THE RELATIVISTIC KEPLER PROBLEM IN THE LOBACHEVSKY SPACE

N.A. Chernikov

Joint Institute for Nuclear Research
Laboratory of Theoretical Physics
141980 Dubna, Russia

(Received October 30, 1992)

Equations of gravitation in the Lobachevsky space are formulated. The problem of the gravitational field of point mass in the Lobachevsky space is solved. In the Newtonian (nonrelativistic) case, this problem was posed and solved by Lobachevsky himself. In the relativistic case, one should first find adequate equations for the metric describing the gravitational field and then find their solutions. These equations are found by the author on the basis of the theory, developed by him, with two affine connections; one called Christoffel and the other, background. The latter is given by the equations of motion of free material particle in the Lobachevsky space. It is independent of the light velocity c. The static spherically symmetric metric found here depends on the ratio of the gravitational radius γMc^{-2} of mass M to the Lobachevsky constant k for the visible world. In the limit $k \to -\infty$ it turns into the well known Schwarzschild metric. The world line of a planet is geodesic with respect to this metric. The relativistic Kepler problem in the Lobachevsky space is reduced to a nonlinear differential equation.

PACS numbers: 04.20. Cv, 04.20. Jb

Lobachevsky, the creator of the non-Euclidean geometry, did not consider the geometry alone. Having no doubt about the self-consistency of the new geometry and being convinced in its validity, he posed the problem of astronomical verification of the geometry of our visible world. Supplementing the data on parallaxes of stars by his own observations, he found out that the constant k specific of the non-Euclidean geometry is larger than the distances from the Earth to the nearest stars [1, p. 207–210]. This unsatisfactory result did not, however, prevent him from posing the problem of what kind of changes will occur after introducing the new geometry in mechanics [1, p. 261]. The second problem inevitably follows the first one as soon as one starts considering celestial bodies under the conditions of the non-Euclidean geometry. But having introduced the new geometry into celestial mechanics, Lobachevsky went further and posed the problem of what changes this introduces into Newton's law of gravity. He himself answered this question giving a fundamental solution to the Poisson equation in the non-Euclidean space [2, p. 158–160].

Those were completely new problems and Lobachevsky's idea that "*one should not doubt that the forces produce all by themselves: the motion, velocity, time, mass, even distances and angles ... when it is true that forces depend on the distance, then lines can also depend on angles*" [2, p. 159] leads us far beyond the scope of the Newtonian mechanics and theory of gravity. With this idea in mind one can go even far beyond Einstein's theories.

In our previous paper [3] the nonrelativistic Kepler problem in the Lobachevsky space was solved. In the present paper, a new approach to the theory of gravity, based on Lobachevsky's geometry is expounded. We start with an elementary presentation of Lobachevsky's differential geometry and end up with the exact solution of the Schwarzschild problem in the Lobachevsky space and consideration of the geodesic lines of the metric found here. The last results allow us to pose the relativistic Kepler problem.

The first of December of this year is the 200-th anniversary of the birthday of N.I. Lobachevsky. The author presents this paper as well as the previous one to celebrate this great day.

1. Introduction of Lobachevsky's theory

Lobachevsky's geometry is completely defined by the metric

$$dl^2 = L_{\alpha\beta} dx^\alpha dx^\beta \,, \tag{1}$$

which in spherical coordinates ρ, θ, φ takes the form

$$dl^2 = d\rho^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \,, \tag{2}$$

where

$$r = k \sinh \tfrac{\rho}{k} \,. \tag{3}$$

Straight lines in the Lobachevsky space are geodesics with respect to affine connection with the components

$$L^\alpha_{\mu\nu} = \tfrac{1}{2} L^{\alpha\sigma} (\partial_\mu L_{\sigma\nu} + \partial_\nu L_{\sigma\mu} - \partial_\sigma L_{\mu\nu}) \,, \tag{4}$$

where $L^{\alpha\sigma}$ is the cometric tensor determined by the metric tensor $L_{\sigma\beta}$ and unit affinor δ^α_β from the condition

$$L^{\alpha\sigma} L_{\sigma\beta} = \delta^\alpha_\beta \,. \tag{5}$$

∂_μ is the partial derivative with respect to the coordinate x^μ. Geodesics are determined from the system of equations

$$\frac{d}{dl}\frac{dx^\alpha}{dl} + L^\alpha_{\mu\nu}\frac{dx^\mu}{dl}\frac{dx^\nu}{dl} = 0\,, \quad \alpha \in \{1, 2, 3\}\,. \tag{6}$$

It is interesting that the components (4) remember all about the metric (1) they have been generated by. Indeed, they compose a tensor

$$L^\alpha_{\mu\nu\beta} = \partial_\mu L^\alpha_{\nu\beta} - \partial_\nu L^\alpha_{\mu\beta} + L^\alpha_{\mu\sigma}L^\sigma_{\nu\beta} - L^\alpha_{\nu\sigma}L^\sigma_{\mu\beta}\,, \tag{7}$$

which equals

$$L^\alpha_{\mu\nu\beta} = (L_{\mu\beta}\delta^\alpha_\nu - L_{\nu\beta}\delta^\alpha_\mu)k^{-2}\,. \tag{8}$$

Consequently,

$$L_{\mu\beta} = \tfrac{1}{2}k^2 L^\alpha_{\mu\alpha\beta}\,. \tag{9}$$

This is nontrivial as in the limit

$$k \to \infty \tag{10}$$

the components (4) forget much about the limit metric

$$E_{\alpha\beta}dx^\alpha dx^\beta = \lim_{k\to\infty} L_{\alpha\beta}dx^\alpha dx^\beta\,. \tag{11}$$

In this case, the limit tensor

$$E^\alpha_{\mu\nu\beta} = \partial_\mu E^\alpha_{\nu\beta} - \partial_\nu E^\alpha_{\mu\beta} + E^\alpha_{\mu\sigma}E^\sigma_{\nu\beta} - E^\alpha_{\nu\sigma}E^\sigma_{\mu\beta}\,, \tag{12}$$

composed, like (7), of the components of the limit connection

$$E^\alpha_{\mu\nu} = \tfrac{1}{2}E^{\alpha\sigma}(\partial_\mu E_{\sigma\nu} + \partial_\nu E_{\sigma\mu} - \partial_\sigma E_{\mu\nu}\,, \tag{13}$$

equals zero. Consequently one can find such coordinates y through which the connection (13) will be represented as

$$E^\alpha_{\mu\nu} = \frac{\partial x^\alpha}{\partial y^\sigma}\,\frac{\partial^2 y^\sigma}{\partial x^\mu \partial x^\nu}\,. \tag{14}$$

In the y coordinate map the components of the metric tensor $E_{\alpha\beta}$ are independent of the coordinates. This is probably all we can say about the metric (11) if only the connection components with the zero curvature tensor (12) are known.

However, we can add that the metric (11) defines the Euclidean geometry as in spherical coordinates it takes the form

$$E_{\alpha\beta}dx^\alpha dx^\beta = d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{15}$$

Meanwhile, the connection (14) is invariant with respect to affine substitutions

$$y^\sigma = A^\sigma_\gamma \hat{y}^\gamma + B^\sigma \tag{16}$$

and defines only the affine geometry rather than a much richer Euclidean geometry.

2. The Lobachevsky geometry and the Lorentz group

Four functions

$$\begin{aligned} x &= k\sinh\tfrac{\rho}{k}\sin\theta\cos\varphi\,, \quad y = k\sinh\tfrac{\rho}{k}\sin\theta\sin\varphi\,,\\ z &= k\sinh\tfrac{\rho}{k}\cos\theta\,, \quad u = \cosh\tfrac{\rho}{k} \end{aligned} \tag{17}$$

of the spherical coordinates ρ, θ and φ define the three-dimensional surface in the four-dimensional centro-affine space with the Cartesian coordinates x, y, z and u. The latter is the hyperboloid cavity

$$k^2u^2 - x^2 - y^2 - z^2 = k^2\,, \tag{18}$$

in which $u > 0$. In the Cartesian coordinates it is defined by the equation

$$u = \sqrt{1 + \frac{(x^2 + y^2 + z^2)}{k^2}}\,. \tag{19}$$

We have obtained the one-to-one (or, as it is called now, bijective) mapping of the Lobachevsky space onto the surface (19). Differentiating the functions (3) and (17) we get

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,,$$

$$k^2du = rd\rho, \qquad dr = ud\rho\,. \tag{20}$$

Hence, we find that in the coordinates x, y and z Lobachevsky's metric (2) equals

$$dl^2 = dx^2 + dy^2 + dz^2 - k^2du^2\,, \tag{21}$$

where du is the differential function (19) so that

$$(k^2 + x^2 + y^2 + z^2)k^2du^2 = (xdx + ydy + zdz)^2\,. \tag{22}$$

Consequently, the internal geometry of the surface (19) in the four-dimensional pseudo-Euclidean space with the metric (21) coincides with the Lobachevsky geometry. Therefore, isometric transformations of the Lobachevsky space are given as linear transformations of the coordinates x, y, z and u conserving the quadratic form in the left-hand side of equality (18) and not changing the sign of the coordinate u. By the Poincaré definition transformations of that type form the Lorentz group. Thus, the Lorentz group is isomorphic with the group of isometries of the Lobachevsky space.

According to (21) and (22) the metric tensor components of the Lobachevsky space in the coordinates x, y, and z equal

$$L_{\alpha\beta} = c_{\alpha\beta} - k^{-2}u^{-2}x_\alpha x_\beta\,, \tag{23}$$

where $c_{\alpha\beta}$ are constants (in the present case equal to 1 for $\alpha = \beta$ and 0 for $\alpha \neq \beta$),

$$x_\alpha = c_{\alpha\sigma}x^\sigma\,. \tag{24}$$

Consequently,

$$\tfrac{1}{2}(\partial_\mu L_{\sigma\nu} + \partial_\nu L_{\sigma\mu} - \partial_\sigma L_{\mu\nu}) = -k^{-2}u^{-2}x_\sigma L_{\mu\nu}\,. \tag{25}$$

The components of the cometric tensor in these coordinates are equal to

$$L^{\alpha\beta} = c^{\alpha\beta} + k^{-2}x^\alpha x^\beta\,, \tag{26}$$

where $c^{\alpha\sigma}c_{\sigma\beta} = \delta^\alpha_\beta$. Therefore,

$$L^{\alpha\sigma}x_\sigma = u^2x^\alpha\,. \tag{27}$$

Consequently,

$$L^\alpha_{\mu\nu} = -k^{-2}x^\alpha L_{\mu\nu}\,. \tag{28}$$

A simple form of the components (23) and (28) allows one to prove easily equality (8).

According to (6) and (28), in the x, y and z coordinates straight lines in the Lobachevsky space are determined from the solution of the system of equations

$$\frac{d}{dl}\frac{dx^\alpha}{dl} - k^{-2}x^\alpha L_{\mu\nu}\frac{dx^\mu}{dl}\frac{dx^\nu}{dl} = 0\,, \qquad \alpha \in \{1, 2, 3\}\,. \tag{29}$$

Calculating the second derivative of the function (19) we get

$$\frac{d}{dl}\frac{du}{dl} - k^{-2}uL_{\mu\nu}\frac{dx^\mu}{dl}\frac{dx^\nu}{dl} = 0\,. \tag{30}$$

Now, denote by

$$r = \{x, y, z, u\} \tag{31}$$

the vector of the four-dimensional centro-affine space in which the surface (19) is the Lobachevsky space. The quadratic form (21) is denoted by $(dr,\ dr)$. In this notation the surface (19) is given by the conditions

$$(r, r) = -k^2 , \qquad u > 0 , \tag{32}$$

and Eqs (29) and (30) are written in the form

$$\frac{d}{dl}\frac{dr}{dl} - k^{-2} r \left(\frac{dr}{dl}, \frac{dr}{dl}\right) = 0 . \tag{33}$$

Hence, it follows that a straight line of the Lobachevsky space lies at the intersection of the surface (19) with the two-dimensional plane of the centro-affine space, passing through the center $r = 0$.

3. Velocity space in the special theory of relativity

In the four-dimensional space-time of the special theory of relativity the velocity of a material point can be represented by a bundle of time-like parallel straight lines. The combination of all these bundles is the three-dimensional space of velocities. In that space, the absolute geometry based on all the Euclidean postulates except for the fifth one is realized.

In the Lorentz case, we arrive at the Lobachevsky space of velocities assuming that the constant k is equal to the light velocity c, and the distance ρ is equal to the rapidity of a particle s. In this case, the quantities (17) are equal to the components of the four-velocity of a particle. The components of the usual velocity of a particle in the Lorentz case equal

$$v_1 = c \tanh \tfrac{s}{c} \sin\theta \cos\varphi , \quad v_2 = c \tanh \tfrac{s}{c} \sin\theta \sin\varphi , \quad v_3 = c \tanh \tfrac{s}{c} \cos\theta . \tag{34}$$

In the Galilean case $c = \infty$ and the space of velocities is Euclidean. Instead of the surface (19) in this case there appears the hyperplane $u = 1$. As for the Galilean group, it is isomorphic to the group of isometries of the Euclidean space.

So the light velocity plays the role of the Lobachevsky constant in the space of velocities. This is the essence of the special theory of relativity. It is interesting that the perimeter of a circle in the Lobachevsky space is a particle momentum; and the area of the circle, its energy.

As in cosmic rays one can observe and at accelerators achieve rapidities much exceeding the light velocity c, in the high energy physics one cannot do without the Lobachevsky geometry.

4. Newton's theory of gravitation in Lobachevsky space

The Lagrange function of a material point, if it is influenced only by the gravity force with the potential U, equals

$$\Lambda = \tfrac{1}{2} L_{\mu\nu} \frac{dx^{\mu}}{dt} \frac{dx^{\nu}}{dt} - U \,. \tag{35}$$

Consequently, the Lagrange equations of motion are

$$\frac{d}{dt}\left(L_{\alpha\nu} \frac{dx^{\nu}}{dt}\right) - \tfrac{1}{2}(\partial_{\alpha} L_{\mu\nu}) \frac{dx^{\mu}}{dt} \frac{dx^{\nu}}{dt} + \partial_{\alpha} U = 0 \,, \tag{36}$$

i.e.

$$\frac{d}{dt} \frac{dx^{\alpha}}{dt} + L^{\alpha}_{\mu\nu} \frac{dx^{\mu}}{dt} \frac{dx^{\nu}}{dt} + L^{\alpha\nu} \partial_{\nu} U = 0 \,. \tag{37}$$

Here we have introduced the absolutely stationary (at rest), according to Newton, Lobachevsky space and the absolute, according to Newton also, time t which is equivalent to denial of the Euclidean postulate of parallel lines in the visible world. In this case, the special relativity principle becomes invalid though the principle of kinematic relativity is conserved.

In the space-time $S \times T$ with the coordinate map x^1, x^2, x^3, x^4, where $x^4 = t$, the absolute time gives the differential form $\Theta = dt$. Writing down this form as

$$\Theta = \theta_a dx^a \,, \tag{38}$$

we introduce in $S \times T$ the covector field with the components

$$\theta_1 = 0, \quad \theta_2 = 0, \quad \theta_3 = 0, \quad \theta_4 = 1 \tag{39}$$

and the factorizing metric

$$\theta_{ab} dx^a dx^b = \Theta\Theta \,. \tag{40}$$

The factorizing time tensor determined by it equals

$$\theta_{ab} = \theta_a \theta_b \,. \tag{41}$$

The introduction of the Lobachevsky geometry defines in $S \times T$ the cometric tensor h^{ab} with the components equal to

$$h^{\alpha\beta} = L^{\alpha\beta} \,, \quad h^{\alpha 4} = 0 \,, \quad h^{4\beta} = 0 \,, \quad h^{44} = 1 \,. \tag{42}$$

As

$$h^{ab}\theta_b = 0 \,, \tag{43}$$

the time tensor and cometric tensor are coupled by the condition

$$\theta_{as} h^{sb} = 0 \,. \tag{44}$$

The equations of motion (37) define in $S \times T$ the affine connection. Indeed, they can be written as equations of geodesics

$$\frac{d}{ds}\frac{dx^a}{ds} + \Gamma^a_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds} = 0 \,, \tag{45}$$

by substituting $s = At + B$, where A and B are constant. Hence, we find

$$\Gamma^\alpha_{\mu\nu} = L^\alpha_{\mu\nu} \,, \quad \Gamma^\alpha_{44} = L^{\alpha\nu}\partial_\nu U \,, \quad \Gamma^\alpha_{\mu 4} = 0 \,, \quad \Gamma^\alpha_{4\nu} = 0 \,, \quad \Gamma^4_{mn} = 0 \,. \tag{46}$$

Consequently, the world trajectory of a material point in the gravitational field U is a geodesic line in the world $S \times T$ with respect to the affine connection (46).

Essentially, the world trajectory of the material point, which is not influenced by any forces, is a geodesic line with respect to the affine connection with the components $\check{\Gamma}^a_{mn}$ are equal to (46) for U =const. Thus, we have two connections at once.

When there are two connections (say Γ and $\check{\Gamma}$), then for each tensor field two covariant derivatives ∇ and $\check{\nabla}$ are to be composed. For the vector and covector fields one assumes that

$$\nabla_m T^a = \partial_m T^a + \Gamma^a_{mn} T^n \,, \quad \nabla_m T_n = \partial_m T_n - \Gamma^a_{mn} T_a \,,$$

$$\check{\nabla}_m T^a = \partial_m T^a + \check{\Gamma}^a_{mn} T^n \,, \quad \check{\nabla}_m T_n = \partial_m T_n - \check{\Gamma}^a_{mn} T_a \,. \tag{47}$$

The difference

$$P^a_{mn} = \check{\Gamma}^a_{mn} - Gama^a_{mn} \tag{48}$$

is called the affine deformation tensor.

In the present case, the affine deformation tensor equals

$$P^a_{mn} = -\theta_m \theta_n h^{as} \partial_s U \,. \tag{49}$$

For each affine connection Γ one can define the torsion tensor

$$S^a_{mn} = \Gamma^a_{mn} - \Gamma^a_{nm} \tag{50}$$

and the curvature tensor

$$R^a_{mnb} = \partial_m \Gamma^a_{nb} - \partial_n \Gamma^a_{mb} + \Gamma^a_{ms}\Gamma^s_{nb} - \Gamma^a_{ns}\Gamma^s_{mb} \,. \tag{51}$$

It is obvious that

$$S^a_{mn} + S^a_{nm} = 0\,, \qquad R^a_{mnb} + R^a_{nmb} = 0\,. \tag{52}$$

All the affine connections considered here (for instance, (46)) satisfy the condition

$$\Gamma^a_{mn} = \Gamma^a_{nm}\,, \tag{53}$$

so that their torsion tensor equals zero, and the curvature tensor obeys the algebraic identity

$$R^a_{mnb} + R^a_{bmn} + R^a_{nbm} = 0 \tag{54}$$

and the differential identity

$$\nabla_k R^a_{mnb} + \nabla_n R^a_{kmb} + \nabla_m R^a_{nkb} = 0\,. \tag{50}$$

Moreover, we will consider only the equiaffine connections. For them the contracted tensor of curvature

$$R_{mn} = R^s_{smn} \tag{56}$$

is symmetric, *i.e.*,

$$R_{mn} = R_{nm}\,. \tag{57}$$

Another contraction of the curvature tensor equals zero:

$$R^s_{mns} = 0\,. \tag{58}$$

In the case (46) the curvature tensor equals

$$\begin{aligned} R^\alpha_{\mu\nu\beta} &= L^\alpha_{\mu\nu\beta}\,, \quad R^\alpha_{\mu\nu 4} = 0\,, \quad R^\alpha_{\mu 4\beta} = 0\,, \\ R^\alpha_{\mu 44} &= \partial_\mu U^\alpha + L^\alpha_{\mu\nu} U^\nu\,, \qquad R^4_{mnb} = 0\,, \end{aligned} \tag{59}$$

where

$$U^\alpha = L^{\alpha\sigma}\partial_\sigma U\,, \tag{60}$$

and the contracted curvature tensor, according to (8) and (9), equals

$$R_{\mu\nu} = -2k^{-2}L_{\mu\nu}\,, \quad R_{44} = \Delta U\,, \quad R_{\mu 4} = 0\,, \quad R_{4\nu} = 0\,, \tag{61}$$

where

$$\Delta = L^{\mu\nu}(\partial_\mu\partial_\nu - L^\sigma_{\mu\nu}\partial_\sigma)\,. \tag{62}$$

Note that the gravitational potential U is a scalar function in the Lobachevsky space with the metric (1), U^α is the vector in this space, $R^\alpha_{\mu 44}$ is the covariant derivative of the vector U^α generated by the connection

(4), $R^{\alpha}_{\mu\nu\beta}$ is the curvature tensor (7) for the connection (4), and Δ is the differential Laplace operator generated by the metric (1).

Now, let us recall the connection resulting from (46) for $U = \text{const.}$ and denoted by $\breve{\Gamma}^a_{mn}$. We will call it the background one. The curvature tensor $\breve{R}^a_{mnb}$ of the background connection and the contracted tensor $\breve{R}_{mn}$ result from (59) and (61) for $U = \text{const}$.

Consequently,

$$R_{mn} - \breve{R}_{mn} = \theta_m \theta_n \Delta U \,. \tag{63}$$

It is obvious that the equation

$$R_{mn} - \breve{R}_{mn} = 4\pi\gamma M_{mn} \,, \tag{64}$$

where M_{mn} is the mass tensor equal to

$$M_{mn} = \rho \theta_m \theta_n \,, \tag{65}$$

and ρ is the mass density, is equivalent to the Poisson equation

$$\Delta U = 4\pi\gamma\rho \tag{66}$$

in the Lobachevsky space.

Note that γ is the gravitational Newton constant. Mass, embedded in a region of the Lobachevsky space, is equal to the integral

$$m = \int\int\int \rho\sqrt{L}\, dx^1 dx^2 dx^3 \tag{67}$$

over the region where L is the determinant of the matrix $(L_{\mu\nu})$.

It is interesting to note that the gravitational equation (64) includes the contracted tensor of curvature $\breve{R}_{mn}$ of the background connection $\breve{\Gamma}^a_{mn}$. In the limit (10) Eq. (64) turns into

$$R_{mn} = 4\pi\gamma M_{mn} \,, \tag{68}$$

in which nothing reminds the background connection. The thing is that in the limit (10) the contracted tensor of curvature $\breve{R}_{mn}$ equals zero. In fact, the complete tensor of curvature $\breve{R}^a_{mnb}$ also equals zero in this limit.

5. Fundamental solution to the Poisson equation in the Lobachevsky space

In the sphere of radius ρ for small values of ρ/k one can approximately use the Euclidean geometry. The mass density M lying at the origin of coordinates x, y, z (see (17)) equals

$$\rho = M\delta(x)\delta(y)\delta(z) \,, \tag{69}$$

where $\delta(x)$ is the Dirac function as is these coordinates

$$\sqrt{L} = \frac{1}{u}, \tag{70}$$

at the origin of coordinates $u = 1$. The solution satisfying the Poisson equation

$$\Delta U = 4\pi\gamma M \delta(x)\delta(y)\delta(z) \tag{71}$$

and the condition

$$\lim_{\rho\to\infty} U = 0, \tag{72}$$

is called the fundamental one.

Lobachevsky has shown [2, p. 159] that the "attractive force" is directed towards the center and is reciprocal to the area of the sphere; moreover, if the radius of the sphere ρ, then its area equals $4\pi r^2$, where the quantity r equals (3). Consequently, in the spherical coordinates the fundamental solution has the following partial derivatives:

$$\frac{\partial U}{\partial \rho} = Ar^{-2}, \quad \frac{\partial U}{\partial \theta} = 0, \quad \frac{\partial U}{\partial \varphi} = 0, \tag{73}$$

where A is some constant. Integrating (73) we get

$$U = -\frac{A}{k}\coth\frac{\rho}{k} + B, \tag{74}$$

where B is an integration constant. From condition (72) it follows that $B = A$. As in the small vicinity of the source, the function turns into the Newton potential $U = -\gamma M \rho^{-1}$, $A = \gamma M$. Consequently,

$$U = \frac{\gamma M}{k}\left(1 - \coth\frac{\rho}{k}\right). \tag{75}$$

Note that outside the source the fundamental solution satisfies the Laplace equation

$$\Delta U = 0. \tag{76}$$

In accordance with (62)

$$\Delta U = \frac{1}{\sqrt{L}}\partial_\mu(\sqrt{L}L^{\mu\nu}\partial_\nu U). \tag{77}$$

In the spherical coordinates the operator (62) equals

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial\rho}r^2\frac{\partial}{\partial\rho} + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right). \tag{78}$$

6. Theory of gravity with two connections

In the Einstein theory of gravity, the principal geometric object is the cometric

$$g^{ab}\partial_a\partial_b \tag{79}$$

defined in space-time X. It is of normal hyperbolic type. In the vicinity of every point $x \in X$ one can choose the coordinates

$$x^1 = x\,, \quad x^2 = y\,, \quad x^3 = z\,, \quad x^4 = t\,, \tag{80}$$

so that at the point x the quadratic form (79) becomes equal to

$$g^{ab}\partial_a\partial_b = \partial_1\partial_1 + \partial_2\partial_2 + \partial_3\partial_3 - c^{-2}\partial_4\partial_4\,, \tag{81}$$

where c is the light velocity. Like in the special theory of relativity, the light velocity c is the Lobachevsky constant in the space of velocities. But in the general case we should now speak about the space of velocities of a particle at the given point $x \in X$.

The time tensor in the Einstein theory equals

$$\theta_{ab} = -c^{-2}g_{ab}\,. \tag{82}$$

It is coupled with the cometric tensor by the condition

$$\theta_{as}g^{sb} = -c^{-2}\delta_a^b\,. \tag{83}$$

Another derivative geometric object of the cometric (79) is the volume tensor

$$dV = \varepsilon_{abmn}dx^a \wedge dx^b \wedge dx^m \wedge dx^n\,, \tag{84}$$

where

$$\varepsilon_{1234} = c^{-1}\sqrt{-g}\,, \tag{85}$$

and g is the determinant of the matrix (g_{ab}).

The third derivative geometric object of the cometric (79) is the Christoffel affine connection

$$\Gamma^a_{mn} = \tfrac{1}{2}g^{as}(\partial_m g_{sn} + \partial_n g_{sm} - \partial_s g_{mn})\,, \tag{86}$$

through which one can determine the Riemann–Christoffel tensor (51), Ricci tensor (56) and then the Hilbert scalar

$$R = g^{ab}R_{ab} \tag{87}$$

and the Einstein tensor

$$G_{mn} = R_{mn} - \tfrac{1}{2} R g_{mn} . \tag{88}$$

The latter together with the mass tensor M_{mn} enters into the gravitational equations

$$G_{mn} = 8\pi\gamma M_{mn} , \tag{89}$$

which are transformed into

$$R_{mn} = 8\pi\gamma \left(M_{mn} - \tfrac{1}{2} M \theta_{mn} \right) , \tag{90}$$

where

$$M = -c^2 g^{ab} M_{ab} . \tag{91}$$

As it is known, Hilbert derived gravitational equations (88) independently of Einstein, taking the variation

$$\delta \mathbb{H} = \int G_{mn} \delta g^{mn} dV \tag{92}$$

of the integral

$$\mathbb{H} = \int R dV . \tag{93}$$

Though independent of the choice of the coordinate map, the Hilbert integral does not contain any information on the gravitational field energy. Therefore, Einstein substituted for it the integral

$$\mathbb{E} = \int \mathcal{L} dV , \tag{94}$$

where

$$\mathcal{L} = g^{mn} (\Gamma^a_{mb} \Gamma^b_{an} - \Gamma^a_{sa} \Gamma^s_{mn} . \tag{95}$$

which satisfies the necessary condition

$$\delta \mathbb{E} = \delta \mathbb{H} . \tag{96}$$

The Einstein integral (94) depends on the choice of the coordinate map, and thus, it differs disadvantageously from the Hilbert integral (91) but contains information on the gravitational field energy. Using this integral Einstein determined the so called pseudotensor of the gravitational field energy which, like the integral that generated it, depends on the choice of the coordinate map, which is in disagreement with his requirement to formulate the laws of Nature independently of the choice of coordinates. This point is in the focus of a seventy years old discussion.

The introduction of the energy pseudotensor cannot be justified also from the point of view of the tensor analysis unless one introduces one more object — the background affine connection $\breve{\Gamma}^a_{mn}$ independent of the cometric tensor g^{ab}. Having introduced this connection, let us replace the Einstein integral (94) by

$$\mathbb{I} = \int \breve{\mathcal{L}} dV \,, \tag{97}$$

where

$$\breve{\mathcal{L}} = g^{mn}(P^a_{mb}P^b_{an} - P^a_{sa}P^s_{mn}) \,, \tag{98}$$

and P^a_{mn} is the affine deformation tensor (48). Variation of the integral (97) at the fixed background connection equals

$$\delta \mathbb{I} = \int (S_{mn} - \tfrac{1}{2} S g_{mn}) \delta g^{mn} dV \,, \tag{99}$$

where

$$S_{mn} = R_{mn} - \tfrac{1}{2}(\breve{R}_{mn} + \breve{R}_{nm}) \,, \tag{100}$$

$$S = g^{ab} S_{ab} \,. \tag{101}$$

(It is assumed here that the background connection is symmetric but not necessarily equiaffine.) Therefore, the gravitational equations (89), and correspondingly (90), are replaced by

$$S_{mn} - \tfrac{1}{2} S g_{mn} = 8\pi\gamma M_{mn} \,, \tag{102}$$

or, which is the same,

$$S_{mn} = 8\pi\gamma(M_{mn} - \tfrac{1}{2} M \theta_{mn}) \,. \tag{103}$$

As one can see, equations (89) are conserved provided that

$$\breve{R}_{mn} + \breve{R}_{nm} = 0 \,. \tag{104}$$

But to return back to the Einstein case itself, the background connection should satisfy a stronger condition than (104). The fact is that the canonical tensor of the gravitational field energy, given by the Lagrangian (98), coincides with the Einstein pseudotensor provided that in a given coordinate map the following equality holds:

$$\breve{\Gamma}^a_{mn} = 0 \,, \tag{105}$$

and, consequently

$$\breve{R}^a_{mnb} = 0 \,. \tag{106}$$

On the contrary, if the condition (106) is fulfilled, then one can find such a coordinate map in which the coordinate equality (105) is valid.

So the theory of gravity with two affine connections developed here includes the Einstein theory as a special case with the condition (105).

Assuming that the action of sources of the gravitational field is independent of the choice of the background connection, we get the equality

$$\nabla_s g^{sm} M_{mn} = 0\,. \tag{107}$$

Therefore, from Eq. (102) we get the corollary

$$\nabla_s g^{sm}(\breve{R}_{mn} + \breve{R}_{nm} - g^{ab}\breve{R}_{ab}g_{mn}) = 0\,. \tag{108}$$

It is interesting that the left-hand side of the last equality can be transformed into

$$(\breve{\nabla}_s - P_s)[g^{sm}(\breve{R}_{mn} + \breve{R}_{nm})] - g^{ab}\nabla_n \breve{R}_{ab}\,, \tag{109}$$

where

$$P_s = P^a_{sa}\,. \tag{110}$$

Therefore, if the background connection satisfies the condition

$$\breve{\nabla}_s(\breve{R}_{mn} + \breve{R}_{nm}) = 0\,, \tag{111}$$

then, according to (108), Eq. (102) results in

$$(\breve{R}_{mn} + \breve{R}_{nm})\Phi^m = 0\,, \tag{112}$$

where

$$\Phi^a = (\breve{\nabla}_s - P_s)g^{sa} = g^{mn}P^a_{mn}\,. \tag{113}$$

The corollary (112) is a great interest in connection with the discussion of harmonic coordinates as the condition of harmonicity can be written as

$$\Phi^a = 0\,. \tag{114}$$

Therefore, the vector Φ^a will be called the anharmonicity vector.

7. Choice of the background connection

In the case when there is no gravity we assume that

$$\Gamma^a_{mn} = \breve{\Gamma}^a_{mn}\,. \tag{115}$$

This is a trivial solution of the gravitational equations (103). Indeed, in this case everywhere on the manifold the mass tensor should be equal to zero, and consequently, these equations should take the form

$$S_{mn} = 0 . \tag{116}$$

The condition (115) means that the background connection can be represented in the form of the Christoffel connection. This connection should necessarily be equiaffine. Consequently, Eq. (116) should take the form

$$R_{mn} = \check{R}_{mn} . \tag{117}$$

The theory expounded in Section 6 does not impose any other conditions on the background connection and admits a large freedom in choosing them.

Let us use this freedom and assume that in the absence of gravity the metric takes the form

$$g_{ab}dx^a dx^b = L_{\alpha\beta}dx^\alpha dx^\beta - c^2 dt^2 , \tag{118}$$

where the components $L_{\alpha\beta}$ are independent of the coordinate $x^4 = t$ and form the metric (1). Consequently, the background connection is assumed to be equal to the Christoffel connection, given by the metric (118), and we find it to be equal to

$$\begin{aligned} &\check{\Gamma}^\alpha_{\mu\nu} = L^\alpha_{\mu\nu} , && && \check{\Gamma}^\alpha_{44} = 0 , \\ &\check{\Gamma}^\alpha_{\mu 4} = 0 , && \check{\Gamma}^\alpha_{4\nu} = 0 , && \check{\Gamma}^4_{mn} = 0 . \end{aligned} \tag{119}$$

It is interesting that the light velocity c did not enter into the background connection in spite of the fact that it explicitly enters into the metric (118). Therefore, the background connection (119) coincides with the previously chosen one, *i.e.*, with the connection (46) for U =const.

Correspondingly, the curvature tensor of the background connection equals the tensor (59) for U =const. *i.e.*,

$$\check{R}^4_{mnb} = 0 , \quad \check{R}^a_{4nb} = 0 , \quad \check{R}^a_{m4b} = 0 , \quad \check{R}^a_{mn4} = 0 ,$$

$$\check{R}^\alpha_{\mu\nu\beta} = L^\alpha_{\mu\nu\beta} = (L_{\mu\beta}\delta^\alpha_\nu - L_{\nu\beta}\delta^\alpha_\mu)k^{-2} . \tag{120}$$

The contracted curvature tensor of the background connection equals the tensor (61) for U =const. *i.e.*,

$$\check{R}_{\mu\nu} = -2k^{-2}L_{\mu\nu} , \quad \check{R}_{44} = 0 , \quad \check{R}_{\mu 4} = 0 , \quad \check{R}_{4\nu} = 0 . \tag{121}$$

It is remarkable that the covariant derivative of the tensor field (120) with respect to the background connection (119) equals zero:

$$\breve{\nabla}_s \breve{R}_{mnb} = 0\,. \tag{122}$$

Therefore, the condition (111) is fulfilled and we have the corollary (112) that in the present case means

$$L_{\mu\nu}\Phi^{\nu} = 0\,. \tag{123}$$

As the determinant of the matrix $(L_{\mu\nu})$ is not equal to zero, of four conditions of harmonicity (114) three of them

$$\Phi^{\alpha} = 0 \tag{124}$$

follow from the gravitational equations (103).

8. Solution of the Schwarzschild problem in Lobachevsky space

Let us solve the equations

$$R_{4n} = 0\,, \quad R_{m4} = 0\,, \quad R_{\mu\nu} = -2k^{-2}L_{\mu\nu}\,, \tag{125}$$

assuming that

$$g_{ab}dx^a dx^b = F^2 d\rho^2 + H^2 d\Omega^2 - V^2 dt^2\,, \tag{126}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2\,, \tag{127}$$

and the functions V, F, H depend only on the coordinate ρ.

In this case, nonzero components of the connection (86) equal

$$\begin{aligned}
&\Gamma^4_{41} = V^{-1}\frac{dV}{d\rho} = \Gamma^4_{14}\,; \quad \Gamma^1_{44} = F^{-2}V\frac{dV}{d\rho}\,,\\
&\Gamma^1_{11} = F^{-1}\frac{dF}{d\rho}\,, \quad \Gamma^1_{22} = -F^{-2}H\frac{dH}{d\rho}\,, \quad \Gamma^1_{33} = \Gamma^1_{22}\sin^2\theta\,;\\
&\Gamma^2_{12} = H^{-1}\frac{dH}{d\rho} = \Gamma^2_{21}\,, \quad \Gamma^2_{33} = -\sin\theta\cos\theta\,;\\
&\Gamma^3_{13} = H^{-1}\frac{dH}{d\rho} = \Gamma^3_{31}\,, \quad \Gamma^3_{23} = \cot\theta = \Gamma^3_{32}\,.
\end{aligned} \tag{128}$$

In the same coordinates nonzero components of the background connection equals

$$\check{\Gamma}^1_{22} = -k\sinh\frac{\rho}{k}\cosh\frac{\rho}{k}\,, \qquad \check{\Gamma}^1_{33} = \check{\Gamma}^1_{22}\sin^2\theta\,;$$

$$\check{\Gamma}^2_{12} = k^{-1}\coth\tfrac{\rho}{k} = \check{\Gamma}^2_{21}\,, \qquad \check{\Gamma}^2_{33} = -\sin\theta\cos\theta\,;$$

$$\check{\Gamma}^3_{13} = k^{-1}\coth\tfrac{\rho}{k} = \check{\Gamma}^3_{31}\,, \qquad \check{\Gamma}^3_{23} = \cot\theta = \check{\Gamma}^3_{32}\,. \tag{129}$$

Hence, we find the anharmonicity vector

$$\Phi^1 = \frac{1}{VFH^2}\left[\frac{d}{d\rho}\left(\frac{VH^2}{F}\right) - kFV\sinh\frac{2\rho}{k}\right],$$
$$\Phi^2 = 0\,, \qquad \Phi^3 = 0\,, \qquad \Phi^4 = 0\,. \tag{130}$$

According to (124), Eqs (125) result in the equality $\Phi^1 = 0$, which is equivalent to the equality

$$\frac{d}{d\rho}(F^{-1}VH^2) = FVk\sinh\frac{2\rho}{k}\,. \tag{131}$$

In the limit (10) Eq. (131) turns into the well known condition of harmonicity ([4, Eq. (57.08)]).

In this case, all nondiagonal components of the tensor R_{mn} are equal to zero, and diagonal ones satisfy the condition

$$R_{33} = R_{22}\sin^2\theta\,. \tag{132}$$

Therefore, of Eqs (125) it remains to satisfy only the following three equations:

$$R_{44} = 0\,, \quad R_{11} = -2k^{-2}\,, \quad R_{22} = -2\sinh^2\frac{\rho}{k}\,. \tag{133}$$

In this case we have

$$R_{44} = F^{-1}H^{-2}V\frac{d}{d\rho}\left(\frac{H}{F}\frac{dV}{d\rho}\right),$$
$$R_{22} = 1 - \frac{1}{VF}\frac{d}{d\rho}\left(\frac{VH}{F}\frac{dH}{d\rho}\right),$$
$$\frac{1}{2}H(R_{11} + V^{-2}F^2R_{44}) = \frac{dH}{d\rho}\frac{1}{VF}\frac{d(FV)}{d\rho} - \frac{d^2H}{d\rho^2}\,. \tag{134}$$

According to (133) and (134) we get the following equations:

$$\frac{d^2H}{d\rho^2} - \frac{H}{k^2} = \frac{dH}{d\rho}\frac{1}{VF}\frac{d(FV)}{d\rho}\,, \tag{135}$$

$$\frac{d}{d\rho}\left(\frac{H^2}{F}\frac{dV}{d\rho}\right) = 0\,, \tag{136}$$

$$\frac{d}{d\rho}\left(\frac{VH}{F}\frac{dH}{d\rho}\right) = FV\cosh\frac{2\rho}{k}\,. \tag{137}$$

Before solving these equations let us prove immediately that they lead to the condition of harmonicity (131). Let us consider the covariant divergence

$$\nabla_b G_a^b = \partial_b G_a^b - \Gamma_{ba}^s G_s^b + \Gamma_{bs}^b G_a^s \tag{138}$$

of the Einstein tensor

$$G_a^b = R_{as} g^{sb} - \tfrac{1}{2} R \delta_a^b \,, \tag{139}$$

which equals zero, as it is known. In the considered case, the tensor (139) is diagonal and independent of angles and time. Therefore, the first component of the covector (138) equals

$$\tfrac{d}{d\rho} G_1^1 + \Gamma_{b1}^b G_1^1 - \Gamma_{11}^1 G_1^1 - \Gamma_{21}^2 G_2^2 - \Gamma_{31}^3 G_3^3 - \Gamma_{41}^4 G_4^4 \,. \tag{140}$$

As in the considered case $\Gamma_{31}^3 = \Gamma_{21}^2$, $G_3^3 = G_2^2$, then

$$\nabla_b G_1^b = \frac{d}{d\rho} G_1^1 + 2\Gamma_{21}^2 (G_1^1 - G_2^2) + \Gamma_{41}^4 (G_1^1 - G_4^4) \,. \tag{141}$$

Then, taking into account the following formulae

$$G_1^1 = B - A \,, \quad G_2^2 = A + C \,, \quad G_4^4 = A + B \,, \tag{142}$$

where

$$\begin{aligned} A &= H^{-1} F^{-2} [H'' - H'(FV)^{-1}(FV)'] \,, \\ B &= V^{-1} F^{-1} H^{-2} (VHF^{-1}H')' - H^{-2} \,, \\ C &= V^{-1} H^{-2} (H^2 F^{-1} V')' \,, \end{aligned} \tag{143}$$

and the formulae

$$\Gamma_{21}^2 = H^{-1} H' \,, \qquad \Gamma_{41}^4 = V^{-1} V' \tag{144}$$

for connection, one can easily see that the combination

$$\nabla_b G_1^b = (B - A)' + 2H^{-1}H'(B - 2A - C) - 2V^{-1}V'A \tag{145}$$

equals zero whatever the functions F, H, V be. If these functions satisfy the gravitational equations (135), (136) and (137), then

$$A = F^{-2} k^{-2} \,, \quad B = 2H^{-2} \sinh^2 \frac{\rho}{k} \,, \quad C = 0 \,. \tag{146}$$

Substituting these expressions into (145) we get

$$\nabla_b G_1^b = 2k^{-2} H^{-2} F^{-1} V^{-1} \Big[kFV \sinh \tfrac{2\rho}{k} - (F^{-1} V H^2)' \Big] \,. \tag{147}$$

Since we have proved that (145) equals zero, we have also proved the corollary (131).

A solution of the system of equations (135), (136), (137) satisfies the condition

$$FV = C\,, \tag{148}$$

where C =const.

Indeed, it follows from Eq. (136) that

$$H^2 \frac{dV}{d\rho} = BF\,, \tag{149}$$

where B =const. Substituting here the condition (148) we get

$$\frac{d}{d\rho}(\tfrac{1}{2}V^2) = BCH^{-2}\,. \tag{150}$$

Substituting the condition (148) into (135) we get the equation

$$\frac{d^2H}{d\rho^2} - \frac{H}{k^2} = 0\,, \tag{151}$$

whose general solution is

$$H = Pk \sinh \frac{\rho + \hat{\rho}}{k}\,, \tag{152}$$

where P and $\hat{\rho}$ are integration constants. Substituting this solution into (150), we get

$$\tfrac{1}{2}V^2 = N - BCP^{-2}k^{-1} \coth \frac{\rho + \hat{\rho}}{k}\,, \tag{153}$$

where N is one more integration constant.

We have to consider only Eq. (137) but instead we can consider the sum

$$\frac{d}{d\rho}\left[(FV)^{-1}\frac{d}{d\rho}\left(\frac{1}{2}V^2H^2\right)\right] = FV \cosh \frac{2\rho}{k} \tag{154}$$

of equations (136) and (137), which is more convenient. Substituting here the condition (148), we get the following equation:

$$\frac{d^2}{d\rho^2}\left(\frac{1}{2}V^2H^2\right) = C^2 \cosh \frac{2\rho}{k}\,. \tag{155}$$

From (152) and (153) it follows that

$$\tfrac{1}{2}V^2H^2 = \left[NP^2k \sinh \frac{\rho + \hat{\rho}}{k} - BC \cosh \frac{\rho + \hat{\rho}}{k}\right] k \sinh \frac{\rho + \hat{\rho}}{k}\,. \tag{156}$$

Differentiating this function we get its derivatives

$$\left(\frac{1}{2}V^2H^2\right)' = NP^2k\sinh\frac{2}{k}(\rho+\hat{\rho}) - BC\cosh\frac{2}{k}(\rho+\hat{\rho}),$$
$$\left(\frac{1}{2}V^2H^2\right)'' = 2NP^2\cosh\frac{2}{k}(\rho+\hat{\rho}) - 2BCk^{-1}\sinh\frac{2}{k}(\rho+\hat{\rho}). \quad (157)$$

Comparing this result with Eq. (155) we find that the integration constants should satisfy the following conditions:

$$2NP^2\cosh\frac{2\hat{\rho}}{k} - 2BCk^{-1}\sinh\frac{2\hat{\rho}}{k} = C^2,$$
$$2NP^2\sinh\frac{2\hat{\rho}}{k} - 2BCk^{-1}\cosh\frac{2\hat{\rho}}{k} = 0. \quad (158)$$

Hence, we find

$$2B = Ck\sinh\frac{2\hat{\rho}}{k}, \quad 2NP^2 = C^2\cosh\frac{2\hat{\rho}}{k}. \quad (159)$$

Substituting this into (156) we get

$$V^2H^2 = C^2k^2\sinh\frac{\rho+\hat{\rho}}{k}\sinh\frac{\rho-\hat{\rho}}{k}. \quad (160)$$

Hence, on the basis of (152) we get

$$V^2 = C^2P^{-2}\frac{\sinh\frac{\rho-\hat{\rho}}{k}}{\sinh\frac{\rho+\hat{\rho}}{k}}, \quad (161)$$

and then on the basis of (148) we get

$$F^2 = P^2\frac{\sinh\frac{\rho+\hat{\rho}}{k}}{\sinh\frac{\rho-\hat{\rho}}{k}}. \quad (162)$$

It is interesting to verify that the condition of harmonicity (131) is fulfilled. Indeed, according to (148) it means that

$$\frac{d}{d\rho}(V^2H^2) = C^2k\sinh\frac{2\rho}{k}, \quad (163)$$

and according to (160)

$$V^2H^2 = \frac{1}{2}C^2k^2\left[\cosh\frac{2\rho}{k} - \cosh\frac{2\hat{\rho}}{k}\right]. \quad (164)$$

Consequently, the condition (163) is fulfilled.

Thus, the metric (126), satisfying Eqs (125), is found in the form

$$P^2k^2\{\Xi^{-1}d\xi^2 + \sinh^2(\xi+\alpha)d\Omega^2\} - C^2P^{-2}\Xi dt^2 , \tag{165}$$

where

$$\Xi = \frac{\sinh(\xi-\alpha)}{\sinh(\xi+\alpha)} , \quad \xi = \frac{\rho}{k} , \quad \alpha = \frac{\hat{\rho}}{k} . \tag{166}$$

At a large distance from the source, *i.e.* at large values of ξ, it should asymptotically approach the metric (118), more exactly the metric

$$k^2\{d\xi^2 + \sinh^2\xi d\Omega^2\} - c^2dt^2 . \tag{167}$$

Hence, it follows that

$$C = c , \quad P = \exp\{-\alpha\} . \tag{168}$$

It remains to elucidate what is the parameter α equal to. We find it from the condition that at large values of ξ the connection (128) should tend to the connection (46). In other words, all the components of the connection (128) should tend to the background connection (129), except for the component Γ^1_{44}, that should tend not to zero but to U', *i.e.*,

$$\Gamma^1_{44} \to U' = \gamma M(k\sinh\xi)^{-2} . \tag{169}$$

Note that the latter includes the very "attractive force" about which Lobachevsky wrote in 1835 (p. 159).

This condition can be fulfilled as

$$\begin{aligned} F^{-1}F' &= -H^{-1}H' = \frac{1}{2k}[\coth(\xi+\alpha) - \coth(\xi-\alpha)] , \\ H^{-1}H' &= k^{-1}\coth(\xi+\alpha) , \\ HH' &= kP^2\sinh(\xi+\alpha)\cosh(\xi-\alpha) , \\ VV' &= \frac{1}{2}kc^2\frac{\sinh 2\alpha}{[kP\sinh(\xi+\alpha)]^2} . \end{aligned} \tag{170}$$

Comparing (169) with (170) we get that

$$\frac{1}{2}\sinh 2\alpha = \frac{\gamma M}{kc^2} . \tag{171}$$

As the gravitational radius γMc^{-2} is much smaller than the constant k we can approximately put

$$\alpha = \frac{\gamma M}{kc^2} , \quad P = 1 . \tag{172}$$

9. The Kepler problem in the Lobachevsky space

Let us consider now the relativistic Kepler problem in the Lobachevsky space. In view of spherical symmetry we can put

$$\theta = \frac{\pi}{2} \tag{173}$$

without loss of generality. The equations of motion of a planet in the case (126) have the angular momentum

$$H^2 \frac{d\varphi}{d\tau} = \mu \tag{174}$$

and the energy

$$\left(\frac{V}{c}\right)^2 \frac{dt}{d\tau} = \varepsilon \tag{175}$$

integrals, where τ is the proper time of the planet. The equation

$$V^2 \left(\frac{dt}{d\tau}\right)^2 - F^2 \left(\frac{d\rho}{d\tau}\right)^2 - \dot{H}^2 \left(\frac{d\varphi}{d\tau}\right)^2 = c^2 \tag{176}$$

defines the derivative $\frac{d\rho}{d\tau}$. Taking into account equations (175) and (176) and a condition $FV = c$, we obtain the equation

$$\left(\frac{d\rho}{d\tau}\right)^2 = \varepsilon^2 c^2 - V^2 \left(\frac{\mu V}{Hc}\right)^2 . \tag{177}$$

Substituting (174) into (177) we get the differential equation for the trajectory $\rho = \rho(\varphi)$ in the form

$$\mu^2 \left(\frac{d\rho}{d\varphi}\right)^2 = \varepsilon^2 c^2 H^4 - V^2 H^4 - \left(\frac{\mu V H}{c}\right)^2 . \tag{178}$$

Substituting (152) and (161) into (178) we finally get

$$\mu^2 \left(\frac{d\rho}{d\varphi}\right)^2 = \varepsilon^2 c^2 P^4 k^4 \sinh^4 \frac{\rho + \hat{\rho}}{k}$$
$$- c^2 P^2 k^4 \sinh \frac{\rho - \hat{\rho}}{k} \sinh^3 \frac{\rho + \hat{\rho}}{k} - \mu^2 k^2 \sinh \frac{\rho - \hat{\rho}}{k} \sinh \frac{\rho + \hat{\rho}}{k} . \tag{179}$$

In the limit (10) Eq. (179) turns into the well known equation [4, Eq. (58.32)].

REFERENCES

[1] N.I. Lobachevsky, *On the Principles of Geometry*, collected works, Vol. 1, Moscow 1946, pp. 185–261.
[2] N.I. Lobachevsky, *New Principles of Geometry with Full Theory of Parallels*, collected works, Vol. 2, Moscow 1949, pp. 147–454.
[3] N.A. Chernikov, *Acta Phys. Pol.* **B23**, 115 (1992).
[4] V.A. Fock, *The Theory of Space Time and Gravitation*, Pergamon Press, London–New York–Paris–Los Angeles 1959.