turns out that a Lorentz metric can locally exist on space-time (modeled by a differential space) only if the differential space in question can locally be immersed in a Minkowski space of a suitably large dimension. It is shown that in such a differential space, provided it is Hausdorff, the regions at which the manifold structure breaks down can have only the form of edges and vertices of a "lattice of crystals" on the faces of which the dimension is constant. The question of how locally defined Lorentz metrics could be "glued together" to form a "global metric field" is also considered. Some suggestions are discussed concerning the transition to the radically non-classical regime.

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1. Introduction

There is a strong belief that in changing from general relativity to quantum gravity a "Draconian step" is required which should take into account

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"not merely a failure of the classical field equations but also the whole edifice of *differential geometry* upon which they are built" [12]. However, before such a Draconian approach is successfully elaborated, less radical changes are tried. For instance, one studies quantum fields on a fixed background space-time with the background metric which can be degenerate on some regions [8, 15], or subject to the signature changes either in the presence of topology changes [11] or in the absence of them [4]. Many interesting results have been obtained with the help of these methods (some of them are discussed in Section 7.

In the present paper I explore a strategy which, although perhaps less Draconian than that alluded to by Isham, provides a rigorous geometric framework to study "non-smooth situations" investigated in the above mentioned works and also many others not taken into account in hitherto researches. To this end I follow our previous proposal to model space time by a differential space rather than by a differentiable manifold [5, 6, 10]. Differential space M is defined in terms of algebra of functions on M which *ex definitione* are assumed to be smooth on M (points of M are maximal ideals of this algebra), and is a vast generalization of the differentiable manifold concept. The aim of the present paper is to study the existence of Lorentz metrics (in general (pseudo)Riemannian metrics) on differential spaces. These metrics are not necessarily assumed to be non-degenerate, their signatures can change owing to either topology changes or other singular situations where by the latter term I understand all situations in which the differentiable manifold structure breaks down. It turns out that in this conceptual framework some methods elaborated by the above mentioned authors are not only recovered (usually as special cases), but also acquire a strict mathematical formulation.

The main idea underlying the present work is motivated by the following picture of the very early cosmic evolution (which, for illustrative purposes, I shall sketch as viewed backwards in time). At later times the Universe is well represented by the standard relativistic model with its space-time being modeled by a Lorentz differentiable manifold. It is worthwhile to notice that such a model of the space-time manifold can be fully reconstructed in terms of the algebra of smooth real functions defined on it [7, 20] which, in fact, is a differential space.

However, when one approaches the threshold of the classical physics applicability the smooth manifold structure of space-time begins to break down. It seems highly plausible that whereas on some space-time domains the classical regime still remains in power, at some other places topological anomalies take over, dimensions of tangent spaces to space-time at certain points change, the metric signature varies, the metric itself becomes degenerate, *etc.* Such a situation could picturesquely be called a geometric space-time foam, and the theory of differential spaces turns out to be a good mathematical tool to model it.

We are entitled to believe that, when approaching closer and closer to the critical quantum threshold, the above picture becomes more and more "foamy", space-time domains modeled by manifolds shrink to zero, and possibly some "radically singular" differential spaces (perhaps non-embeddeable in any (pseudo)Euclidean space) control the situation. And finally, the "quantum phase transition" takes place. It could be imagined to consist in changing from a commutative algebra defining the corresponding differential space to a non-commutative algebra assuming the responsibility for a suitable theory of quantum gravity and other physical fields (for an interesting proposal of such a "phase transition" see [16]).

In the present paper I study the "geometric space-time foam" stage of the above sketched picture of the early world's evolution (as regarded backwards in time). The attention is focused on the Lorentz metric and its foamy behaviour at the epoch during which the smooth manifold structure of space-time begins to break down. The theory of differential spaces allows us to offer a set of strict results (in the form of proved theorems) regulating this behaviour. It goes without saying that these results could be important as far as the search for the full theory of quantum gravity is concerned.

An important result is that a (pseudo)Riemannian metric can locally exist on space-time modeled by a differential space only if the local dimension of the latter (*i.e.* the dimension of the tangent space at a given point) is finite and, consequently, if the differential space in question can locally be immersed in a (pseudo)Euclidean space of a sufficiently large dimension. These conditions cover many intuitively non-smooth situations in which the manifold structure of space-time is violated; we shall call them *singularities* (or *singular regions*). The Lorentz metric in singular regions is defined by pulling it back from the Minkowski space in which the given differential space is locally immersed. If such a differential space is Hausdorff it is called *d-locally Minkowskian differential space.* We prove that the set of non-singular points of any d-locally Minkowskian differential space is open and dense in it (in the natural topology of differential spaces), and that its singularities can have only the form of edges and vertices of a "lattice of crystals", on the faces of which the dimension is constant (strict description of this rather intuitive image is given in Section 5). These geometric facts could be important as far as the quantization of the Lorentz metric on space-time is concerned. In more radical situations (*e.g.*, when the local dimension of the differential space becomes infinite) no metric (in the above sense) exists and, if necessary, one would have to look for other ways of gravity quantization.

The organization of our material is the following: The first two sections are of an introductory character: in Section 2, the fundamentals of the theory of differential spaces are briefly reviewed and, in Section 3, C^k functions and C^k r-forms on differential spaces are introduced. Sufficient conditions of the local existence of a Lorentz metric on differential spaces are established in Section 4, and the properties of d-locally Minkowskian differential spaces are discussed in Section 5. The question of how locally defined metrics could be "glued together" to form "global metric fields" is answered in Section 6. Finally, Section 7 contains a discussion of the physical significance of our results.

2. Preliminaries of differential spaces

Theories of differential spaces are based on the following algebraic strategy. Let $\mathcal{A}$ be a non-zero commutative algebra over a fixed field $\mathbb{K}$ (in the following always equal to $\mathbb{R}$ or $\mathbb{C}$). Let further $\mathcal{A}^*$ be a dual of $\mathcal{A}$ as a vector space (over $\mathbb{K}$), and $\hat{\mathcal{A}} \subset \mathcal{A}^*$ the algebraic dual of $\mathcal{A}$, *i.e.*, the set of all homomorphisms $\{\phi : \mathcal{A} \to \mathbb{K}\}$. Every algebra $\mathcal{A}$ admits a representation as a function algebra, the so-called Gelfand representation. A representation of $\mathcal{A}$, $\rho^{\mathcal{A}} : \mathcal{A} \to \mathbb{K}^{\hat{\mathcal{A}}}$, is called *Gelfand representation* of $\mathcal{A}$ if it is given by $\rho^{\mathcal{A}}(x)(\phi) = \phi(x)$, $x \in \mathcal{A}$, $\phi \in \hat{\mathcal{A}}$. The set on which functions are defined can be reconstructed from the algebra $\mathcal{A}$ as a set $M \subset \hat{\mathcal{A}}$ of its maximal ideals. The Gelfand representation of $\mathcal{A}$ is *universal* in the sense that any other representation of $\mathcal{A}$ is equivalent to its subrepresentation [19]. Different such subrepresentations lead to different theories of differential spaces [9]. The rest of the present work is based on the theory of differential spaces proposed by Sikorski [26, 27, 28].

Let us consider a non empty family C of real functions on a set M with the weakest topology τ_C in which these functions are continuous. A real-valued function f, defined on $A \subset M$, is said to be a *local C-function* if, for every $p \in A$, there exist a neighbourhood B in the topological space (A, τ_A), where τ_A is the topology in A induced by τ_C, and a function $g \in C$ such that $g|B = f|B$. The set of all local C-functions will be denoted by C_A. It is obvious that $C \subset C_A$. If $C = C_A$, the family C is said to be *closed with respect to localization.*

A family C of real functions on M is said to be *closed with respect to superposition with smooth Euclidean functions* if, for any $n \in \mathbb{N}$ and any function $\omega \in C^\infty(\mathbb{R}^n)$, $f_1, \ldots, f_n \in C$ implies $\omega \circ (f_1, \ldots, f_n) \in C$.

A family C of real functions on M which is both closed with respect to localization and closed with respect to superposition with Euclidean functions is called a *differential structure* on M. A pair (M, C), where C is a differential structure on M, is called a *differential space* (d-space for short).

The differential structure C on M is treated, *ex definitione*, as the set of all smooth functions on M. It can be easily seen that C is an algebra, in fact a subrepresentation of the Gelfand representation of $\mathcal{A}$.

For a given set C_0 of real functions on a set M, there is the smallest differential structure C on M such that $C_0 \subset C$ and $\tau_{C_0} = \tau_C$. C is said to be *generated* by C_0; one writes $C = \text{Gen } C_0$. If C_0 is a finite set of real functions, C is said to be *finitely generated.*

Differential geometry is developed on differential spaces in a similar manner as it is done on differentiable manifolds. Let (M, C) be a d-space. A *tangent vector* to (M, C) at a point $p \in M$ is defined to be a linear mapping v: $C \to \mathbb{R}$ satisfying the Leibniz condition

$$v(\alpha\beta) = \alpha(p)v(\beta) + \beta(p)v(\alpha),$$

for any α, $\beta \in C$. The set of all tangent vectors to (M, C) at p is called the *tangent space* to (M, C) at p and denoted T_pM. *Vector field* on a d-space (M, C) is a mapping

$$V : p \mapsto V(p) \in T_pM \subset \bigcup_{q \in M} T_qM .$$

Vector field V on (M, C) is smooth if, for every $f \in C$, the real function $V(\cdot)(f) : p \mapsto V(p)(f)$ is an element of C.

The concept of dimension is not a part of the d-space definition, but one can sensibly speak of the dimension of the tangent space T_pM (in the usual sense). In the theory of d-spaces it is sometimes called *local dimension* of (M, C) at p, and, of course, it can vary from point to point. The *global differential dimension* of (M, C) is defined to be the real number n such that *(i)* $n = \dim T_pM$ for every $p \in M$, and *(ii)* for every $p \in M$ and every vector $v \in T_pM$, there is a smooth tangent vector field on (M, C) such that $V(p) = v$. The theory of dimension of d-spaces in terms of the algebra C has been fully elaborated by Multarzyński and Sasin [17, 18].

Differential spaces form a very rich family of objects (every subset of $\mathbb{R}^n$ is a d-space but there are many d-spaces which cannot be embedded in any $\mathbb{R}^n$, irrespectively of n), and in practice one should often constraint oneself to a subfamily of d-spaces. Such an important subfamily, which will be used below, is formed by d-spaces of class D_0. A d-space (M, C) is said to be *of class* D_0 if, for every point $p \in M$, there exists an open neighbourhood U of p (in τ_C topology) and a differentiable manifold N such that: *(i)* $U \subset N$, *(ii)* $\dim T_pM = \dim N$, *(iii)* $C^\infty(U) = (C^\infty(N))_U$. This subfamily was originally introduced by Walczak [30] as consisting of those d-spaces which satisfy the diffeomorphism theorem[1]; it has turned out to be important in many applications.

3. C^k functions and r-forms on d-spaces

Lorentz metric on a manifold M can be defined as a 2-form (two-covariant tensor field) on M with a suitable signature. One can define Lorentz metric on a d-space (M, C) in a similar way. For doing so, let us first review basic notions and facts concerning r-forms on d-spaces. The concept of smooth r-form on a d-space (in its local interpretation) was first introduced by Kowalczyk [13], and that of C^k r-form by Sasin [22]. We shall need a few introductory concepts.

Let (M, C) be a d-space. The disjoint union of tangent spaces T_pM, $p \in M$, is denoted by TM, and the differential structure on TM generated by the set

$$\{\alpha \circ \pi : \alpha \in C\} \cup \{d\alpha : \alpha \in C\}$$

by TC, where, as usual, $\pi : TM \to M$ is the projection, and $d\alpha$: $TM \to \mathbb{R}$ is given by $d\alpha(v) = v(\alpha)$, $v \in T_pM$, $p \in M$, for every $\alpha \in C$. The pair (TM, TC) is called the *tangent d-space* (or *tangent bundle*) to the d-space (M, C).

Now, we define the r^{th} *Whitney sum* of tangent d-spaces to a d-space (M, C) to be the pair (T^rM, T^rC), where

$$T^rM := \{(v_1, \ldots, v_r) \in TM \times \underset{r \text{ times}}{\ldots} \times TM : \pi(v_1) = \ldots = \pi(v_r)\},$$

$$T^rC := \{TC \times \underset{r \text{ times}}{\ldots} \times TC\}_{T^rM}.$$

It is assumed that $T^0M = M$ and $T^0C = C$.

A function $f : M \to \mathbb{R}$ on a d-space (M, C) is defined to be *of class* C^k if, for any $p \in M$, there exist an open set $U \in \tau_c$, $p \in U$, and functions $f_1, \ldots, f_n \in C$, $\sigma : \mathbb{R}^n \to \mathbb{R}$, σ being of class C^k, such that

$$f|U = \sigma \circ (f_1, \ldots, f_n)|U.$$

It can be checked that the set $\mathcal{F}^k(M)$ of all real C^k functions on a d-space (M, C) is a linear algebra. A real valued function f on (M, C) is smooth if $f \in C$.

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[1] Which says: If (M, C) and (N, D) are d-spaces and $f : M \to N$ is a smooth mapping, $p \in M$, and df is an isomorphism, then f is a local diffeomorphism in a neighbourhood of p.

Let $(M,\ C)$ be a d-space generated by C_0, $C = \text{Gen } C_0$. It can be shown [22] that a real valued function f on $(M,\ C)$ is of class C^k on $(M,\ C)$ iff for any $p \in M$ there exist a neighbourhood $U \in \tau_c$ of p, functions $f_1, \ldots f_n \in C_0$, and a C^k function $\sigma : \mathbb{R}^n \to \mathbb{R}$, $n \in \mathbb{N}$, such that

$$f|U = \sigma \circ (f_1, \ldots, f_n)|U \,.$$

Now, we are ready to define r-form on a d-space $(M,\ C)$.

Definition 3.1. A C^k function $\omega :\ T^r M \to \mathbb{R}$ is called a C^k r-form on a d-space $(M,\ C)$ if the mapping

$$\omega_p := \omega|(T_pM \times \underset{r \text{ times}}{\ldots} \times T_pM)$$

is a r-linear, for any $p \in M$. An r-form ω on a d-space $(M,\ C)$ is smooth if $\omega \in T^r C$.

For further use let us define a C^k r-form on a d-space $(M,\ C)$ to be *non-degenerate* if for any $p \in M$, $v \in T_pM$ and $1 \le j \le r$, the relationship

$$\omega(v_1, \ldots, v_{j-1},\ v,\ v_j, \ldots, v_{r-1}) = 0 \,,$$

for any $(v_1, \ldots, v_{r-1}) \in T_pM$, implies $v = 0$.

For any mapping $F :\ (M,\ C) \to (N,\ D)$ and a smooth r-form ω on $(N,\ D)$, $F^*\omega$ is the smooth r-form on $(M,\ C)$ defined by

$$F^*\omega(v_1, \ldots, v_r) = \omega(F_* v_1, \ldots, F_* v_r) \,,$$

for any $(v_1, \ldots v_r) \in T^r M$.

An important fact is that every C^k r-form on $(M,\ C)$ can be regarded as a C^k r-form on the Euclidean d-space $(\mathbb{R}^n, \mathcal{E}_n)$ pulled back to an open set in M. This fact can be given the form of the following

Proposition 3.2. Let $(M,\ C)$ be a d-space generated by a family of functions C_0, $C = \text{Gen}\, C_0$, $p \in M$, and let ω be a C^k r-form on $(M,\ C)$, $r \le k$. There exist:

(i) a smooth mapping $F\colon (M,\ C) \to (\mathbb{R}^n,\ \mathcal{E}_n)$ with coordinates $F_1, \ldots, F_n \in C_0$, $n \in \mathbb{N}$;

(ii) a C^k r-form $\theta :\ T^r\mathbb{R}^n \to \mathbb{R}^n$ on $(\mathbb{R}^n,\ \mathcal{E}_n)$, and

(iii) an open set $U \in \tau_c$, $p \in U$,

such that

$$\omega|\pi_0^{-1}(U) = F^*\theta|\pi_0^{-1}(U) \,,$$

where $\pi_0 : T^r M \to M$ is defined by $(v_1, \ldots, v_n) \mapsto p = \pi(v_1) = \ldots = \pi(v_n)$. ■

Sketch of the proof. The d-space $T^r(\mathbb{R}^n,\ \mathcal{E}_n)$ can be identified with the d-space $(\mathbb{R}^{(r+1)n},\ \mathcal{E}_{(r+1)n})$ by

$$(v_1, \ldots, v_r) \mapsto (\pi_1(p), \ldots, \pi_n(p),\ d\pi_1(v_1), \ldots, d\pi_n(v_1)\,, \\ \ldots,\ d\pi_1(v_r), \ldots, d\pi_n(v_r))\,,$$

where $\pi_i : T^r\mathbb{R}^r \to \mathbb{R}^n$, is defined by $\pi_i(v_1, \ldots, v_r) = v_i$, $i = 1, \ldots, r$.

ω is a C^k function on T^rM, and consequently there exists a neighbourhood $U \in \tau_c$ of p, and a function $\sigma : \mathbb{R}^{(r+1)n} \to \mathbb{R}$ such that

$$\omega|\pi_0^{-1}(U) = \sigma(F_1 \circ \pi_0, \ldots, F_n \circ \pi_0, dF_1 \circ \pi_1, \ldots, dF_n \circ \pi_1, \ldots, dF_1 \circ \pi_r, \ldots, dF_n \circ \pi_r)|\pi_0^{-1}(U).$$

The C^k r-form $\theta : T^r\mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\theta = \sum_{i_1,\ldots,i_r=1}^{n} \frac{\partial^r \sigma}{\partial x_{n+i_1} \partial x_{2n+i_2} \ldots \partial x_{rn+i_r}} \circ \iota_{n,(r+1)n} dx_{i_1} \otimes \ldots \otimes dx_{i_r},$$

where $\iota_{n,(r+1)n} : \mathbb{R}^n \to \mathbb{R}^{(r+1)n}$ is given by

$$\iota_{n,(r+1)n}(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0),$$

satisfies the conditions of the proposition (for details see the work by Sasin [22]). □

From above proposition it follows an important

Corollary 3.3. If there exists a non-degenerate C^k r-form ω ($r \le k$) on a d-space (M, C), then there is an open neighbourhood $U \in \tau_c$ of any point $p \in M$ such that the d-space (U, C_U) can be immersed in a Euclidean space. Moreover, $\dim T_p\,(M, C) < +\infty$, for any $p \in M$. ■

Proof. The mapping $F|U$ of the above proposition is a smooth immersion. Indeed, since ω is non-degenerate, and

$$\omega|\pi_0^{-1}(U) = F^*\theta|\pi_0^{-1}(U),$$

therefore F_{*q} is a monomorphism for every $q \in U$ (otherwise $F^*\theta|\pi_0^{-1}(U)$ would be degenerate). Hence,

$$F_{*q} : T_q(M, C) \to T_{F(q)}(\mathbb{R}^n, \mathcal{E}_n)$$

is an isomorphism onto the image, and consequently

$$\dim T_q(M, C) = \dim F_{*q}(T_q(M, C)) \le \dim T_{F(q)}(\mathbb{R}^n, \mathcal{E}_n) = n. \quad □$$

4. Local existence of (pseudo)Riemannian metric on d-spaces

Definition 4.1. A C^k 2-form $g : T^2M \to \mathbb{R}$ on a d-space (M, C) is said to be a *(pseudo)Riemannian metric* on (M, C) if, for any $p \in M$, the 2-form

$$g_p = g|T_pM \times T_pM$$

is symmetric, non-degenerate and has the signature $(m,\ n)$. If the signature is $(1,\ \dim T_pM-1)$, the (pseudo)Riemannian metric is said to be the *Lorentz metric*. If a (pseudo)Riemannian metric is defined on $U \in \tau_c$, $p \in U \subset M$, it is said to be a *pseudo(Riemann) metric (defined locally)* at p. Correspondingly, we shall also speak of *(locally) (pseudo)Riemannian* or *Lorentz d-spaces*.

It is evident that if $(M,\ C)$ is a differentiable manifold the above definition reproduces the usual (pseudo)Riemannian metric on a differentiable manifold. On differentiable manifolds such a metric locally always exists. The sufficient condition of the local existence of a (pseudo)Riemannian metric on a d-space is given by the following:

Theorem 4.2. Let $(M,\ C)$ be a Hausdorff d-space, $U \in \tau_c$, $p \in U$. A C^k (pseudo)Riemannian metric g exists (locally) at p if a d-subspace $(U,\ C_U)$ is finitely generated, *i.e.*, $C = \mathrm{Gen}\{f_1, \ldots, f_n\}$. ■

Proof. A d-space $(U,\ C_U)$ is finitely generated iff there is a smooth mapping

$$\phi : (U,\ C_U) \to (\phi(U),\ (\mathcal{E}_n)_{\phi(U)}),$$

where $\mathcal{E}_n$ denotes $C^\infty(\mathbb{R}^n)$ and $\phi := (f_1, \ldots, f_n) : U \to \mathbb{R}^n$ is defined by $\phi(p) = (f_1(p), \ldots, f_n(p))$, such that

$$\phi^* : (\mathcal{E}_n)_{\phi(U)} \to C$$

is an isomorphism of linear algebras [25]. Since $(\mathbb{R}^n,\ \mathcal{E}_n)$ is paracompact, a (pseudo)Riemannian metric η always exists on $(\mathbb{R}^n,\ \mathcal{E}_n)$ and $g = \phi^*\eta$ is the (pseudo)Riemannian metric on $(U,\ C_U)$. Indeed, for any $p \in U$, there exists a set of generators $\phi = (f_1, \ldots, f_n)$ such that $(\phi_*)_p : T_pM \to T_{\phi(p)}M$ is a bijection, and consequently $g = \phi^*\eta$ is non-degenerate on $(U,\ C_U)$. (Notice that the signature can change when pulling back the metric from $(\mathbb{R}^n,\ \mathcal{E}_n)$ to $(U,\ C_U)$.) □

From Corollary 3.3 it immediately follows that if a (pseudo)Riemannian metric exists locally at $p \in U \subset M$ then $\dim T_qM < \infty$, for any $q \in U$. One should also not forget that the mapping $\phi : (U,\ C_U) \to (\phi(U),\ (\mathcal{E}_n)_{\phi(U)})$ of theorem 4.2 is smooth in the sense of the d-space theory, and consequently the theorem refers to many situations which would intuitively be qualified as non-smooth. This is illustrated by the following

Example 4.3. Let $(\overline{M},\ \overline{C})$ be a d-space such that $\overline{M} = \{(x, y, z) \in \mathbb{R}^3 : x = 0 \vee y = 0\}$ and $\overline{C}$ is the differential structure on $\overline{M}$ induced from $(\mathbb{R}^3,\ \mathcal{E}_3)$. One can easily see that

$$\bar{g} = \iota^*(-(d\pi_3)^2 + (d\pi_1)^2 + (d\pi_2)^2),$$

where ι is the inclusion and π_i, $i = 1,2,3$, are projections onto coordinate axes, is a Lorentz metric on $(\overline{M},\ \overline{C})$ which is a disjoint union of two 2-dimensional Minkowski space-times: (M_1, g_1) and (M_2, g_2), where

$$\begin{aligned} M_1 &= \{(x,y,z) \in \overline{M} : y = 0\}\,, \\ M_2 &= \{(x,y,z) \in \overline{M} : x = 0\}\,, \\ g_1 &= -(d\pi_3)^2 + (d\pi_2)^2\,, \\ g_2 &= -(d\pi_3)^2 + (d\pi_1)^2\,. \end{aligned}$$

It can be proved [22] that if $(M,\ C)$ is a connected d-space of constant differential dimension, and g is a (pseudo)Riemannian metric on $(M,\ C)$, then the signature of g is constant on $(M,\ C)$. This is evidently not true for a d-space of non-constant differential dimension (*i.e.*, such that $\dim T_pM$ is not the same for all $p \in M$), although in such a case the signature of g can still remain Lorentzian [*i.e.*, $(1, \dim T_pM - 1)$].

In the case of differentiable manifolds any (pseudo)Riemannian metric can always be locally reduced to the Minkowski (or Euclidean) form. This fact, interpreted as the statement that any gravitational field can be locally transformed away (the so-called principle of equivalence), has important consequences for relativistic physics. Since, in the general case of d-spaces, a Lorentz metric can exist also on intuitively "non-smooth situations" (see above example 4.3), the question arises what could be saved from the "locally Minkowski" property. A partial answer to this question is the following:

Let (U, C_U) be a d-subspace of (M, C) such that $C_U = \mathrm{Gen}\{f_1, \ldots, f_n\}$, $U \in \tau_c$, $p \in U$, and let a C^k Lorentz metric g exists at p. Since $(U,\ C_U)$ is finitely generated, the smooth mapping ϕ appearing in the proof of theorem 4.2 must be a diffeomorphism and $(U,\ C_U)$ of class D_0, provided ϕ is Hausdorff [25]. Moreover, if $\dim T_pM = n$ and g is of the Lorentz signature $(1,\ n-1)$, then there exists an open neighbourhood $V \in \tau_c$ of p, $V \subset U$ and the Minkowski metric η (of signature $(1,\ n-1)$) on an open subspace of $(\mathbb{R}^n,\ \xi_n)$ such that

$$g|\pi_0^{-1}(V) = \phi^*\eta|\pi_0^{-1}(V)\,,$$

where $\phi = (f_1, \ldots, f_n)$ [22]. In the sense of this equality we can say that any locally Lorentz, Hausdorff, finitely generated d-space retains locally some Minkowskian properties; such d-spaces will be called *d-locally Minkowskian d-spaces.* It has been shown [25] that every Hausdorff, locally finitely generated d-space is of class D_0 and, consequently, every d-locally Minkowskian d-space is of class D_0. In the next section we shall study the structure of d-locally Minkowskian d-spaces.

5. *d*-locally Minkowskian *d*-spaces

An interesting question arises: to which extent d-locally Minkowskian d-spaces differ from smooth manifolds? If we agree, for the purposes of the present study, to call all those regions of a d-space in which the manifold structure breaks down singular regions (or singularities), we want to know which kind of singularities can occur in a d-locally Minkowskian d-space. To make our question more precise, let us introduce the following:

Definition 5.1. Let (M, C) be a d-space. A point $p \in M$ is called a *regular point* of (M, C) if there is a neighbourhood $U \in \tau_c$ of p such that, for every $q \in U$, $\dim T_qM = \dim T_pM$. A point $p \in M$ is called *singular* point of (M, C) if it is not a regular point of (M, C) .

Let ∂M denote the set of all singular points of (M, C), and M_i the set of all regular points of (M, C) such that the dimension of tangent spaces to (M, C) at these points equals i (for details see the works by Domitrz [3] and Kowalczyk [14]).

Lemma 5.2. Let (M, C) be a d-locally Minkowskian d-space. The set of all regular points of (M, C) is open and dense in M (in the τ_c topology). ■

Proof. From definition 5.1 it follows that if a point $p \in M_i$ then an open neighbourhood $U \in \tau_c$ of p is contained in M_i. Therefore, M_i is open in M.

The remaining part of the lemma will be proved by induction. Let $p \in M$ and $\dim T_pM = 0$. The set $\{p\} \in \tau_c$ is a neighbourhood of p; therefore p is a regular point.

Now, let us assume that the set of all regular points, such that the dimension of tangent spaces to (M, C) at these points is less than n, is dense in M. Let $p \in M$ be a point such that $\dim T_pM = n$, and let $U \in \tau_c$, $p \in U$. From the fact that (M, C) is of class D_0 it follows that there exist a neighbourhood V of p and a differentiable manifold (N, D) such that for any $q \in V$ one has $\dim T_qM \leq \dim T_pM = n$. If p is a regular point the theorem is proved. Let $p \in M$ be a singular point. Definition 5.1 tells us that there exists a point $q \in U \cap V$ such that $\dim T_qM < n$ and, by assumption, q belongs to the closure of the set of all regular points in (M, C) . Therefore, $V \cap U \in \tau_c$ is a neighbourhood of q. We have shown that a regular point belongs to U. □

Taking into account that if for every $p \in M$, $\dim T_pM = n$, then (M, C) is a d-space of a constant differential dimension [10], one can easily prove [3] the following:

Theorem 5.3. If (M, C) is a d-space of class D_0, then M can be presented as a disjoint sum

$$M = \bigcup_{i=0}^{\infty} M_i \cup \partial M ,$$

where $i = 0, 1, 2, \ldots$. Each M_i is a d-space of class D_0 of constant differential dimension i and open in M; ∂M is a boundary set of M. □

This theorem tells us that each d-locally Minkowskian d-space has locally the structure of a lattice of "crystals" the faces of which are D_0 d-spaces of constant differential dimensions, and singularities have form of edges and vertices. In this sense example 4.3 presents a "typical" situation.

In a d-locally Minkowskian space-time (*i.e.*, space-time modeled by a d-locally Minkowskian d-space) singularities (in the sense of definition 5.1) could prevent the gravitational field from being locally transformed away, but since such a space-time is of class D_0 it can always locally be regarded as a part of a manifold (in the sense defined at the end of Section 2) in which, as usually, the gravitational field can locally be transformed away. This could be considered as the equivalence principle generalized to the intermediate phase between the classical and quantum regimes.

6. Global existence of Lorentz metric on d-spaces

For physical reasons explained in the introduction, we are interested in patching together (pseudo)Riemannian metrics of different signatures, possibly even degenerate ones (in some regions), rather than in creating a "smooth field" of the Lorentz metric on a given d-space. Consequently, we shall first discuss "gluing together" r-forms of class C^k defined locally on a d-space.

Lemma 6.1. Let $(A,\ C_A)$ be a d-subspace of a finitely generated Hausdorff d-space $(M,\ C)$, $p \in A$, and $i : A \hookrightarrow M$ the inclusion map. The existence of a C^k r-form g on $(A,\ C_A)$, implies the existence of a C^k r-form $\tilde{g}$ on $(U,\ C_U)$, where $U \in \tau_c$, $U \subset M$, $p \in U$, such that

$$i^*\tilde{g}|\pi_0^{-1}(A \cap U) = g|\pi_0^{-1}(A \cap U) \,. \blacksquare$$

Proof. The set $\{\alpha|A :\ \alpha \in C\}$ generates C_A, and on the strength of proposition 3.2 there exist a smooth mapping $F :\ (U,\ C_U) \to (\mathbb{R}^n, \mathcal{E}_n)$, a C^k 2-form θ on $(\mathbb{R}^n,\ \mathcal{E}_n)$ and $U \in \tau_c$ such that

$$\tilde{g}|\pi_0^{-1}(A \cap U) = (F|(A \cap U))^*\theta|\pi_0^{-1}(A \cap U) \,.$$

By putting $\tilde{g} = F^*\theta$ the conclusion follows. □

Theorem 6.2. Let $(M,\ C)$ be a d-space such that to any of open coverings of any of its open d-subspaces a C^k partition of unity can be subordinated. If a C^k r-form g exists on a d-subspace $(A,\ C_A)$ of $(M,\ C)$, then there exists a C^k r-form $\tilde{g}$ on an open d-subspace $(U,\ C_U)$ of $(M,\ C)$, $A \subset U$, such that $g = i^*\tilde{g}$ where $i :\ A \hookrightarrow U$ is the inclusion map. If A is closed in

M, the C^k Lorentz metric exists globally on M (*i.e.*, one can take $U = M$). ■

Proof is a combination of lemma 6.1 with the standard procedure of using partitions of unity [13] (for the partition of unity definition adapted to the theory of d-spaces see Appendix). □

Of course, theorem 6.2 applies, in particular, to a C^k 2-form $\tilde{g}$ which could be assumed to play the role of a Riemann metric on (M, C). Let us notice, however, that the theorem does not guarantee that $\tilde{g}$ will be nondegenerate. By abusing terminology, we shall continue to call it a metric on (M, C) (we shall also use the term *degenerate (pseudo)Riemannian metric* on (M, C)).

Let us assume that a (proper) Riemannian metric exists on a d-subspace (U, C_U) of a d-space (M, C), and let T_pU be a tangent space to (U, C_U) at a point $p \in U$. A one-dimensional vector subspace Q_p of T_pU is called a *direction* in T_pU. The function Q: $p \mapsto Q_p$, $p \in U$, is called a *direction field* on (U, C_U). If one additionally assumes that a C^k direction field exists on (U, C_U), one can show, by using standard methods, that a C^k Lorentz metric exists on (U, C_U) [5]. In particular, this remains true if instead of (U, C_U) we take (M, C).

Let us finally mention that (pseudo)Riemannian metrics on a d-space (M, C), understood locally in the sense of definition 4.1, can be collected together to form a sheaf of metric tensor fields (M, C) [23, 24]. If (M, C) is of constant differential dimension this sheaf will be a locally free sheaf of modules.

7. Discussion and comments

In the foregoing sections rigorous geometric results have been presented referring to the transition period between classical and quantum regimes when the (Lorentz) metric structure still makes some sense but the smooth manifold description of space-time breaks down. Indeed, such a picture slowly emerges from partial results obtained by various authors.

First, let us notice that if gravity quantization involves topology changes (and this is nowadays commonly believed) some metric singularities are unavoidable (under reasonable conditions). Indeed, from Tipler's [29] theorem it follows that if a space-time interpolates between two compact spacelike hypersurfaces of different topology, there is no smooth (in fact even C^2) Lorentz metric on this space-time satisfying the conditions of the theorem. Consequently, the Lorentz metric must be somewhat badly behaved [11].

Moreover, in a Lorentzian approach to gravity quantization there seems to be no reason in restricting the action integral to smooth Lorentz metrics. Horowitz [11] convincingly argues that "at least all metrics with finite action (and probably more singular metrics as well)" should be included into it.

It is also worthwhile to notice that the Ashtekar formulation of general relativity makes sense if degenerate metrics are admitted. In such a case the Ashtekar theory [1, 2] admits all solutions that are admitted by general relativity plus additional ones in which the metric is degenerate. As remarked by Rovelli [21]: "The possibility of having a degenerate metric in the theory is a crucial ingredient of the quantization attempts". During the phase transition the geometric framework which has been studied in the foregoing sections, generally relativistic physics should meet quantum physical methods, and indeed something of this kind seems to occur. Let us quote only almost at random chosen example. On the one hand, as shown by Horowitz [11], in a first-order formulation the field equations of general relativity remain well defined (in the limit) if the metric becomes degenerate. On the other hand, Dray et al. [4] attempted to define the wave equation on a manifold with the Lorentz metric admitting some sorts of singularities and demonstrated that the signature changes are connected with the production of particles.

The study of the phase transition period between the classical and quantum gravity regimes, both from the mathematical and physical point of view, might be crucial for finding the correct version of the theory of quantum gravity. The approach adopted in the present work could help in tracing the correct path. As we have seen in Section 2, differential geometry can be formulated in terms of operations on an algebra of real functions; similarly, one could use complex-valued functions to obtain a corresponding d-space theory [9, 24]. It is tempting to develop a non-commutative differential geometry by replacing the algebra of functions defining the differential structure by an associative but non-commutative algebra, and construct with the help of it a quantum gravity field theory. There are some attempts leading in this direction; I shall only mention an interesting model worked out by Madore [16] since it beautifully fits into the conceptual framework of the differential space theory.

Let us consider a d-space $(S^2,\ C)$, where $S^2 \subset \mathbb{R}^3$ is a 2-sphere $g_{ab}x^a x^b = r^2, a, b = 1, 2, 3,\ g_{ab} = \delta_{ab}$ being the Euclidean metric in $\mathbb{R}^3$, and C the differential structure generated by complex-valued functions which have the polynomial expansion

$$f(x^a) = f_0 + f_a x^a + \tfrac{1}{2} f_{ab} x^a x^b + \dots ,$$

where $a = 1, 2, 3$. Now, we truncate all these functions to the first constant term; the corresponding d-space generated by constant complex-valued functions is just a point. We can say that in the zero order the sphere S^2 is approximated by a single point. If we truncate all functions to the first two terms we obtain a four-dimensional vector space. We change it into a non-commutative algebra (by suitably defining product of its elements)

M_2 of complex 2×2 matrices. To this end we make the transformation $x^a \mapsto x'^a = \kappa\sigma^a$, where σ^a are the Pauli matrices and κ is related to r by $3\kappa^2 = r^2$. As it is evident from this equation, the algebra M_2 approximates S^2 in a very fuzzy manner: only two poles are distinguished. In a similar way we continue to construct the series of non-commutative matrix algebras M_3, $M_4, \ldots$ better and better approximating the sphere S^2. For M_n the relationship between κ and r becomes $r^2 = (n^2 - 1)\kappa^2$, and for large n one has $\kappa \cong r/n$. Therefore, $\kappa \to 0$ as $n \to \infty$. One can define the constant $k = 4\pi\kappa r$ which has the dimension [length]^2 and bears a clear resemblance to the Planck constant in quantum mechanics. If $k \to 0$ one recovers the d-space (S^2, C) as the commutative limit of the above non-commutative matrix algebras. The field theory developed by Madore on a "fuzzy sphere" indicates what can be obtained by exploring possibilities offered by the theory of differential spaces.

In the present work I have discussed the existence of (both local and global) (pseudo)Riemannian metrics on d-spaces. In a similar way connection, curvature and torsion forms, and consequently a generalization of Einstein's equations, on d-spaces can be analyzed; this will be a subject-matter of a forthcoming paper.

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Appendix

Partition of unity on differential spaces

Usually, one "glues together" local quantities to form a global structure with the help of partition of unity. The following definition adapts this concept to the theory of d-spaces.

Definition. Let (M, C) be a d-space, and $(A_s, s \in S)$ an open covering of the topological space (M, τ_c). The family of functions $\{\alpha_i : t \in T, \alpha_t \in \mathcal{F}^k(M)\}$, where $\mathcal{F}^k(M) \subset C$ is the set of all C^k functions on M, is said to be a C^k *partition of unity subordinated to the covering* $(A_s, s \in S)$ if

(i) $\alpha_t \geq 0$ on M, for any $t \in T$,

(ii) the family of sets $\{cl\ \alpha_t^{-1}[0, \infty]\}$ is locally finite and each of the sets $cl\ \alpha_t^{-1}[0, \infty]$ is contained in a set A_s belonging to the covering $(A_s, s \in S)$,

(iii) $\sum\limits_{t \in T} \alpha_t = 1$ on M.

It can be shown that if a topological space (M, τ_c) is locally compact and paracompact then, for any open covering of this space, there exists a smooth partition of unity subordinated to it (see the book by Walczak and Waliszewski [31]).

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