Vol. 24(1993) *ACTA PHYSICA POLONICA B* No 6

DYNAMICAL SYSTEMS WITH IMPULSES: STROBOSCOPIC MAPS APPROACH

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(Received January 29, 1993)

A stroboscopic maps approach for continuous dynamical systems has been elaborated and generalized. For an impulse perturbation, the map equivalent to a given equation can be found analytically for a large class of systems, in the limit of the impulse width going to 0. This technique can be used for analyzing dynamical systems in which the perturbation has an impulse form, or can be joined with the Generalized Modulated Kicks Approximation to model a system with continuous or noisy perturbation. The validity of a "naive" method of finding the map is discussed.

PACS numbers: 03.40.-t

1. Introduction

The paper is devoted to an analysis of linear and nonlinear dynamical systems with a time dependent perturbation in a form of a series of short impulses. The choice is well supported by both natural and mathematical examples (*cf. eg.* [1]). Many physical, chemical or biological systems exhibit pulse nature of evolution or can be acted upon by a perturbation of such form. In some cases, investigation of a response of the systems with respect to the periodic or aperiodic impulse perturbation may lead to a definition of new characteristics [2–5]. It is also useful to approximate time-dependent perturbations, originally not having the impulse form, by a series of "kicks" (δ-functions) with finite or zero distance between the impulses [6, 7]. This technique may be applied either to regular or to stochastic perturbations. The first case, when a regular, time-dependent perturbation is replaced by series of kicks, is known in quantum optics as the Modulated Kicks Approximation (shortly MKA) [8–10]. For a stochastic perturbation, this approach may be dated back to Langevin and Haken [11–14].

In the paper we are building, in a systematic way, a formalism of a stroboscopic map approach to the systems with impulses [15], based on the continuous impulses approximation scheme. The stroboscopic maps are well known and widely applied in the MKA as well as in the dynamical systems theory ("kicked rotator" [16–18], and others), but it can be also applied to the systems with noise (*cf.* [19–21]). For many systems the method simplifies the simulations by making part of the work analytically done, even if the original system cannot be analyzed by other methods (including Fokker–Planck equation). The method of the continuous approximation of the delta function forms a standard approach in the theory of special functions, and alternative approaches for one-dimensional special cases are standard in the theory of stochastic processes. Nevertheless, up to the author's knowledge there has been no attempt to discuss the correctness of the way the stroboscopic map is constructed. As it will be shown below, the "naive" way of solving such problems may lead to significant errors.

The problems appear because impulses are usually considered to be of an infinitely short duration (δ-functions). However, from physical point of view, δ-function is only an approximation of a continuous process of finite duration. The validity of the approximation depends on the relation between the natural time scales of the system and of the perturbation. From mathematical point of view, calculations with δ-functions may lead to some problems when not properly carried on, because some expressions with special functions (like δ) are not well defined. In both cases (physical and mathematical) it is natural to consider impulses as continuous functions, and — after calculations are done — to perform the limit procedure of the impulse width going to 0. The problem discussed here is very similar to the one encountered in the theory of Brownian motion [12, 13].

In the paper we discuss the formalism for one- and multi-dimensional systems, show how to solve the problems with the zero-width impulses, and finally touch the case of the finite-width impulses.

2. One-dimensional case

Consider a 1-dimensional dynamical system described by a variable x and an equation of motion

$$\dot{x} = f(x) + c(t)g(x) , \tag{1}$$

where $f(x)$ determines a free (unperturbed) evolution and $c(t)g(x)$ is a time dependent perturbation (regular, chaotic, stochastic, continuous or impulse). Eq. (1) is general and includes many applications. Moreover, the results obtained here can be generalized to the cases when (1) is not valid, at least in some approximation (*e.g.* adiabatic approximation). f and g are assumed to be sufficiently smooth (analytical).

Suppose that $c(t)$ has a form

$$c(t) = \sum_{n=0}^{\infty} c_n \delta_{\Delta}(t - t_n) . \tag{2}$$

c_n are numbers and $\delta_{\Delta}(t)$ is a universal shape function of the impulses, equal 0 outside the interval $(-\Delta/2, \Delta/2)$, continuous with continuous first derivative on $[-\Delta/2, \Delta/2]$ and normalized to 1.

t_n are moments in time, ordered ($t_n > t_{n-1}$ for all n), $t_n - t_{n-1} = T_n$ may be a constant ($T_n \equiv T$), a regular or a stochastic function of n. Impulses are assumed to be well separated, *i.e.* $T_n > \Delta$ for all n. Although we assume that the shape function does not depend on n, our method can be applied to a more general case.

Eq. (2) may be considered as a continuous version of a singular perturbation

$$c(t) = \sum_{n} c_n \delta(t - t_n) , \tag{3}$$

$\delta(t)$ being Dirac function. This form is commonly used in MKA and in the shot noise theory, as well as in theory of dynamical systems identification.

The impulses form the reference system of moments in time and make it natural to consider values of the dynamical variable x at times correlated with impulses. This leads us to a concept of a stroboscopic map. Denote

$$\begin{cases} x_n^+ = x(t_n + \Delta/2) , \\ x_n^- = x(t_n - \Delta/2) . \end{cases} \tag{4}$$

The continuity of δ_{Δ} guarantees that

$$x(t_n \pm \Delta/2 + 0) = x(t_n \pm \Delta/2 - 0).$$

The stroboscopic map is given by subsequent relations: $\ldots \rightarrow x_{n-1}^+ \rightarrow x_n^- \rightarrow x_n^+ \rightarrow \ldots$. The first step can be solved easily. The evolution from $t_{n-1} + \Delta/2$ till $t_n - \Delta/2$ is given by the unperturbed equation

$$\dot{x} = f(x) . \tag{5}$$

We assume that (5) can be solved analytically or numerically to get a relation

$$x_n^- = F(x_{n-1}^+; t_{n-1}, t_n) , \tag{6}$$

with F being the solution of (5) at time $t_n - \Delta/2$ if $x(t_{n-1} + \Delta/2) = x_{n-1}^+$.

The full evolution may be considered on a single impulse level; the equation from $t = t_n - \Delta/2$ to $t_n + \Delta/2$ is given by

$$\dot{x} = f(x) + c_n \delta_\Delta(t - t_n) g(x) . \tag{7}$$

For most nonlinear f, g and general $\delta_\Delta(t)$, Eq. (7) cannot be solved analytically. However, in some cases it is possible to simplify it, when the limit $\Delta \to 0$ is considered (approximations of δ-function).

Rewrite Eq. (7) in a integral form

$$x(t) = x_n^- + \int_{t_n - \Delta/2}^{t} \left[f\big(x(\tau)\big) + c_n \, \delta_\Delta(\tau - t_n) g\big(x(\tau)\big) \right] d\tau . \tag{8}$$

This equation can be formally solved by an iteration method. The first approximation is $x(t)_{(0)} = x_n^-$. The subsequent approximation is obtained by putting the previous one into the right-hand-side of (8) and evaluating the equation, *i.e.*

$$x(t)_{(r+1)} = x_n^- + \int_{t_n - \Delta/2}^{t} \left[f\big(x(\tau)_{(r)}\big) + c_n \, \delta_\Delta(\tau - t_n) g\big(x(\tau)_{(r)}\big) \right] d\tau , \tag{9}$$

with r indexing the subsequent iteration steps. This procedure converges for f, g and δ_Δ fulfilling the regularity conditions stated above.

In the limit $\Delta \to 0$ some terms on the right-hand-side of (9) may be neglected, being of order of $\Delta^1, \Delta^2, \ldots$, and other terms may be put together to get the formula

$$x_n^+ = G(x_n^-, c_n) + \mathcal{O}(\Delta^1) , \tag{10}$$

where $\mathcal{O}(\Delta^1)$ denotes terms of order Δ^1 and higher, and G is given by the implicit relation

$$\int_{x_n^-}^{G(x_n^-, c_n)} \frac{dx}{g(x)} = c_n . \tag{11}$$

Equivalently, G is a solution of the equation

$$\dot{x} = c_n \delta_\Delta(t - t_n) g(x) , \tag{12}$$

at $t_n + \Delta/2$ with an initial value x_n^- at $t_n - \Delta/2$. In (12) the variables can be separated and the equation is easier to handle than Eq. (7), as there are no singularities in it ($\Delta \to 0$), and only one characteristic time scale (of g) instead of two (Δ times g and f) in Eq. (7).

In the limit $\Delta \to 0$, the precise form of $\delta_\Delta(t)$ does not play any role. Therefore, if $\delta_\Delta(t)$ is treated as an approximation of δ-function, the result (10) is independent of the particular approximation, and therefore correct and well defined from the mathematical point of view. This will be discussed in details in the next section.

The results just obtained are valid in the limit $\Delta \to 0$ which must be taken first, at the level of a single impulse. In particular that means that $t_n - t_{n-1} = T_n$ is bounded from below by Δ for any n. Further in this paper we will show how to find the corrections of order of $\Delta^1, \Delta^2, \ldots$ However, if the limit $\Delta \to 0$ is taken first, the iterations of the stroboscopic map match exactly the solution of the equation (1) (if no other approximations are made), even for long times. On the other hand, if Δ is finite or some approximations are used (as in the next section), the map has a finite time range of validity.

The same result may be obtained easier in the 1-dimensional case, if f, g, and c fulfill additional assumptions. If *e.g.* (11) can be solved analytically (as here) and the solution is invertible, one can change the variables in (1) in such a way that the dependence of the second term on x disappears, and then solve the equation. The method presented here is more general, and includes *e.g.* the case of $g(x)$ periodic in x.

The relations (10) and (6) form the stroboscopic map for (1). The above results can be easily generalized to the case when more than one perturbation is present in the system, each in a form of (2), *i.e.*

$$\dot{x} = f(x) + \sum_j c^{(j)}(t) g^{(j)}(x)\,, \tag{13}$$

where

$$c^{(j)}(t) = \sum_n c_n^{(j)}\, \delta_\Delta(t - t_n^{(j)})\,. \tag{14}$$

3. Singular perturbation

For continuous impulses Eq. (1) is not singular. However, if we use δ-functions, as in standard MKA, we may encounter mathematical problems. Consider Eq. (7) with $\delta(t)$ instead of $\delta_\Delta(t)$ (we assume $c_n = 1$ and n fixed)

$$\dot{x} = f(x) + \delta(t - t_n) g(x)\,, \tag{15}$$

around t_n ($t \neq t_n$). To solve the equation we must determine the integral

$$x(t_n + 0) - x(t_n - 0) = \lim_{\Delta \to 0} \int_{t_n - \Delta/2}^{t_n + \Delta/2} \delta(t - t_n) g(x)\, dt\,, \tag{16}$$

with $x(t)$ being the solution of Eq. (15). However, as $x(t)$ has a discontinuity at $t = t_n$, and so has $g(x(t))$ (except for the case when $g(x) = \text{const}$), the right-hand-side of (16) cannot be simply evaluated.

Eq. (10) gives us the result

$$x(t_n + 0) - x(t_n - 0) = G\Big(x(t_n - 0), 1\Big) - x(t_n - 0)\,. \tag{17}$$

(17) was obtained by approximating $\delta(t)$ by a series of smooth functions $\delta_\Delta(t)$ parametrized by Δ with $\delta_\Delta(t) \to \delta(t)$ when $\Delta \to 0$ (in a weak sense of the limit). The result is independent of the particular choice of the approximating functions.

In most cases, (17) gives different results than the ones obtained by a "trivial" integration (16), *i.e.* under the assumption that in the limit $\Delta \to 0$ the impulse becomes so short that $g(x)$ does not change. Then we have

$$\dot{x} = f(x) + \delta(t - t_n) g\big(x(t_n - 0)\big)\,, \tag{18}$$

instead of Eq. (15) and

$$x(t_n + 0) - x(t_n - 0) = g\big(x(t_n - 0)\big)\,. \tag{19}$$

Although (19) is wrong for $g(x)$ being non-constant, it is so much simpler than (17) in applications, that it seems to be interesting to find whether (19) may be sometimes used as an approximation for (17). For $g(x) = \text{const}$ the results are identical. If g is small for all x, namely if such a scaling constant ϵ may be found that $g(x) = \epsilon \bar{g}(x)$ with $\epsilon \ll 1$, corrections are of order of ϵ^2 and higher (this can be proved by evaluating the integral in (16) and estimating the terms). These corrections are not important on the level of one impulse, but they become important for a long time analysis. When n ("time") becomes of order of $1/\epsilon^2$, the map constructed by the exact (in the limit $\Delta \to 0$) prescription (17) and the map given by (19) may produce different results.

4. Two- and more-dimensional systems

The mathematical difficulties mentioned in the previous sections of the paper do not appear in a certain class of higher-dimensional systems. Fortunately, all up-to-date applications of MKA belong to this class. Consider the standard system to which the MKA is usually applied

$$\begin{cases} \dot{x} = y\,, \\ \dot{y} = f(x,y) + c(t)\,g(x)\,. \end{cases} \tag{20}$$

The system (20) is of a great importance, because it forms the Hamiltonian equations for a particle moving in one dimension under the influence of two forces, one of which $(c(t)\,g(x))$ depends explicitly on time and on the "coordinate" x, but does not depend on the "momentum" y . Eq. (20) is also equivalent to a Schrödinger equation for an atom in an external field varying in time according to $c(t)$, being the standard application in quantum optics. Usually $f(x,y) = -\gamma y + u(x)$, $c(t) = A\cos(\omega t)$ and $g(x)$ is given by a nonlinear potential $V(x)$: $g(x) = -\partial/\,\partial x\, V(x)$. Applying MKA to such a system we approximate $c(t)$ by a series of kicks like (3) and find an approximate stroboscopic map [6, 7]. The latter is usually done (implicitly) by the method analogous to the "naive" integration. The correctness of this approach in the case of Eq. (20) can be verified in the same way as in Section 2.1 (iteration method). In the limit of infinitely short impulses, *i.e.* for $\Delta \to 0$, we get the part of the stroboscopic map for the perturbed evolution as follows

$$\begin{cases} x_n^+ = x_n^-\,, \\ y_n^+ = y_n^- + c_n\,g(x_n^-)\,. \end{cases} \tag{21}$$

This can be supplemented by the appropriate map obtained by the solution of the unperturbed equations. For this system the corrections of order $\Delta^1, \Delta^2, \ldots$ can be also easily found (see below).

On the other hand, if g in Eq. (20) also depends on y, we are back in the more complicated case which we studied in Section 2.1 of the paper. Then, "naive" integration leads to results different from exact ones (if the latter can be obtained), and the continuous approximation method must be used. This result may be generalized as follows.

If in a set of equations

$$\dot{\xi}_{(r)} = f\Big(\{\xi_{(s)}\}\Big) + \sum_n c_n^{(r)}\delta(t-t_n)g^{(r)}\Big(\{\xi_{(s)}\}\Big)\,, \tag{22}$$

$(r, s = 1, \ldots, N)$, the condition

$$\frac{\partial g^{(r)}\Big(\{\xi_{(s)}\}\Big)}{\partial \xi_{(r)}} = 0\,, \tag{23}$$

holds for all $r = 1, \ldots, N$ and almost everywhere on the interesting set of possible $\xi_{(s)}$, then the "naive" integration gives the same results as the continuous approximation method (and as the exact solution of the equations with δ-functions, if such a solution can be found). However, if (23) is not fulfilled, we cannot use the simple method based on the "naive" integration, and the exact results are given by solving the set of equations differing from Eq. (22) by setting all $f \equiv 0$ (which is the essence of the result presented in Section 2.1 of the paper).

This theorem shows that in some cases the simple method of solving singular equations might be incorrect (like in the case of a generalization of Eq. (20) to the case when g depends also on the "momentum" y).

5. Finite impulse width

The iteration method presented above can be applied not only in the limit $\Delta \to 0$, *i.e.* if the terms of order $\Delta^1, \Delta^2, \ldots$ are neglected, but also in the case of a finite impulse width. However, the applications of the results are strongly limited by the fact that the stroboscopic map is obtained in an approximation, and therefore valid for finite times only (of order $1/\Delta^{M+1}$ where M is the highest power of Δ kept in the expansion).

Suppose now that we treat (2) "literally", *i.e.* not as the approximation of a process, but rather as a perturbation with finite impulse width and finite time interval between impulses. The following results can be then obtained with the use of the iteration method.

Let us start with the more complicated, one-dimensional case. Suppose that there exists an invertible transformation $x \to z(x)$ such that

$$z(x) = \int^{x} \frac{dx'}{g(x')} \,. \tag{24}$$

Existence of this transformation is closely related to the possibility of solving the integral equation (11); however, the integral equation may be solvable, but the transformation may not be invertible (when *e.g.* g is periodic).

The method presented here is a special case of a more general method which has already been mentioned. The idea is to change the variables in the equation Eq. (2) in such a way that the resulting equation can be easily solved, even with a time dependent term (here our goal is to make the time- and coordinate-dependent term independent of x). However, it should be stressed that this method works under several assumptions (*e.g.* g — positive) only, and cannot be easily generalized to higher dimensions. The iteration method, shown in the previous sections, does not require such assumptions.

The transformation (24) linearizes the second term in Eq. (1) and the equation becomes

$$\dot{z} = \eta(z) + c(t) \,, \tag{25}$$

where

$$\eta(z) = \left.\frac{f(x)}{g(x)}\right|_{x=z^{-1}(z)} . \tag{26}$$

The equation (25) is needed only for finding the part of the stroboscopic map describing the evolution during the impulses (*i.e.* the $x_n^- \to x_n^+$ relation). The "free" part may be obtained by solving the equation (5).

The iteration method based on the assumption that impulses are short and high gives the result

$$z_n^+ = z_n^- + c_n + \Delta\left[\frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} d\tau\eta\Big(z_n^- + c_n\int_{-\Delta/2}^{\tau} d\tau_1\delta_\Delta(\tau_1)\Big)\right] + \mathcal{O}(\Delta^2), \tag{27}$$

which might be translated into x after evaluation of the integral. Terms of order Δ^2 and higher can also be found by the same method.

For two-dimensional systems considered in the previous section of the paper, the procedure of finding corrections of order Δ^1 and higher is much simpler. Then, we do not need to transform the equation into a linear form and the following result can be obtained for x-variable

$$\begin{aligned} x_n^+ &= x_n^- + \Delta\Big(y_n^- + d_n g(x_n^-)\Big) + \mathcal{O}(\Delta^2), \\ y_n^+ &= y_n^- + c_n g(x_n^-) \\ &\quad + \Delta\left[\frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} d\tau f\Big(x_n^-, y_n^- + g(x_n^-)\int_{-\Delta/2}^{\tau} d\tau_1 c_n\delta_\Delta(\tau_1)\Big)\right. \\ &\quad \left. + \left(\frac{dg}{dx}\right)_{x=x_n^-}\Big(c_n^{(1)} y_n^- + e_n g(x_n^-)\Big)\right] \\ &\quad + \mathcal{O}(\Delta^2), \end{aligned} \tag{28}$$

for the system of equations (20). The numbers $c_n^{(1)}, d_n, e_n$ are given by integrals of τ and $\delta_\Delta(\tau)$, *e.g.*

$$c_n^{(1)} \equiv c_n \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} d\tau\,(\tau + \Delta/2)\,\delta_\Delta(\tau), \tag{29}$$

and for the symmetric shape functions $\delta_\Delta(t) = \delta_\Delta(-t)$, can be evaluated as follows: $c_n^{(1)} = c_n/2$, $d_n = c_n/2$, $e_n = c_n/6$. Moreover, for the special case $f(x,y) = -\gamma y + u(x)$, which is often used, the only remaining integral of $\delta_\Delta(\tau_1)$ in (28) may be evaluated in terms of d_n, and the result depends on general parameters $c_n^{(1)}, d_n, e_n, \ldots$ only and not on a particular shape of $\delta_\Delta(t)$. Again, corrections of higher order of Δ may be found.

The range of applications of the above results is rather limited, first by the time limit (length of iterations of order $1/\Delta^2$), and second, by complications in evaluating (27) or (28). However, it should be stressed that the remark concerning the limited range of time interval where the approximation methods may be applied, is valid for most of the numerical methods (see [22, 23]).

Concerning the theorem given in Section 2.2 of the paper, we may add that the result (27) and the method of obtaining it are typical for systems for which the condition (23) does not hold. In this case the transformation of the variables is needed and in many cases the corrections cannot be found. On the other hand, when (23) holds, the correction can be obtained without additional assumptions, like for Eq. (20).

The iteration method can also be applied to a wider class of equations with impulse-like perturbation (resulting from either physical reasons or from GMKA), *e.g.* for integro-differential equations. In this case the resulting stroboscopic map has "memory" properties [24].

6. Final remarks

The approach to an analysis of dynamical systems with time-dependent perturbations as presented here is formed by two basic techniques:

- The singular approximation: a given time-depending signal is approximated by a series of impulses with suitably chosen weights. The technique has been known and applied in different areas, especially in quantum optics, but the range of applications may be widened to include stochastic and chaotic perturbations.
- After the perturbation is represented in the form of δ-like impulses, the stroboscopic maps formalism can be used to transform a continuous dynamical system into a discrete map. It is shown that for a large class of systems this transformation can be carried out in the analytical way. The stroboscopic map can be then analyzed by standard methods of discrete dynamical systems or easily simulated on the computer. It should be stressed that the formalism presented above is not limited to the case of equidistributed impulses (as the standard stroboscopic maps approach), but can also be applied to the case when the interval between impulses is a dynamical variable and changes in time.

This work was supported by the Polish KBN Grant 2.0387.91.01. The support of the Forschungszentrum Jülich GmbH (Germany) and the University of Cambridge (England) is also acknowledged. Finally, I would like to express my gratitude to the referees of the earlier versions of the paper, and to Andrzej Fuliński and Tamás Tél for their comments and suggestions.

REFERENCES

[1] V. Lakshmikautham, D.D. Bainov, P.S. Simeonov, *Series in Modern Applied Mathematics* vol. 6, World Scientific, Singapore, New Jersey, London, Hong Kong 1989.
[2] A. Fuliński, *Acta Phys. Pol.*, **A74**, 193 (1988).
[3] A. Fuliński, *Acta Phys. Pol.*, **A75**, 785 (1989).
[4] A. Fuliński, *Physica* **A178**, 493 (1991).
[5] A. Fuliński, L. Kozarzewski, *Chaos, Solitons & Fractals* **2**, 175 (1992).
[6] K. Życzkowski, J. Zakrzewski, *J. Phys. A* **21**, L371 (1988).
[7] K. Życzkowski, *Phys. Rev.* **A35**, 3546 (1987).
[8] A. Carnagie, *J. Phys. B* **17**, 3435 (1984).
[9] J.G. Leopold, D. Richards, *J. Phys. B* **18**, 3369 (1985).
[10] R. Blümel, R. Meir, *J. Phys. B* **18**, 2835 (1985).
[11] E. Nelson, *Dynamical Theories of Brownian Motion*, Princeton University Press, Princeton, NJ 1967.
[12] N.G. van Kampen, *Stochastic Processes in Physics and Chemistry*, North Holland, Amsterdam, New York, Oxford 1981.
[13] C. Gardiner, *Handbook of Stochastic Methods*, Springer Verlag, Berlin, Heidelberg, New York, Tokyo 1983; and references therein.
[14] H. Haken, *Synergetics: An Introduction*, Springer Series in Synergetics. Springer Verlag, Berlin, Heidelberg, New York, Tokyo 1984.
[15] H.G. Schuster, *Deterministic Chaos. An Introduction*, Physik Verlag, Weinheim 1984; and references therein.
[16] R.S. McKay, J.D. Meiss, I.C. Percifal, *Physica*, **D13**, 55 (1984).
[17] D. Bensimon, L.P. Kadanoff, *Physica* **D13**, 82 (1984).
[18] I. Dana, S. Fishman, *Physica* **D17**, 63 (1985).
[19] E. Gudowska-Nowak, A. Kleczkowski, G. O. Williams, *J. Stat. Phys.* **54**, 539 (1989).
[20] A. Fuliński, E. Gudowska-Nowak, *Acta Phys. Pol.* **B22**, 457 (1991).
[21] E. Gudowska-Nowak, P.P. Szczęsny, *Acta Phys. Pol.* **B23**, 3 (1992).
[22] B.-L. Hao, S.Y. Zhang, *Phys. Lett.* **87A**, 267 (1982).
[23] B.-L. Hao, S.Y. Zhang, *J. Stat. Phys.* **28**, 769 (1982).
[24] A. Fuliński, A. Kleczkowski, *Phys. Scr.* **35**, 119 (1987).