THE ORIGIN OF THE SUPERTRACE SUM RULE IN THE N=4 SUPER YANG–MILLS THEORY

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The $N = 4$ Super Yang–Mills theory remains finite if mass terms for the scalars and fermion are introduced explicitly by hand provided these terms satisfy the mass-squared supertrace sum rule

$$\sum_{J=0,\frac{1}{2}} (-1)^{2J+1} M_J^2 (2J+1) = 0.$$

It is shown that the sum rule emerges from the elimination of quadratic divergences in the one-loop effective potential of the theory.

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After the proof [1] that the $N = 4$ supersymmetric Yang–Mills theory is finite to all orders in perturbation theory, it was demonstrated that the theory also remains finite if mass terms for the scalars and fermion are introduced explicitly by hand [2]. These explicit mass terms come in several different forms. One curious feature of these mass terms is that they all satisfy the mass squared supertrace sum rule [2]

$$\sum_{J=0,\frac{1}{2}} (-1)^{2J+1} (2J+1) M_J^2 = 0\,. \tag{1}$$

The resulting potential, in the presence of a certain class of such mass terms, is bounded from below [3] and after spontaneous symmetry breaking the sum rule Eq. (1) also includes the gauge boson sector, *i.e.* $J = 0, {}^1/_2, 1$. In this note we demonstrate that the mass squared sum rule, which the particles satisfy before and after spontaneous symmetry breaking, is a consequence of eliminating the quadratic divergences encountered in the one-loop effective potential of the theory.

The Lagrangian in components and with explicit mass terms for the scalars and fermions of the theory is given by

$$
\begin{aligned}
L = 2\,\mathrm{Tr}\Big[&-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\Psi}^{\alpha} i \nabla\!\!\!\!/\, \Psi_\alpha + \tfrac{1}{4}\nabla^\mu \overline{H}^{\alpha\beta}\nabla_\mu H_{\alpha\beta} \\
&- \frac{ig}{\sqrt{2}}\overline{H}^{\alpha\beta}[\Psi_\alpha^T c^{-1}, \Psi_\beta] + \mathrm{h.c.} - \tfrac{1}{2} m\eta_\alpha \Psi_\alpha^T c^{-1}\Psi_\alpha + \mathrm{h.c.} \\
&- \tfrac{1}{2}\left(m_{\alpha\beta}^2 + \overline{m}_{\alpha\beta}^2\right)\left(H_{\alpha\beta}^{\mathrm{R}}\right)^2 - \tfrac{1}{2}\left(m_{\alpha\beta}^2 - \overline{m}_{\alpha\beta}^2\right)\left(H_{\alpha\beta}^{\mathrm{I}}\right)^2 \\
&+ \sqrt{2} ig \sum_\alpha m\eta_\alpha \epsilon^{\alpha\beta\gamma\delta} H_{\beta\alpha}[H_{\gamma\alpha}, H_{\delta\alpha}] + \mathrm{h.c.} \\
&- \frac{g^2}{16}[\overline{H}^{\alpha\beta}, \overline{H}^{\gamma\delta}][H_{\alpha\beta}, H_{\delta\gamma}]\Big]\,,
\end{aligned}
\tag{2}
$$

where $H_{\alpha\beta} = {}^1\!/_2\epsilon_{\alpha\beta\gamma\delta}H^*_{\gamma\delta}$; $\alpha, \beta = 1, 2, 3, 4$; $H^{\mathrm{R}}_{\alpha\beta} = \sqrt{2}\,\mathrm{Re}\, H_{\alpha\beta}$ and $H^{\mathrm{I}}_{\alpha\beta} = \sqrt{2}\,\mathrm{Im}\, H_{\alpha\beta}$. The covariant derivatives are defined as

$$
\nabla_\mu H_{\alpha\beta} = \partial_\mu H_{\alpha\beta} - ig[A_\mu, H_{\alpha\beta}]\,, \tag{3}
$$

$$
\nabla\!\!\!\!/\,\Psi_\alpha = \gamma^\mu\left(\partial_\mu \Psi_\alpha - ig[A_\mu, \Psi_\alpha]\right). \tag{4}
$$

In what follows we take for illustrative purpose the internal symmetry gauge group to be SU(2). Also the particle masses are taken to be equal. Thus each fermion has a mass $m_\eta = \sqrt{3}M_0$ and each scalar has a mass $m_s = 2M_0$ leading to the relation

$$
m_s = \frac{2}{\sqrt{3}} m_\eta \tag{5}
$$

and this clearly satisfies the sum rule Eq. (1). Let $H_{j4} = (1/\sqrt{2})(A_j + iB_j)$ $j = 1, 2, 3$. The potential of the theory in terms of the real scalar (A_i) and the pseudoscalar (B_i) fields is

$$
\begin{aligned}
&V(A,B) = \\
&2\,\mathrm{Tr}\Big[\tfrac{1}{2}m_s^2\left(A_1^2 + A_2^2 + A_3^2 + B_1^2 + B_2^2 + B_3^2\right) + i4gm_\eta A_1[A_2, A_3] \\
&+ \frac{g^2}{2}\Big\{[A_1, B_1]^2 + [A_2, B_2]^2 + [A_3, B_3]^2 + [A_1, A_2]^2 \\
&+ [A_1, A_3]^2 + [A_2, A_3]^2 + [B_1, B_2]^2 + [B_1, B_3]^2 + [B_2, B_3]^2 \\
&+ [A_1, B_2]^2 + [A_1, B_3]^2 + [A_2, B_1]^2 + [A_1, B_3]^2 + [A_3, B_1]^2 + [A_3, B_2]^2\Big\}\Big]\,,
\end{aligned}
\tag{6}
$$

where

$$
A_i = A_i^{(1)}\frac{\sigma^1}{2} + A_i^{(2)}\frac{\sigma^2}{2} + A_i^{(3)}\frac{\sigma^3}{2}\,, \qquad [\sigma^\gamma, \sigma^s] = 2i\epsilon^{\gamma st}\sigma^t
$$

and $\mathrm{Tr}\,\sigma^{\gamma}\sigma^{s} = 2\delta^{\gamma s}$. Indices in parenthesis are group indices. The gauge symmetry is broken spontaneously by choosing the following vacuum expectation values for the scalar fields

$$\langle A_1^{(1)}\rangle = a\,\frac{\sigma^1}{2}\,,$$
$$\langle A_2^{(2)}\rangle = b\,\frac{\sigma^2}{2}\,,$$
$$\langle A_3^{(3)}\rangle = c\,\frac{\sigma^3}{2}\,. \qquad (7)$$

These vacuum expectation values are solutions of the following equations that minimize the potential $V(A,B)$ of Eq. (6):

$$m_s^2 a - 4gm_\eta bc + ag^2(b^2+c^2) = 0\,, \qquad (8)$$
$$m_s^2 b - 4gm_\eta ac + g^2 b(a^2+c^2) = 0\,, \qquad (9)$$
$$m_s^2 c - 4gm_\eta ab + g^2 c(a^2+b^2) = 0\,. \qquad (10)$$

In the absence of the cubic term, the only vacuum expectation values which give a local minimum of the potential are $\langle A_j^{(i)}\rangle = \langle B_j^{(i)}\rangle = 0$, $i,j = 1,2,3$. The resulting mass terms contributing to the Lagrangian are L_{VB}, L_{S} and L_{F} for vector bosons, scalars and fermions where

$$L_{\mathrm{VB}} = \tfrac{1}{2}g^2(b^2+c^2)w_1^2 + \tfrac{1}{2}g^2(a^2+c^2)w_2^2 + \tfrac{1}{2}g^2(a^2+b^2)w_3^2\,, \qquad (11)$$

$$\begin{aligned} L_{\mathrm{S}} = {} & \tfrac{1}{2}(m_s^2+g^2b^2+g^2c^2)\big[A_1^{(1)}A_1^{(1)} + B_1^{(1)}B_1^{(1)} + B_2^{(2)}B_2^{(2)} + B_3^{(3)}B_3^{(3)}\big] \\ & + \tfrac{1}{2}(m_s^2+g^2a^2+g^2c^2)\big[A_2^{(2)}A_2^{(2)} + B_1^{(2)}B_1^{(2)} + B_2^{(2)}B_2^{(2)} + B_3^{(2)}B_3^{(2)}\big] \\ & + \tfrac{1}{2}(m_s^2+g^2a^2+g^2b^2)\big[A_3^{(3)}A_3^{(3)} + B_1^{(3)}B_1^{(3)} + B_2^{(3)}B_2^{(3)} + B_3^{(3)}B_3^{(3)}\big] \\ & + \tfrac{1}{2}(m_s^2+g^2a^2)\big[A_2^{(3)}A_2^{(3)} + A_3^{(2)}A_3^{(2)}\big] \\ & + \tfrac{1}{2}(m_s^2+g^2b^2)\big[A_1^{(3)}A_1^{(3)} + A_3^{(1)}A_3^{(1)}\big] \\ & + \tfrac{1}{2}(m_s^2+g^2c^2)\big[A_1^{(2)}A_1^{(2)} + A_2^{(1)}A_2^{(1)}\big] \\ & - g^2ab\big[A_1^{(2)}A_2^{(1)}\big] - g^2ac\big[A_1^{(3)}A_3^{(1)}\big] - g^2bc\big[A_2^{(3)}A_3^{(2)}\big]\,. \end{aligned} \qquad (12)$$

$$\begin{aligned} L_{\mathrm{F}} = {} & \tfrac{1}{2}m_\eta \Psi_\alpha^{(i)T} c^{-1}\Psi_\alpha^{(i)} + ag\epsilon^{1jk}\left[\Psi_1^{(j)T}c^{-1}\Psi_4^{(k)} + \Psi_2^{(j)T}c^{-1}\Psi_3^{(k)}\right] \\ & + bg\epsilon^{2jk}\left[\Psi_2^{(j)T}c^{-1}\Psi_4^{(k)} + \Psi_3^{(j)T}c^{-1}\Psi_1^{(k)}\right] \\ & + cg\epsilon^{3jk}\left[\Psi_3^{(j)T}c^{-1}\Psi_4^{(k)} + \Psi_1^{(j)T}c^{-1}\Psi_2^{(k)}\right]\,, \end{aligned} \qquad (13)$$

$$\alpha = 1,2,3,4;\quad i,j,k, = 1,2,3\,,\quad \epsilon^{123} = +1\,.$$

A remarkable result is that after spontaneous symmetry breaking the eigenvalues of the mass matrices from L_{VB}, L_{S} and L_{F} still satisfy the supertrace mass squared sum rule

$$\sum_{J=0,\frac{1}{2},1} (-1)^{2J+1}(2J+1)\,\mathrm{Tr}\,M_J^2 = 0\,, \tag{14}$$

where $\mathrm{Tr}\,M_J^2$ represents the trace of the mass squared matrix of spin J particles: From Eqs (11), (12) and (13),

$$\mathrm{Tr}\,M_0^2 = 18m_s^2 + 10g^2(a^2+b^2+c^2)\,, \tag{15}$$

$$\mathrm{Tr}\,M_{\frac{1}{2}}M_{\frac{1}{2}}^{\dagger} = 12M_\eta^2 + 8g^2(a^2+b^2+c^2)\,, \tag{16}$$

$$\mathrm{Tr}\,M_1^2 = 2g^2(a^2+b^2+c^2) \tag{17}$$

and these clearly satisfy Eq. (14) after using the relation between scalar boson and fermion masses Eq. (5).

We now compute the one-loop effective potential. Let us denote the eigenmodes of the mass matrices L_{VB}, L_{S}, L_{F} by $M_1^{2(\lambda)}$, $M_0^{2(\mu)}$, $M_{1/2}^{(\sigma)}$; $\lambda = 1,2,3$; $\mu = 1,2,\ldots,18$; $\sigma = 1,2,\ldots,12$. The one-loop effective potential including the zero point energy is given by

$$V_1 = -\tfrac{1}{2} \sum_{J=0,\frac{1}{2},1} (-1)^{2J+1} N_J \int \frac{d^3|k|}{(2\pi)^3}\sqrt{|k|^2 + M_J^{2(\delta)}}\,, \tag{18}$$

where $M_J^{2(\delta)} = M_1^{2(\lambda)}, M_0^{2(\mu)}, M_{1/2}^{2(\sigma)}$; $|k| = \sqrt{k_1^2+k_2^2+k_3^2}$ and N_J is the number of degrees of freedom of the particle with spin J. The effective potential exhibits three divergences in terms of the momentum cut-off Λ. There are proportional to

$$\Lambda^4 \sum_{J=0,\frac{1}{2},1} (-1)^{2J+1} N_J\,, \tag{19}$$

$$\Lambda^2 \sum_{J=0,\frac{1}{2},1} (-1)^{2J+1}(2J+1) M_J^2\,, \tag{20}$$

$$\ln \Lambda^2 \sum_{J=0,\frac{1}{2},1} (-1)^{2J+1}(2J+1)(M_J^2)^2\,. \tag{21}$$

The degrees of freedom N_J in Eq. (19) are $N_1 = 2$, $N_0 = 1$, $N_{1/2} = 2$ since this divergence is associated with massless particles. The absence of divergence proportional to Λ^4 requires the bosonic and fermionic degrees of freedom to be equal. This divergence is absent since the $N = 4$ Yang–Mills theory is supersymmetric. The absence of the divergence proportional to Λ^2 Eq. (20) requires the mass-squared supertrace sum rule Eq. (14). Since the particle masses satisfy this sum rule, the quadratic divergence is also absent from the one-loop effective potential. Finally the divergence proportional to $\ln \Lambda^2$ induces shifts in the vacuum energy of the theory. There are indications that this divergence is gauge dependent. Hence, its absence does not imply a new quantic mass sum rule.

The one-loop radiatively corrected potential $V' = V + V_{(1)}$ will again induce further spontaneous symmetry breaking provided there are solutions to the equations

$$\frac{\partial V'}{\partial a} = 0\,, \quad \frac{\partial V'}{\partial b} = 0\,, \quad \frac{\partial V'}{\partial c} = 0\,.$$

In terms of these new solutions, which we denote by a', b', c', the resulting mass spectrum of the particles will again satisfy the sum rule of Eq. (14) with $M_J(a, b, c)$, replaced by $M_J(a', b', c')$.

In conclusion, the $N = 4$ Super Yang–Mills theory remains finite if mass terms for the scalars and fermions are explicitly introduced into the theory. These mass terms are required to satisfy the supertrace sum rule. The supertrace sum rule is shown to emerge from the elimination of quadratic divergences in the one-loop effective potential of the theory. The sum rule is shown to be satisfied after spontaneous symmetry breaking and also in the presence of radiative corrections.

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