ALGEBRAIC SPINOR SPACES IN THE CLIFFORD ALGEBRAS OF MINKOWSKI SPACES*

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Algebraic spinor spaces in the Clifford algebras of two- and four-dimensional Minkowski spaces are considered. Their description in terms of primitive idempotens and their classification with respect to the action of the Lorentz group are given.

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1. Introduction

According to the standard approach, a spinor space is introduced as an irreducible representation space of the Clifford algebra or of the group Pin related to that algebra. In a slightly different approach, proposed in the mathematical literature by C. Chevalley [1] and in the physical literature by F. Sauter [2] and A. Sommerfeld [3], spinors are regarded as elements of the Clifford algebra itself. More precisely, an *algebraic spinor space* is introduced as a minimal left ideal (MLI) in the Clifford algebra and the multiplication from the left provides an irreducible representation.

As emphasized by Graf [4], it is possible to consider, in contrast to ordinary spinor spaces, a non-trivial equivalence relation between various algebraic spinor spaces contained in the same Clifford algebra. Namely, two algebraic spinor spaces are equivalent if they can be mapped one onto the other by an orthogonal transformation prolongated to the Clifford algebra (and sometimes called in this case the Clifford automorphism [4]). Nonequivalent MLIs could in principle correspond to different fermions within a multiplet [5, 6].

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In [6], I considered all spinorial (*i.e.* primitive) idempotents in the Clifford algebras of two- and four-dimensional Minkowski spaces; their detailed classification, taking into account the action of the Lorentz group, was given. The aim of the present paper is to give a classification of MLIs in the Clifford algebras of these Minkowski spaces taking the classification of [5] as the starting point. A formula (n) of [6] will be quoted as (nK) and Sec. N of [6] as Sec. NK.

Sec. 2 contains a general description of tools needed to consider idempotents generating the same MLI. In Sec. 3, I consider MLIs in two-dimensional complexified Minkowski space. They appear to be in a one–to–one correspondence with points of a 2-dimensional sphere. The discussion of orbits of the Lorentz group exhibits a distinguished role of poles of the sphere. These pole-type MLIs can be generalized to all complex, complexified or real with the neutral signature Clifford algebras; then they will correspond to maximal totally null subspaces and to simple spinors. Sec. 4 is devoted to MLIs in the four-dimensional Minkowski space. In Secs 4.1. and 4.2., the two real cases corresponding to signatures (3,1) and (1,3), respectively, are considered. For the signature (3,1), the situation is particularly simple, since there are only two orbits of the special ortochronous Lorentz group acting on the space of all MLIs; both have the same topology. For the signature (1,3), the situation is more involved; there exist here, like in the 2-dimensional case, two types of orbits of different topology; the lower dimensional orbit is single, unlike in the 2-dimensional case. In Sec. 4.3., the complexified case is considered. The space of orbits of the special ortochronous Lorentz group acting on the space of all MLIs consists of three types of orbits. These types correspond to three types of totally null subspaces of the complexified Minkowski space: 1-dimensional real, 1-dimensional complex and 2-dimensional maximal.

2. Notation and some remarks about idempotents

I shall use the following notation:
$\vee$ — Clifford product,
$\wedge$ — exterior product, $u \wedge v = \frac{1}{2}(u \vee v - v \vee u)$,
$\cdot$ — scalar product, $(u_1 \wedge \ldots \wedge u_p) \cdot (v_1 \wedge \ldots \wedge v_p) = \det(u_i \cdot v_j)$.
The convention for the Hodge dual is

$$*\omega = \omega \vee \eta, \quad \eta = \text{volume element}. \tag{1}$$

For the 4-dimensional Minkowski space I have $**\omega = -\omega$ regardless of the degree of the multivector ω.

For a decomposable bivector $f = u \wedge v$ and a vector w, I define

$$f \lfloor w = u(v \cdot w) - v(u \cdot u),$$

and extend the above operation by linearity to an arbitrary bivector f. Then I have

$$w \vee f = w \wedge f - f \lfloor w, \qquad f \vee w = f \wedge u + f \lfloor w. \tag{2}$$

For a decomposable bivector $\varphi = \alpha \wedge \beta$, I can define the bivector

$$f \lfloor \varphi = (f \lfloor \alpha) \wedge \beta - (f \lfloor \beta) \wedge \alpha = -\varphi \lfloor f,$$

and extend this operation to an arbitrary bivector φ. For two bivectors, I get

$$\varphi \vee f = \varphi \wedge f - f \lfloor \varphi - \varphi \cdot f. \tag{3}$$

The formulas (1)–(3) are used for calculations of Clifford squares in Sec. 4.

Let p and p' be two idempotents, *i.e.* $p^2 = p$ and $p'^2 = p$. Let moreover

$$p' \vee p = p', \tag{4}$$

i.e. p' belongs to the MLI $\mathcal{S}_p$ generated by p. For any element c of the Clifford algebra I have $c \vee p' \vee p = c \vee p'$, therefore $c \vee p' \in \mathcal{S}_p$. This means that $\mathcal{S}_{p'} \subset \mathcal{S}_p$. If p is moreover primitive and $p' \neq 0$, then $\mathcal{S}_{p'} = \mathcal{S}_p$. In this case p' is also primitive and $p \vee p' = p$.

Let $q = p' - p$, where p and p' are idempotents such that $\mathcal{S}_{p'} = \mathcal{S}_p$. Then

$$q^2 = 0, \tag{5}$$

thus q is nilpotent, and moreover

$$q \vee p - p \vee q = q. \tag{6}$$

In the opposite direction: let p be a primitive idempotent and let q satisfy (5) and (6). Then, multiplying (6) from the left and from the right by p, I get $p \vee q \vee p - p \vee q = p \vee q$ and $p \vee q \vee p = 0$. Therefore $p \vee q = 0$ and $q \vee p = q$. Defining

$$p' = p + q,$$

I get $p'^2 = p$ and $p' \vee p = p'$. Therefore p' is a primitive idempotent such that $\mathcal{S}_{p'} = \mathcal{S}_p$.

The method of finding idempotents generating the same MLI as p based on (5) and (6) appeared to be more efficient than the method based on (4) in the 4-dimensional case (Sec. 4).

3. Minimal left ideals in two-dimensional Minkowski Space

The complexified Clifford algebra will be considered only, since the real case can be easily deduced from the complexified case.

Following Sec. IIIK, if (k, l) is a null basis of the 2-dimensional Minkowski space, $k^2 = l^2 = 0$, $k \cdot l = {}^1\!/_2$, then any primitive idempotent has the form

$$p = \tfrac{1}{2} + bk + cl + dk \wedge l, \qquad bc = \tfrac{1}{4}(1 - d^2). \tag{7}$$

If

$$p' = \tfrac{1}{2} + b'k + c'l + d'k \wedge l, \qquad b'c' = \tfrac{1}{4}(1 - d'^2)$$

is another primitive idempotent, then (4) gives

$$2b'c + 2c'b + d'd = 1\,, \tag{8}$$

$$(1 + d)b' - d'b = b\,, \tag{9}$$

$$(1 - d)c' + d'c = c\,, \tag{10}$$

$$-2b'c + 2c'b + d' = d\,. \tag{11}$$

If p and p' belong to the continuous type (*cf.* Sec. IIIK), $d \neq \pm 1$ and $d' \neq \pm 1$, then (9) and (10) give:

$$\frac{b'}{1 + d'} = \frac{b}{1 + d}\,, \tag{12}$$

$$\frac{c'}{1 - d'} = \frac{c}{1 - d}\,. \tag{13}$$

If $d = \pm 1$ or $d' = \pm 1$ then at least one of the relations (12) and (13) holds. In any case (12) and (13) imply (8) and (11). The complex numbers of the compactified plane,

$$e^{i\phi} \cot\frac{\theta}{2} := \frac{2b}{1 + d} = \frac{1 - d}{2c} \tag{14}$$

are in one–to–one correspondence with MLIs. They are described in (14) by the standard spherical coordinates θ, ϕ.

If $\theta \neq 0, \pi$ then the idempotent (7) belongs either to the continuous type ($d \neq \pm 1$) and is arbitrary within this type, or to the bridge-type ($d = \pm 1$ and b, c do not vanish simultaneously) and is not arbitrary: $d = 1$ and $c = 0$ or $d = -1$ and $b = 0$. The poles on the sphere are distinguished. If $\theta = \pi$ then the idempotent (7) is such that $d = 1$ and $b = 0$, whereas c is arbitrary; it can belong either to the bridge-type ($c \neq 0$) or to the multicross-type ($c = 0$). If $\theta = 0$ then the idempotent (7) is such that $d = -1$ and $c = 0$, whereas b is arbitrary; it can belong either to the bridge-type ($b \neq 0$) or to the multicross-type ($b = 0$).

The orbits of the 2-dimensional special ortochronous Lorentz group $SO^{\uparrow}(1,1)$ acting on the space of MLIs can be described as follows. The azimuthal angle $\phi \bmod 2\pi$ is invariant and the action is transitive on any open semicircle $\phi = \text{const}$. The poles $\theta = 0$ and $\theta = \pi$ form separate orbits consisting of single MLIs. Under the action of the full Lorentz group $O(1,1)$, the azimuthal angle is invariant $\operatorname{mod} \pi/2$ only and the poles form one orbit.

4. Minimal left ideals in four-dimensional Minkowski Space

Let q be an element of the Clifford algebra,

$$q = \alpha + \mu + \varphi + \nu \vee \eta + \beta\eta\,,$$

where α is a scalar, μ a vector, φ a bivector, ν a pseudovector and β a pseudoscalar. Eq. (5) leads to $\alpha = 0$ and to

$$-\beta^2 + \mu^2 + \nu^2 - \varphi^2 = 0\,, \tag{15}$$

$$\varphi \wedge \nu = 0\,, \tag{16}$$

$$\beta\varphi + \mu \wedge \nu = 0\,, \tag{17}$$

$$\varphi \wedge \mu = 0\,, \tag{18}$$

$$\varphi \cdot *\varphi = 0\,. \tag{19}$$

If $\beta \neq 0$ then (17) determines φ, whereas (16), (18) and (19) are identically satisfied. Eq. (15) gives then the biquadratic equation for β, assuming μ and ν are given:

$$\beta^4 - (\mu^2 + \nu^2)\beta^2 + \mu^2\nu^2 - (\mu \cdot \nu)^2 = 0\,. \tag{20}$$

If $\beta = 0$ then $\dim \operatorname{span}\{\mu, \nu\} \leq 1$. If this dimension is 1 then I get

$$\varphi \wedge \mu = \varphi \wedge \nu = 0\,, \quad \varphi^2 = \mu^2 + \nu^2\,. \tag{21}$$

If this dimension is 0 then I get

$$\varphi^2 = \varphi \cdot *\varphi = 0\,. \tag{22}$$

Let

$$p = \tfrac{1}{2} + u + f + v \vee \eta + b\eta$$

be one of primitive idempotents described in Sec. 4K. Eq. (6) leads to

$$-b\nu + \beta v - f \lfloor \mu + \varphi \lfloor u = \frac{\mu}{2}\,, \tag{23}$$

$$\mu \wedge u + \nu \wedge v + \varphi \lfloor f = \frac{\varphi}{2}\,, \tag{24}$$

$$b\mu - \beta u + \varphi \lfloor v - f \lfloor \nu = \frac{\nu}{2}, \tag{25}$$

$$\mu \cdot v - \nu \cdot u = \frac{\beta}{2}. \tag{26}$$

Assuming $\beta \neq 0$ and using (17), I can eliminate φ; then (23) and (25) give:

$$f \lfloor \mu = \left(\frac{\mu \cdot u}{\beta} - b\right) \nu - \left(\frac{\nu \cdot u}{\beta} + \tfrac{1}{2}\right) \mu + \beta v, \tag{27}$$

$$f \lfloor \nu = \left(b - \frac{\nu \cdot v}{\beta}\right) \mu + \left(\frac{\mu \cdot v}{\beta} - \tfrac{1}{2}\right) \nu - \beta u. \tag{28}$$

Eq. (24) is identically satisfied. It remains to solve (26), (27), (28) and (20).

4.1. Signature (3,1)

For this signature, according to (6K)–(8K),

$$f = \frac{1}{4u^2} * (u \wedge v) - \frac{b}{u^2} u \wedge v \tag{29}$$

and

$$u^2 = v^2 = b^2 + 1/16, \quad u \cdot v = 0.$$

Following the conventions of Sec. 1, I have

$$*(u \wedge v) \lfloor \mu = - * (u \wedge v \wedge \mu). \tag{30}$$

Substituting (29) and (30) into (27) and (28), I get

$$- \frac{1}{4u^2} * (u \wedge v \wedge \mu) - \frac{b}{u^2} (\mu \cdot v) u + \frac{b}{u^2} (\mu \cdot u) v = \left(\frac{\mu \cdot u}{\beta} - b\right) \nu - \left(\frac{\nu \cdot u}{\beta} + \tfrac{1}{2}\right) \mu + \beta v, \tag{31}$$

$$- \frac{1}{4u^2} * (u \wedge v \wedge \nu) - \frac{b}{u^2} (\nu \cdot v) u + \frac{b}{u^2} (\nu \cdot u) v = \left(b - \frac{\nu \cdot v}{\beta}\right) \mu + \left(\frac{\mu \cdot v}{\beta} - \tfrac{1}{2}\right) \nu - \beta u. \tag{32}$$

Multiplying (31) and (32) by u and v, I get

$$\mu \cdot u = \nu \cdot v = b\beta \tag{33}$$

and $(\mu \cdot v)^2 = (\nu \cdot u)^2 = \beta^2/16$. Eqs. (33) are consistent with (26) if and only if

$$\mu \cdot v = -\nu \cdot u = \frac{\beta}{4}. \tag{34}$$

Substituting (33) and (34) into (31) and (32), I get

$$*(u \wedge v \wedge \mu) = u^2 \mu - \beta b u - \frac{\beta v}{4}, \tag{35}$$

$$*(u \wedge v \wedge \nu) = u^2 \nu - \beta b v + \frac{\beta u}{4}. \tag{36}$$

The spatial vectors u and v determine uniquely the null direction dir κ such that

$$*(u \wedge v \wedge \kappa) = u^2 \kappa. \tag{37}$$

All solutions of (35) and (36) have the form

$$\mu = \frac{\beta}{u^2}\,(bu + \frac{v}{4}) + c\kappa,$$

$$\nu = \frac{\beta}{u^2}\,(bv - \frac{u}{4}) + d\kappa,$$

where c and d are arbitrary real numbers. The pseudoscalar β is also arbitrary since (20) is identically satisfied.

The relation between two primitive idempotents p and p' which generate the same MLI becomes

$$u' = \frac{bb' + \frac{1}{16}}{b^2 + \frac{1}{16}}\,u + \frac{b' - b}{b^2 + \frac{1}{16}}\,\frac{v}{4} + c\kappa, \tag{38}$$

$$v' = \frac{bb' + \frac{1}{16}}{b^2 + \frac{1}{16}}\,v - \frac{b' - b}{b^2 + \frac{1}{16}}\,\frac{u}{4} + d\kappa. \tag{39}$$

Therefore, each MLI can be described by means of a pair of vectors (u_0, v_0) such that

$$u_0^2 = v_0^2 = \tfrac{1}{16}, \qquad u_0 \cdot v_0 = 0. \tag{40}$$

Any idempotent p which generates the same MLI as the pair (40) is given by

$$u = u_0 + 4bv_0 + c\kappa, \tag{41}$$

$$v = v_0 - 4bu_0 + d\kappa, \tag{42}$$

where b, c and d are arbitrary and κ is determined by (37). In particular, the MLI determines the pair (40) up to the equivalence relation

$$(u_0, v_0) \sim (u_0 + c\kappa, v_0 + d\kappa). \tag{43}$$

The space of all MLIs is homeomorphic to the space of all pairs

$$(\operatorname{dir}\kappa, [(4u_0, 4v_0)]),$$

where $\operatorname{dir}\kappa$ is a null direction and $[(4u_0, 4v_0)]$ is an orthonormal basis in the screen space $(\operatorname{span}\kappa)^{\perp}/\operatorname{span}\kappa$. There are two orbits of the special orthochronous Lorentz group $SO^{\uparrow}(1,3)$ acting on the space of all MLIs. They correspond to two orientations of bases $[(4u_0, 4v_0)]$. The topology of each orbit is $\mathbf{S}_2 \times \mathbf{S}_1$.

4.2. Signature (1,3)

Types II_T^+, II_T^-, $I_{T,\mathrm{ext}}^+$, $I_{T,\mathrm{ext}}^-$, $I_{T,\mathrm{circle}}$, $I_{T,\mathrm{int}}$, I_S *and* $I_N^{\downarrow}$. Assume initially that the idempotent p belongs to the types II_T or $I_{T,\mathrm{ext}}$ (*cf.* Sec. IVBK), *i.e.* $b \neq 0$ and

$$f = -\frac{1}{b} u \wedge v. \tag{44}$$

Substituting (44) into (27) and (28), I get

$$\left(\frac{\mu \cdot u}{\beta} - b\right) \nu - \left(\frac{\nu \cdot u}{\beta} + \tfrac{1}{2}\right) \mu = \left(\frac{\mu \cdot u}{b} - \beta\right) v - \frac{\mu \cdot v}{b} u,$$

$$\left(b - \frac{\nu \cdot v}{\beta}\right) \mu + \left(\frac{\mu \cdot v}{\beta} - \tfrac{1}{2}\right) \nu = \left(\beta - \frac{\nu \cdot v}{b}\right) u + \frac{\nu \cdot u}{b} v.$$

The general solution of the above equations is

$$\mu = \frac{\beta}{b} u + n \cos\psi, \tag{45}$$

$$\nu = \frac{\beta}{b} v + n \sin\psi, \tag{46}$$

and (26) is satisfied if

$$\beta = 2n \cdot (v \cos\psi - u \sin\psi).$$

Eq. (20) is satisfied identically. The vector n in (45) and (46) is arbitrary, whereas the angle ψ is constrained by

$$\cot\psi = \frac{u^2 - b^2}{u \cdot v + \frac{b}{2}} = \frac{u \cdot v - \frac{b}{2}}{v^2 - b^2}. \tag{47}$$

The last equality is equivalent to (11K).

Any idempotent p' such that $\mathcal{S}_{p'} = \mathcal{S}_p$ is determined by

$$u' = \frac{b'}{b}\, u + n \cos\psi\,,$$

$$v' = \frac{b'}{b}\, v + n \sin\psi\,.$$

The vector

$$w = \frac{v\cos\psi - u\sin\psi}{b} \tag{48}$$

and $\cot\psi$ are invariant under the change of idempotent generating the same MLI, $w' = w$ and $\cot\psi' = \cot\psi$. The vector w is timelike and normalized, $w^2 = 1$, and I can choose the convention that w is past oriented and the angle ψ is determined by (47) and (48) $\bmod 2\pi$. Each MLI which contains idempotents of the types II_T and $\mathrm{I}_{T,\mathrm{ext}}$ is determined by the pair (w, ψ).

If the arbitrary vector n is given then I have the relation

$$b' = b(1 + 2n \cdot w)\,.$$

Each value of b' can be achieved from $b \neq 0$ including $b' = 0$ which gives

$$n \cdot w = -\tfrac{1}{2}\,.$$

The general solution of this equation is

$$n = n_\perp - \tfrac{w}{2}\,,$$

where $n_\perp$ is an arbitrary vector orthogonal to w. The coordinates t', r' and ϕ' of Sec. 4BK for the idempotent p' are given by

$$t' = n_\perp^2 + \tfrac{1}{4}\,,$$

$$r' e^{i\phi'} = (n_\perp^2 + \tfrac{1}{4})e^{2i\psi}\,.$$

Thus for $n_\perp = 0$ I get an arbitrary idempotent of the type $\mathrm{I}_{T,\mathrm{circle}}$, for $0 > n_\perp^2 > -{}^1\!/_4$ an arbitrary idempotent of the type $\mathrm{I}_{T,\mathrm{int}}$ and for $n_\perp^2 < -{}^1\!/_4$ an arbitrary idempotent of the type I_S.

If $n_\perp^2 = -{}^1\!/_4$ the angle ϕ' is undetermined; in this case,

$$u' + v' = e^{i\psi}\, n$$

and the complex coordinate z of Sec. IVBK is given by $z = e^{i\psi}$. The transformation law for the bivector f is

$$f' = -\frac{1}{b'} u' \wedge v' = \frac{b'}{b}\, f + n \wedge x\,.$$

For $b' = 0$ it gives

$$f' = w \wedge n .$$

Thus in the case $n_\perp^2 = -1/4$, I get an arbitrary idempotent of the type $\mathrm{I}_N^\downarrow$. The vectors n and w of Sec. IVBK are the same as the vectors n and w of the present paper up to positive coefficients.

The space of MLIs of the above class is homeomorphic to the space of all pairs $(\operatorname{dir} w, \psi)$ such that $\operatorname{dir} w$ is a timelike direction and the angle ψ is given $\bmod 2\pi$. Its topology is $\mathbf{R}^3 \times \mathrm{S}_1$. The topology of any orbit is $\mathbf{R}^3$. The angle ψ $(\bmod 2\pi)$ is the unique invariant function under the action of the group $\mathrm{SO}^\uparrow(1,3)$. If the full Lorentz group $\mathrm{O}(1,3)$ is taken into account then the angle ψ is invariant $\bmod \pi/2$ only.

Types 0 and $I_N^\uparrow$. Let now begin with the idempotent p belonging to the type 0. Then $u = v = 0 = b$ and

$$f = w \wedge n, \quad w^2 = n^2 = 0, \quad w \cdot n = \tfrac{1}{2} .$$

I can regard w and n as future oriented, then they are determined by p up to a single factor.

Eqs. (23)–(26) lead to $b' = \beta = 0$ and to

$$2f \lfloor u' = -u' , \quad f \lfloor v' = -v' .$$

The solution is

$$u' = cn , \qquad v' = dn ,$$

where c and d are arbitrary. If c and d do not vanish simultaneously, I can gauge the vectors n and w so that $c^2 + d^2 = 1$. In this case, the last consequence of (23)–(26) reads

$$f' = f + \rho \wedge n , \tag{49}$$

where the vector ρ satisfies

$$\rho \cdot n = 0 . \tag{50}$$

Without loss of generality, I can put also

$$\rho \cdot w = 0 . \tag{51}$$

In the case $c = d = 0$, the vectors n and w can also be gauged so that (49) holds together with the conditions (50) and (51).

The bivector (49) can be written as

$$f' = w' \wedge n , \quad \text{where} \quad w' = w + \rho - \rho^2 n .$$

Any null vector w' which satisfies $w' \cdot n = {}^1/_2$ has this form. Therefore, starting from an idempotent of the type 0, I can get all idempotents of the types 0 and $\mathrm{I}_N^{\uparrow}$ with the fixed vector n.

The space of MLIs of the above class is homeomorphic to the space of all real null directions $\operatorname{dir} n$. Therefore, these MLIs constitute a single orbit of the groups $\mathrm{SO}^{\uparrow}(1,3)$ and $\mathrm{O}(1,3)$ and its topology is S_2.

4.3. Complex minimal left ideals

The considerations of Sec. 4.1. can be extended to the complexified case provided u, v, b and κ are now complex.

According to (38) and (39), each idempotent such that $b \neq \pm i/4$ can be replaced (preserving the MLI) by an idempotent such that $b = 0$. Next, each idempotent with $b = 0$ can be replaced by an idempotent with $b = \pm i/4$. The following question arises: when does an idempotent with $b = \pm i/4$ can be achieved from $b = 0$?

Let p_0 be an idempotent described by $(u_0, v_0, b = 0)$ and $p_\pm$ an idempotent described by $(u_\pm, v_\pm, b = \pm i/4)$. Then, according to (41) and (42), I get

$$u_\pm = u_0 \pm i v_0 + c\kappa \,, \tag{52}$$

$$v_\pm = v_0 \mp i u_0 + d\kappa \,. \tag{53}$$

If $u_\pm$ and $v_\pm$ are given, then κ can be extracted from

$$u_\pm \mp i v_\pm = (c \mp id)\kappa \,. \tag{54}$$

Next, I have to look for u_0 and v_0 such that (37), (40), (52) and (53) hold. Notice that the coefficients c and d in (52) and (53) are arbitrary (now complex) numbers. Consistently with the equivalence relation (43), their choice has no effect on the choice of MLI.

Throughout this Subsection, I shall use the null basis $(k, l, m, \bar{m})$ of the complexified Minkowski vector space with signature (1,3) such that: k and l are real future oriented, m is complex, the non-vanishing scalar products are

$$k \cdot l = \tfrac{1}{2}, \qquad m \cdot \bar{m} = -\tfrac{1}{2} \,,$$

and the spacetime orientation of the basis is fixed by the requirement

$$k \wedge l \wedge m \wedge \bar{m} = \frac{i}{4}\eta \,.$$

The above conventions are consistent with those of Sec. IVCK. I shall use also the real vector space

$$V = \operatorname{span}\{\operatorname{Re} u, \operatorname{Im} u, \operatorname{Re} v, \operatorname{Im} v\} \,.$$

Types: III_{N+} — *specific case* $\mu_{1+} = 0$, III_N^-, II_S, II^0_{S++}, III_{N-} — *specific case* $\mu_{2-} = 0$, II^0_{S--}. For this specific case within the type III_{N+}, according to Sec. IVCK, I have

$$u_+ = \tfrac{1}{2}e^{-i\psi}l + i\mu_{2+}\bar{m}, \tag{55}$$

$$v_+ = \tfrac{i}{2}e^{-i\psi}l + \mu_{2+}\bar{m}, \tag{56}$$

where $\mu_{2+} > 0$. For $\kappa = l$ and

$$u_0 = \frac{i}{8\mu_{2+}}m + \tfrac{i}{2}\mu_{2+}\bar{m},$$

$$v_0 = -\frac{1}{8\mu_{2+}}m + \tfrac{1}{2}\mu_{2+}\bar{m},$$

the conditions (37), (40), (52), (53) and (54) are satisfied. From (41) and (42), for an arbitrary b, I get

$$u = -\frac{b - \frac{i}{4}}{2\mu_{2+}}m + (b + \frac{i}{4})2\mu_{2+}\bar{m} + cl, \tag{57}$$

$$v = -i\frac{b - \frac{i}{4}}{2\mu_{2+}}m - i(b + \frac{i}{4})2\mu_{2+}\bar{m} + dl. \tag{58}$$

Expressions (57) and (58) determine all idempotents generating the same MLI as that described by (55) and (56). Notice that the idempotent p determines the vector m in (57) and (58) up to a vector proportional to the vector l and | itself is determined up to a real positive factor.

If $b \neq \pm i/4$ then the real vector space V is either 3-dimensional null (if c and d cannot be gauged away) or 2-dimensional spacelike (if they can). In the first case, I get one of the types III_N^+ or III_N^-, since only one (namely l) of two (*i.e.* k and l) null real vectors orthogonal to m appears in (57) and (58); let me call this type III_N^-. In the second case, I get the type II_S.

Suppose $b = i/4$. If $u \neq iv$ then the space V is 3-dimensional null and p remains in the type III_{N+}. If $u = iv$ then this space is 2-dimensional spacelike and I have the type II^0_{S++}.

Suppose $b = -i/4$. For $v \neq iu$ I get the type III_{N-} in the specific case $\mu_{2-} = 0$ (and $\mu_{1-} = 1/(4\mu_{2+})$). For $v = iu$ I get the type II^0_{S--}.

The space of MLIs of the above class is homeomorphic to the space of all pairs $(\mathrm{dir}\, l, \mu_{2+})$, where $\mathrm{dir}\, l$ is a real null direction and $\mu_{2+} > 0$; its topology is $\mathbf{S}_2 \times \mathbf{R}$. The topology of any orbit is $\mathbf{S}_2$. The scalar μ_{2+} is the unique invariant function under the action of both groups $\mathrm{SO}^\uparrow(1,3)$ and $\mathrm{O}(1,3)$.

Types: III_{N+} *— generic case* $\mu_{1+} > 0$, IV^+, IV^-, III_T, III_N^+, III_S, II_T, II_{S++}^1, I_{N++}, III_{N-} *— generic case* $\mu_{2-} > 0$, II_{S--}^1 *and* I_{N--}. For this generic case of the type III_{N+} (24K) gives

$$u_+ = \frac{i}{2}\mathrm{e}^{-i\psi} l + \left(\mu_{1+} - \mu_{2+}\mathrm{e}^{i\phi_2}\right)\bar{m}\,, \tag{59}$$

$$v_+ = \tfrac{1}{2}\mathrm{e}^{-i\psi} l + i\left(\mu_{1+} + \mu_{2+}\mathrm{e}^{i\phi_2}\right)\bar{m}\,. \tag{60}$$

For $\kappa = \bar{m}$ and

$$u_0 = -\tfrac{i}{4}\mathrm{e}^{i\psi} k + \tfrac{i}{4}\mathrm{e}^{-i\psi} l\,,$$
$$v_0 = \tfrac{1}{4}\mathrm{e}^{i\psi} k + \tfrac{1}{4}\mathrm{e}^{-i\psi} l\,,$$

the conditions (37), (40), (52), (53) and (54) are satisfied. For an arbitrary b, I get from (41) and (42)

$$u = (b - \tfrac{i}{4})\mathrm{e}^{i\psi} k + (b + \tfrac{i}{4})\mathrm{e}^{-i\psi} l + c\bar{m}\,, \tag{61}$$

$$v = i(b - \tfrac{i}{4})\mathrm{e}^{i\psi} k - i(b + \tfrac{i}{4})\mathrm{e}^{-i\psi} l + d\bar{m}\,. \tag{62}$$

The directions of the real null vectors k and l are uniquely determined by the idempotent p described by (61) and (62).

Suppose $b \neq \pm i/4$. The space V is either 4-dimensional (types IV^+ and IV^-), 3-dimensional spacelike (type III_T), 3-dimensional null (type III_N^+ — since the other case was exhausted earlier), 3-dimensional spacelike (type III_S) or 2-dimensional timelike (type II_T — occurs only if $c = d = 0$). In order to demonstrate the above statements in the case c or $d \neq 0$, it is convenient to shift $\bar{m}$ adding a multiple either of k or of l.

Suppose $b = i/4$. The general idempotent equivalent to that described by (59) and (60) can be written as

$$u'_+ = \tfrac{i}{2}\mathrm{e}^{-i\psi} l + c\bar{m} = \tfrac{i}{2}\mathrm{e}^{-i\psi} l + \left(\mu'_{1+} - \mu'_{2+}\mathrm{e}^{i\phi'_2}\right)\bar{m}'\,,$$
$$v'_+ = \tfrac{1}{2}\mathrm{e}^{-i\psi} l + d\bar{m} = \tfrac{1}{2}\mathrm{e}^{-i\psi} l + i\left(\mu'_{1+} + \mu'_{2+}\mathrm{e}^{i\phi'_2}\right)\bar{m}'\,,$$

where $m' \sim m$ was used. Thus for a given MLI, I can achieve arbitrary values of $\mu'_{1+} \geq 0$, $\mu'_{2+} \geq 0$ and ϕ'_2. If $\mu'_{1+} > 0$ then the idempotent remains within the generic case of the type III_{N+}. If $\mu'_{1+} = 0$ then I get $u'_+ = iv'_+$ and: if $\mu'_{2+} > 0$ then the idempotent belongs to the type II_{S++}^1, if $\mu'_{2+} = 0$ then the idempotent belongs to the type I_{N++}.

The idempotents with $b = -i/4$ equivalent to that described by (59) and (60) are given by

$$u_- = -\tfrac{i}{2}e^{i\psi}k + c\bar{m} = -\tfrac{i}{2}e^{i\psi}k + i\left(\mu_{2-} + \mu_{1-}e^{i\phi_{2-}}\right)\bar{m}',$$
$$v_- = \tfrac{1}{2}e^{i\psi}k + d\bar{m} = \tfrac{1}{2}e^{i\psi}k + \left(\mu_{2-} - \mu_{1-}e^{i\phi_{2-}}\right)\bar{m}'.$$

The restrictions on the parameters introduced above are $\mu_{2-} \geq 0$ and $\mu_{1-} \geq 0$. If $\mu_{2-} > 0$ then the idempotent remains within the generic case of the type III_{N-}. If $\mu_{2-} = 0$ then I get $v_- = iu_-$ and: if $\mu_{1-} > 0$ then the idempotent belongs to the type II^1_{S--}, if $\mu_{1-} = 0$ then the idempotent belongs to the type I_{N--}.

The space of MLIs of the above class is homeomorphic to the space of all pairs $(\operatorname{dir}\bar{m}, \psi)$, where $\operatorname{dir}\bar{m}$ is an essentially complex (*i.e.* $\bar{m} \not\sim m$) null direction and ψ is given mod 2π. The topology of this space is $\mathbf{S}_2 \times \mathbf{S}_1 \times \mathbf{R}$. The topology of any orbit is $\mathbf{S}_2 \times \mathbf{R}$; it has the highest dimension among all types considered in Sec. 4.3. The angle ψ (mod 2π) is the unique invariant function under the action of the group $\mathrm{SO}^{\uparrow}(1,3)$. The group $\mathrm{O}(1,3)$ leaves ψ invariant but mod $\pi/2$.

Types: 0_+, II^1_{S+-}, II^0_{S+-} *and* I_{N+-}. *Types:* 0_-, II^1_{S-+}, II^0_{S-+} *and* I_{N-+}. I look now for idempotents generating the same MLIs as those generated by the idempotents of the types $0_\pm$ for which

$$u = v = 0 = b, \qquad f = -\tfrac{1}{2}k \wedge l \mp \tfrac{1}{2}m \wedge \bar{m}. \tag{63}$$

I have to solve (23)–(26), remembering that $u' = \mu$, $v' = \nu$ and $b' = \beta$. Eq. (26) gives $b' = b$. The solution for u' and v' is

$$v' = \pm iu', \qquad u' = ck + \begin{cases} d\bar{m} \\ dm \end{cases},$$

where c and d are arbitrary. If $d \neq 0$, I can introduce new m',

$$|d|m' = \begin{cases} \bar{d}m + \bar{c}k \\ dm + ck \end{cases},$$

and new l',

$$|d|^2 l' = |d|^2 l + |c|^2 k + \begin{cases} c\bar{d}m + \bar{c}d\bar{m} \\ \bar{c}dm + c\bar{d}\bar{m} \end{cases}$$

so that the expression (55) for f remains untouched. The solution of (21) and (24) for φ is

$$\varphi = ek \wedge \begin{cases} \bar{m}' \\ m' \end{cases}, \tag{64}$$

where e is arbitrary. Therefore, for $e \neq 0$ I get the types II^1_{S+-} and II^1_{S-+} respectively (the consistency of (64) with (27K) one obtains after the replacements $m' \leftrightarrow \bar{m}$ and $l' \leftrightarrow k$). For $e = 0$, I get the types II^0_{S+-} and II^0_{S-+}. If $d = 0$ and $c \neq 0$ then the solution of (21) and (24) is also given by (64). For any e, I get the types I_{N+-} and I_{N-+} respectively. If $d = c = 0$ the solution of (22) and (24) is still given by (64); adding (63) and (64), I get f' with arbitrary l' and m', whereas $k' = k$ is untouched.

The MLIs of each of the above two classes are therefore in one–to–one correspondence with the real null directions dir$k$. They constitute two orbits under the action of $\mathrm{SO}^{\uparrow}(1,3)$, each one is homeomorphic to S_2, and a single orbit under the action of $\mathrm{O}(1,3)$.

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