RELATIVISTIC BOUND STATES OF ELEMENTARY PARTICLES IN LIGHT-FRONT HAMILTONIAN APPROACH TO QUANTUM FIELD THEORY*†

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This paper explains the light-front Hamiltonian approach to relativistic bound state dynamics of elementary particles in quantum field theory. The paper is an introduction to this emerging theoretical area and reports on the recent progress. A selected set of papers is reviewed. Special attention is paid to renormalization which assumes a central role in the light-front Hamiltonian formalism for quantum chromodynamics.

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That may partially be a mistake. But we must start somewhere and we have chosen to start here [99].

1. Introduction

Theory of relativistic bound states of elementary particles is considered partly for application in high energy nuclear physics and partly for the purpose of describing hadrons in quantum chromodynamics (QCD).

Our common understanding of bound state dynamics is based on the knowledge of systems like a hydrogen atom. That picture is not sufficient to understand dynamics of strongly interacting relativistic bound states. The reason is that the electromagnetic forces are weak and bound states in quantum electrodynamics (QED) are nonrelativistic. Charged constituents

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of atoms resemble very closely free particles, move slowly and interact practically only through the Coulomb potential which is instantaneous. But already in nuclear physics interactions are so strong that the relativistic effects of retardation and changing number of particles have to be seriously considered. For example, in high energy heavy ion collisions relativistic effects pose practical problems which demand solution. Still more demanding and unsolved problems appear in nucleons and other hadrons which are highly relativistic bound states of light quarks, u, d or s, and gluons. Relativistic effects are also important in hadrons built from light and heavy quarks.

The bound state problems in relativistic nuclear physics may be viewed as not quite fundamental. Since nucleons and mesons are built from quarks and gluons the theory of bound states of the nucleons and mesons should follow from the fundamental theory of their smaller constituents. However, the dynamics of hadrons is so complicated that it would be naive to expect quick and simple explanation of nuclear dynamics on the basis of elementary interactions of quarks and gluons. Many theorists consider relativistic nuclear physics to be a challenge by itself, requiring rigorous methods of describing relativistic nuclear bound states on its own.

Some aspects of relativistic dynamics of nucleons and mesons are attributed to partially releasing quarks and gluons from binding in hadrons. Sound methods of distinguishing nuclear and subnuclear effects are not yet available. Investigations of nuclear form factors at high momentum transfers, semiinclusive processes, structure functions of nuclei and a host of problems in nuclear collisions all require understanding of relativistic bound state dynamics. Smashing heavy ions with energies of hundreds GeV per nucleon in order to create quark-gluon plasma is a good example. There is a little hope for distinguishing adequate explanations of future experiments without well founded description of relativistic quark and nucleon bound states.

Construction of a theory of relativistic bound states for application in hadronic physics is essential. Every calculation for processes involving hadrons relies on a model for how hadrons are built from quarks and gluons. The primary example of such models is Feynman's parton model. The parton model replaces unknown theory of hadronic bound states, while such a theory should in principle follow from QCD. Perturbation theory based on asymptotic freedom — the basis of success of QCD — does not apply in the case of bound state problems and the theory is in a corner. It can describe interactions of free quarks and gluons at short distances but cannot rigorously relate these predictions to experimental data for hadrons.

An important additional feature which obscures theory of bound state dynamics of quarks and gluons, is the unknown structure of the QCD vacuum. The vacuum is a see of quarks and gluons. Quark and gluon condensates are formed. The vacuum is responsible for confinement and spontaneous chiral symmetry breaking. Unfortunately, no rigorous relativistic description of the vacuum state is available.

So far, attempts to formulate a theory of relativistic bound states in the instant form of dynamics have not led to quantitative results. One can investigate alternative formulations. Dirac had initiated work in this direction a long time ago [1]. He used the name instant form of dynamics for the standard equal-time (ET) description of physical states, and their dynamical changes from one to another instant of time. Dirac had considered more ways of constructing relativistic quantum mechanics. He pointed out that there exists an alternative form of dynamics which he called the front form. Dirac discovered that the front form may turn out to be in some respects superior to the ET form. It is true that we are mostly familiar with the ET form. This is due to our intuitive and nonrelativistic notion of absolute time. But the ET dynamical scheme is by no means unique. Dirac observed that the front form of dynamics is unfamiliar to us and deserves particular attention. It may offer new paths towards solution of profound problems of combining relativity with quantum mechanics. About 20 years had passed before Dirac's idea became popular among some other physicists.

The idea of considering alternative forms of dynamics for their special properties (which might simplify a theory) has been brought up again in a particularly elegant way by Weinberg in the context of quantum field theory [2]. Weinberg considered scalar field theory in the infinite momentum frame (IMF). He observed simplifications in the old-fashioned perturbation theory due to the absence of spontaneous particle creation from the vacuum in the IMF. Then Bardakci and Halpern [3], Susskind [4], Chang and Ma [5], Flory [6], Drell, Levi and Yan [7], Kogut and Soper [8], Bjorken, Kogut and Soper [9], Soper [10] and others have opened a series of studies of various theories in the IMF. The IMF rose to a prominent place because very fast moving hadrons were simply described in this frame using Feynman's parton model. In the IMF, internal hadronic clocks were so slow that the bound state interactions were frozen and hadrons could be seen by external probes as beams of essentially free partons.

Dirac's front form of dynamics in application to quantum fields and particle physics became popular when it was realized that it is closely related to the IMF limit in the ET form [11–15]. (These references do not include all important works but certainly illustrate the early developments.)

The front form of dynamics is also called light-front (LF) or even light-cone dynamics, although it has little to do with Minkowski's light cone. Let's denote four space-time coordinates in the Minkowski space by x^0, x^1, x^2 and x^3. In LF dynamics (in the convention which we adopt here) the role of time is assumed by $x^+ = x^0 + x^3$. Then, x^1 and x^2 describe the transverse part of space, often denoted together as $x^\perp$, and $x^- = x^0 - x^3$ is the longitudinal space coordinate. The role of energy is assumed by p^-. $p^\perp$ and p^+ are the spatial momentum variables. The notation $\pm = 0 \pm 3$ is adopted for the $\pm$ components of all tensors.

About two decades of successful phenomenological and theoretical studies have passed. A major work has been published by Brodsky and Lepage [16], which stimulated a lot of activity in relativistic nuclear physics and building models of hadrons. New technics have been developed. The reader may find more information in two recent reviews by Namysłowski and by Brodsky, McCartor, Pauli and Pinsky, and also by Coester [17, 18]. It is also useful to look up Ref. [19]. Despite all these intense efforts the precise formulation of the relativistic bound state problem for elementary particles has not yet been achieved.

The major problem is to formulate quantum field theory so that physical results for processes involving bound states obey basic requirements of special relativity. For the purpose of this introduction two kinds of relativistic effects can be distinguished.

The first may be called *internal relativity*. Vaguely speaking, one may consider a bound state at rest and ask how fast the constituents move. For example, positronium is an internally nonrelativistic bound state. Binding energy is a tiny fraction, order 10^{-5}, of the electron mass and the electron velocities are about 10^{-2} of the speed of light. Nuclear bound states are partly relativistic; binding energies of nucleons in nuclei are on the order of 10^{-2} of the nucleon mass and the nucleon velocities are about 0.1 of the speed of light. Finally, bound states of light quarks and gluons are very relativistic. Binding is so strong that no free colored particles are observed (confinement) and quark velocities in hadrons are comparable with the speed of light.

The second kind of effects may be called *external relativity*. It means that a bound state may move very fast and may be probed with highly relativistic probes, *e.g.* hard photons. Again, no practical need arises for considering collisions of fast moving atom-like systems in QED. Binding of electrons in atoms is so small that when atoms are probed with energetic projectiles the binding is practically irrelevant. On the other hand, nuclei are accelerated to large velocities and violently collided and their contraction when accelerated to large momentum cannot be neglected. When nuclei are probed with energetic probes, as, for example, in the deep inelastic lepton-nucleus scattering, binding effects are not negligible. Finally, hadrons in modern experiments take part in extremely high energy collisions and are probed with most relativistic objects available.

LF bound-state dynamics is interesting and worth investigation as an alternative to ET dynamics in cases where relativistic effects are important and our Galilean intuitions are of little use. LF dynamics may help in understanding phenomena whose description is so far out of reach in ET dynamics.

Let us begin with the issue of internal relativity. When fast moving constituents appear in a bound state at rest and their individual energies are large in comparison to their masses, then, according to the uncertainty principle, creation of more constituents has to be taken into account. As long as the number of constituents is not too large one may hope to compute the static bound state properties. However, there is an apparently unsurmountable difficulty involved in ET dynamics of quantum fields. Namely, there is a spontaneous creation of particles from the bare vacuum. Problems with the unlimited creation of new constituents and formation of a nontrivial physical vacuum state so far prevented construction of a satisfactory description of relativistic bound states in quantum field theory [20]. The vacuum problem is fundamental to our understanding of elementary particles as well as cosmology [21].

In LF approach to quantum field theory, the spontaneous creation of particles from a bare vacuum does not occur, if the theory is cut off by an infrared cutoff in momentum space. This feature greatly distinguishes LF dynamics from ET dynamics [22]. In ET dynamics the spontaneous particle creation cannot be removed by imposing cutoffs on momenta. Spontaneous symmetry breaking in LF dynamics is intriguing since it cannot happen in the same way as in ET dynamics. Investigations of the LF vacuum problem are a subject of interest of many authors. Examples of recent literature can be found in Ref. [23]. In this paper we consider aspects of the LF vacuum effects which are related to hadronic bound state dynamics.

In LF dynamics of quantum fields the bound state problem requires precise formulation. In order to make a theory finite one has to introduce cutoffs, and among them the infrared one. Analysis of the infrared cutoff dependence becomes essential to the whole theory, since this cutoff excludes the spontaneous particle creation from the bare vacuum and interferes with the ground state formation. If the vacuum structure was known the infrared cutoffs would not be necessary. One could compute dynamics of the vacuum degrees of freedom and include them in a complete theory. Moreover, the arbitrarily chosen splitting between degrees of freedom associated with the vacuum and those associated with a bound state could not matter. In actual situation, when we know nothing about the vacuum and consider only degrees of freedom above the infrared cutoff, all our results depend on this cutoff. This feature is viewed in this paper as a great advantage of LF dynamics. It may provide an opportunity for learning about the unknown vacuum structure with minimal guessing.

Cutoff dependence is usually removed by renormalization. There is a chance to reconstruct, at least partly, the missing vacuum effects by removing the infrared cutoff dependence in renormalization procedure. One may use existing knowledge about renormalization of Hamiltonians for quantum field theory [24] and hope that a consequent application of the renormalization ideas may shed some light on the vacuum effects in QCD. We shall describe initial steps made in theory in this direction. The unusual property of LF dynamics is that the vacuum problem is mixed with the cutoff dependence problem and, therefore, one can try to use known methods for dealing with the former and learn about the latter. One may not assume that a theory constructed that way must be complete. However, the part involving the observable bound states may have all properties required in order to describe static experimental data.

We should stress that in order to define the bound state calculations the renormalization theory has to be considered quite independently of the vacuum problems. Even if the infrared vacuum cutoff is somehow made irrelevant, there are ultraviolet singularities which need special care. The relativistic bound state theory cannot be formulated without understanding complicated renormalization effects. The bound states are assumed to be eigenstates of LF Hamiltonians. We shall explain the current status of the renormalization theory for LF Hamiltonians.

Next we consider the issue of external relativity. Here LF becomes overwhelmingly useful. The LF form of dynamics attracted Dirac's attention thanks to this special property. Namely, the formal algebra of ten Poincaré generators is represented in LF dynamics in such a way that as many as seven generators are independent of interactions. Only three generators are interaction dependent. It is useful to formally compare situations in ET and LF dynamics.

In ET dynamics only six generators are independent of interactions. These are three momentum operators and three angular momentum operators. The generator of time evolution, *i.e.* the Hamiltonian, and three generators of Lorentz boosts, depend on the interaction. Therefore, it is easy, and customary in ET dynamics, to consider bound states at rest and classify them using angular momentum quantum numbers. Questions about what happens when a bound state is moving are complicated and rarely considered in literature.

In LF dynamics as many as seven generators, two transverse momenta, $P^\perp$, the longitudinal momentum, P^+, angular momentum about 3rd axis, $J^3 = M^{12}$, two mixed rotation-boost generators, $M^{+\perp}$, and the generator of boosts along the 3rd axis, M^{+-}, are the same in free and interacting theories. Only the Hamiltonian P^- and two mixed boost-rotation generators, $M^{-\perp}$, depend on the interaction. (Early literature about the LF Poincaré algebra and related issues in particle theory can be found in papers from Ref. [25].) Therefore, it is easy, and customary in LF scheme, to consider models of arbitrarily moving bound states. However, problems with description of good angular momentum states in the LF quantum field theory are not solved.

The net result is that in ET dynamics one may construct theories of some static nonrelativistic bound states of good angular momentum, while in LF dynamics one has a big freedom of making studies of moving relativistic bound states but without precise knowledge of how to specify their angular momenta.

One can state this result somewhat differently as follows. The development of ET quantum theory of relativistic bound states is blocked by the lack of knowledge of boost generators. This is a disaster. No relativistic experiment can be done with static objects. Motion, acceleration and recoil effects have to be considered in practice. Our nonrelativistic intuitions based on studies of systems like atoms or heavy quarkonia are not sufficient. Quite unexpectedly, in LF dynamics unlimited motion of bound states is described by kinematics and many studies may be attempted. The remaining problem related to rotations which change the 3rd axis, does not prevent applications. It rather forces phenomenological input for the structure of a bound state at rest.

An important issue in external relativity is the coupling of a bound state to external particles and fields. Here again LF dynamics offers an unexpected help, in a similar way as the IMF did. For example, the coupling of photons to charged fields can be to large extent described using a good component of the current operator, j^+, in processes where the longitudinal momentum transfer, q^+, vanishes. Thus, at least some physical events can be modelled in a LF theory without losing control on how well one obeys first principles of quantum mechanics and special relativity.

We should stress two additional aspects of LF dynamics which are rarely considered.

- Firstly, one has to expect that introducing cutoffs may ruin kinematical symmetries of the theory, if the cutoffs do not obey the symmetries. For example, imposing a cutoff on absolute longitudinal momenta, k^+, in the Fourier expansion of fields, violates the celebrated longitudinal boost invariance. Then, special procedures in renormalization theory have to be introduced in order to restore the symmetry along with achieving the cutoff independence.
- Secondly, this cutoff intrusion into the theory is actually not as much of a problem as a key source of practical predictive power. The argument is following.

Cutoff dependence of a theory is removed by defining necessary new terms in its Hamiltonian. These terms are called counterterms. The counterterms depend on the cutoffs in such a way that the full theory is cutoff independent. However, the new terms contain finite parts of precisely the same structure as the diverging parts. We cannot decide how big are the finite parts without invoking experimental data. The finite parts should be adjusted so that results of the theory agree with the data. For example, all particle data confirm symmetries of special relativity. Then, the counterterms must possess enough many free parameters in order to fit special relativity. Renormalization theory for LF Hamiltonians presented in this paper points in this direction.

This paper is organized as follows.

In Section 2 we briefly describe LF QED and discuss electron-positron bound state dynamics in the simplest approximation. We describe how the center of mass Coulomb bound state equation for positronium follows from LF QED. We stress the apparent difference between the nonrelativistic nature of the positronium structure function and the highly relativistic character of the structure functions of hadrons. The QED example is also useful in later discussion of bound state equations in QCD and serves as an opportunity to introduce the LF formalism on a well know ground.

Section 3 describes an example of LF Hartree theory for nuclei. We focus our attention on relativistic effects of mass shifts for nucleons in nuclear matter. Nuclear matter is treated as a giant many-body bound state. We consider a simplified version of Walecka's model for illustration. The model involves a new interaction term which changes the nucleon mass. A similar term will be shown later to be necessary in renormalization theory. A brief discussion of applications of the LF many body theory is included. Special attention is paid to the nuclear binding effects in the deep inelastic lepton scattering off nuclei. Formation of the LF Fermi sphere is discussed in detail since it may be relevant to a future LF theory of the QCD ground state.

A special class of phenomenological LF quark models of nucleons is discussed in Section 4. We introduce LF quark wave functions based on models of hadronic interpolating fields which are used in QCD sum rules. We show how the idea that quark mass terms may be induced by the quark condensate can be incorporated in the LF models.

Hamiltonian approach to LF QCD including vacuum fields is presented in Section 5. We show how vacuum condensates known from the sum rules may appear in the LF QCD Hamiltonian. We write the corresponding quark-antiquark bound state eigenvalue equations and describe effects caused by gluon condensate in these equations. We present the idea that the gluon condensate may induce a strong Gaussian bound on the transverse momenta of partons in hadrons. Finally, we discuss divergences which appear in the eigenvalue equations to conclude that little progress is possible without understanding renormalization theory for LF Hamiltonians in quantum field theory.

In Section 6 we discuss in detail divergences which appear in the relativistic bound state eigenvalue problem in QCD. We consider singular seagull terms, self energies of quarks and gluons, one massive gluon exchange, wave function singularities and vertex corrections.

Section 7 introduces elements of the required renormalization theory. We present the LF theory of fixed sources and Wilson's model of coupling constant renormalization. Then, the idea of renormalization of LF Hamiltonians is explained using a simplified example in Yukawa theory. We describe renormalization of transverse overlapping divergences to all orders of perturbation theory and beyond. The model shows how the degeneracy of the bound state spectrum which is required by rotational invariance can be restored using the freedom of choice of the finite parts of the renormalization counterterms. We also briefly describe results of numerical studies of cutoff dependence and renormalization effects in this model. Section 7 is concluded by an elementary discussion of general covariance conditions to be satisfied by measurable matrix elements, requiring adjustments of free parameters in the renormalized Hamiltonians.

A special example of renormalized LF Hamiltonian theory for fermions interacting with scalar bosons is described in Section 8. We present exact solutions of the model. The fermion–boson scattering amplitude exhibits full covariance. One obtains this result within a very limited model Fock space thanks to a special choice of the counterterms in the Hamiltonian. We describe solutions for the fermion–boson bound state and study its structure. We describe properties of the bound state form factors and structure functions and discuss their residual cutoff dependence due to lack of an asymptotic freedom in the model.

Advanced new renormalization theory for Hamiltonians is sketched in Section 9. As an example of application, it is shown how the new theory solves the problem of longitudinal logarithmic divergence in the one gluon exchange interaction between quarks.

Section 10 concludes the paper by a brief outline of prospects of the LF Hamiltonian approach to the theory of the relativistic bound states of elementary particles.

Some parts of this article are extensive quotations from the reviewed papers. The quotations are provided for completeness where it is not possible to shorten presentation of the ideas without losing clarity.

2. Bound states in QED

For the purpose of this paper it is instructive to explain how the standard picture of bound states emerges in LF QED. We consider the well known example of positronium. In order to derive positronium wave functions in LF QED one has to make special assumptions, choose small cutoffs and suppress effects which otherwise would spoil the nonrelativistic positronium picture.

2.1. Light-front QED

The Lagrangian density of electrodynamics is

$$\mathcal{L}_{ED} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi . \tag{2.1}$$

Equations of motion for the electromagnetic field, $F^{\mu\nu} = -ie^{-1}[D^\mu, D^\nu] = \partial^\mu A^\nu - \partial^\nu A^\mu$, and the electron field, ψ, are

$$\partial_\mu F^{\mu\nu} = j^\nu = e\bar{\psi}\gamma^\nu\psi , \tag{2.2}$$

$$(i\not{D} - m)\psi = 0 . \tag{2.3}$$

The fermion field equation of motion in LF coordinates can be rewritten as

$$\left[\tfrac{1}{2}(i\partial^+ - eA^+)\gamma^- + \tfrac{1}{2}(i\partial^- - eA^-)\gamma^+ - (i\partial^\perp - eA^\perp)\gamma^\perp - m\right]\psi = 0 , \tag{2.4}$$

where $\partial^\pm = 2(\partial/\partial x^\mp)$. One can introduce projection operators $\Lambda_\pm = {}^1\!/_4\gamma^\mp\gamma^\pm = {}^1\!/_2\gamma^0\gamma^\pm$ and fields $\psi_\pm = \Lambda_\pm\psi$. Eq. (2.4) reads

$$(i\partial^+ - eA^+)\psi_- + (i\partial^- - eA^-)\psi_+ - \alpha^\perp(i\partial^\perp - eA^\perp)\psi - \beta m\psi = 0 , \tag{2.5}$$

and splits into two coupled equations,

$$(i\partial^+ - eA^+)\psi_- - \left[\alpha^\perp(i\partial^\perp - eA^\perp) + \beta m\right]\psi_+ = 0 , \tag{2.6}$$

$$(i\partial^- - eA^-)\psi_+ - \left[\alpha^\perp(i\partial^\perp - eA^\perp) + \beta m\right]\psi_- = 0 . \tag{2.7}$$

Eq. (2.6) contains no time derivative (*i.e.* no derivative over x^+) and is a constraint. It is visible that one can express ψ_- in terms of other fields by integration over x^- and solve this constraint when one chooses the gauge $A^+ = 0$ [13]. This choice of gauge is adopted for all gauge fields in this paper. Thus, the field ψ_- is not a dynamical degree of freedom. From Eq. (2.2) for $\nu = +$ it follows that

$$-\partial^+\left(\tfrac{1}{2}\partial^+ A^- - \partial^\perp A^\perp\right) = j^+ , \tag{2.8}$$

where j^+ denotes the fermion current. This equation does not contain time derivatives and is also a constraint, which can be used to find the field A^- in terms of other fields by integration over x^-. Thus, A^- is not a dynamical degree of freedom.

The dynamical degrees of freedom are fields ψ_+ and $A^\perp$. One can express the density of a canonical energy-momentum tensor, $T^{\mu\nu}$, in terms of these variables and their conjugate momenta and evaluate the canonical LF Hamiltonian for electrodynamics at a time $x^+ = 0$,

$$H(x^+ = 0) = \int dx^- d^2x^\perp \mathcal{H}(x^+ = 0, x^-, x^\perp), \tag{2.9}$$

where $\mathcal{H} = {}^1\!/\!_2 T^{+-}$ is the LF Hamiltonian density,

$$\begin{aligned}\mathcal{H} &= \psi_+^\dagger \sigma \frac{1}{i\partial^+} \sigma \psi_+ + \tfrac{1}{2} \partial^i A^j \partial^i A^j \\ &\quad + e\bar{\tilde{\psi}} \tilde{\not{A}} \tilde{\psi} + e^2 \bar{\tilde{\psi}} \tilde{\not{A}} \frac{\gamma^+}{2i\partial^+} \tilde{\not{A}} \tilde{\psi} + \frac{e^2}{2} \bar{\tilde{\psi}} \gamma^+ \tilde{\psi} \frac{1}{(i\partial^+)^2} \bar{\tilde{\psi}} \gamma^+ \tilde{\psi}.\end{aligned} \tag{2.10}$$

The first two terms represent LF energy density of noninteracting fermion and electromagnetic fields, $\sigma = i\partial^i \alpha^i + \beta m$ and the new fields, $\tilde{\psi}$ and $\tilde{A}$, are defined in terms of the independent degrees of freedom, ψ_+ and $A^\perp$, so that they would represent full fermion field and the electromagnetic potential in the free field theory [13, 16]. Namely,

$$\tilde{\psi} = \psi_+ + \tilde{\psi}_-, \tag{2.11}$$

where

$$\tilde{\psi}_- = \frac{1}{i\partial^+} \sigma \psi_+, \tag{2.12}$$

$$\tilde{A}^i = A^i, \tag{2.13}$$

and

$$\tilde{A}^- = \frac{2}{i\partial^+} i\partial^\perp A^\perp. \tag{2.14}$$

This way of writing the energy density in Eq. (2.10) is useful in electrodynamics since the free field approximation is a very good one for a small coupling constant e.

In order to introduce particle interpretation of this field theory we quantize Fourier components of the fields ψ_+ and $A^\perp$ as follows. Fourier expansions of free LF quantum fields are

$$\psi_{\text{free}}(x) = \int [dk] \sum_\lambda \left[b_{k\lambda} u_{mk\lambda} e^{-ikx} + d_{k\lambda}^\dagger v_{mk\lambda} e^{ikx} \right], \tag{2.15}$$

and

$$A^{\mu}_{\rm free}(x) = \int [dk] \sum_{\lambda} \left[a_{k\lambda} \varepsilon^{\mu}_{k\lambda} e^{-ikx} + a^{\dagger}_{k\lambda} \varepsilon^{\mu *}_{k\lambda} e^{ikx} \right] , \qquad (2.16)$$

where we use the abbreviated notation for integration over the three LF momentum variables,

$$\int [dk] = \int \frac{dk^{+} \theta(k^{+}) d^2 k^{\perp}}{16\pi^3 k^{+}} . \qquad (2.17)$$

The commutation relations for LF creation and destruction operators are,

$$\left\{ b_{k\lambda}, b^{\dagger}_{q\sigma} \right\} = \left\{ d_{k\lambda}, d^{\dagger}_{q\sigma} \right\} = 16\pi^3 k^{+} \delta^3(k-q) \delta_{\lambda\sigma} , \qquad (2.18)$$

$$\left[a_{k\lambda}, a^{\dagger}_{q\sigma} \right] = 16\pi^3 k^{+} \delta^3(k-q) \delta_{\lambda\sigma} , \qquad (2.19)$$

and all other commutators or anticommutators vanish. $b_{k\lambda}$ annihilates a bare electron with the three momentum components k^{+}, k^1 and k^2, and the spin projection on the z-axis equal $\lambda/2$. $b^{\dagger}_{k\lambda}$ creates a corresponding electron. Similarly, $d_{k\lambda}$ annihilates a bare positron and $d^{\dagger}_{k\lambda}$ creates a positron. $a_{k\lambda}$ annihilates a bare photon of polarization $\lambda = \pm 1$ along the z-axis, $a^{\dagger}_{k\lambda}$ creates a corresponding photon. The fermion spinors are

$$u_{mk\lambda} = \frac{1}{\sqrt{mk^{+}}} \left[\Lambda_{+} k^{+} + \Lambda_{-}(m + \alpha^{\perp} k^{\perp}) \right] u_{m\lambda} , \qquad (2.20)$$

where $u_{m\lambda}$ is a spinor for a fermion at rest; $u^T_{m1} = \sqrt{2m}(1,0,0,0)$ and $u^T_{m-1} = \sqrt{2m}(0,1,0,0)$. For positrons u is replaced in Eq. (2.20) by v and the positron spinor at rest is obtained from the corresponding electron spinor by charge conjugation. The matrix multiplying the spinor of a fermion at rest on the rhs of Eq. (2.20) represents the kinematical LF rotation-boost operator. The photon polarization vectors are

$$(\varepsilon^{\mu}_{\lambda}) = \left(\varepsilon^{+}_{\lambda} = 0,\ \varepsilon^{-}_{\lambda} = 2\frac{k^{\perp}\varepsilon^{\perp}_{\lambda}}{k^{+}},\ \varepsilon^{\perp}_{\lambda} \right) , \qquad (2.21)$$

where $\varepsilon^{\perp}_{1} = -(1/\sqrt{2})(1,i)$ and $\varepsilon^{\perp}_{-1} = (1/\sqrt{2})(1,-i)$.

It is important that the momentum variable k^{+} in the free field expansions is not allowed to be negative. This is a consequence of the condition that a free particle is not allowed to have an imaginary mass. Namely, $k^2 = k^+k^- - k^{\perp 2} = m^2 \geq 0$ implies that $k^+ = k^0 + k^3 \geq 0$, where $k^0 = \sqrt{(k^\perp)^2 + (k^3)^2 + m^2}$. This fact has profound consequences to the LF quantum field theory.

Using these definitions one can write Fourier expansions of the independent LF quantum field degrees of freedom at $x^+ = 0$ as follows,

$$\psi_+(x^-, x^\perp) = \Lambda_+ \psi_{\mathrm{free}}(x^-, x^\perp)\,, \tag{2.22}$$

$$A^\perp(x^-, x^\perp) = A^\perp_{\mathrm{free}}(x^-, x^\perp)\,, \tag{2.23}$$

and using Eqs (2.12) and (2.14) one obtains expressions

$$\tilde{\psi}(x^-, x^\perp) = \psi_{\mathrm{free}}(x^-, x^\perp) \tag{2.24}$$

and

$$\tilde{A}^\mu(x^-, x^\perp) = A^\mu_{\mathrm{free}}(x^-, x^\perp)\,, \tag{2.25}$$

for $x^+ = 0$.

Substituting expansions from Eqs (2.22) to (2.25) into Eqs (2.10) and (2.9) one obtains the Hamiltonian of LF QED. As Hamiltonians for almost all local quantum field theories, quite independently of the form of dynamics, this Hamiltonian is diverging and needs elaborate analysis before one can use it in practical computations. We describe below how one may use this Hamiltonian to derive an approximate description of positronium.

2.2. Positronium

Creation and annihilation operators introduced in the previous Section can be used to build the LF Fock space representation of eigenstates of the QED Hamiltonian. The bare vacuum state $|0\rangle$ is annihilated by all annihilation operators. A bare electron-positron pair is represented as $b^\dagger_{p_2\lambda_2} d^\dagger_{p_1\lambda_1}|0\rangle$. Such a pair may be accompanied by a bare photon, and the three particle state is represented by $b^\dagger_{p_2\lambda_2} d^\dagger_{p_1\lambda_1} a^\dagger_{q\epsilon}|0\rangle$. Infinitely many bare states are coupled by interaction terms in the Hamiltonian of QED. In order to approximately describe positronium one can use the intuition that it is a state built mostly from a bare electron and a bare positron. One has to include the state with one photon in addition to a pair of fermions in order to incorporate all leading order interactions which combine together into the Coulomb potential. All other possible bare states are neglected in this approximation and the full Hamiltonian is projected on the space spanned by linear combinations of various bare electron-positron states $|e^+e^-\rangle$ and bare electron-positron-photon states $|e^+e^-ph\rangle$. This approximation is called LF Tamm–Dancoff approximation [26, 27].

Positronium of the total momentum P is represented in the LF Tamm–Dancoff approximation by the following state,

$$|P\rangle = \sum_{\lambda_1\lambda_2} \int [dp_2][dp_1]\Psi^P_{\lambda_2\lambda_1}(p_2,p_1)b^\dagger_{p_2\lambda_2}d^\dagger_{p_1\lambda_1}|0\rangle + \sum_{\lambda_1\lambda_2\epsilon} \int [dp_2][dp_1][dq]\Psi^P_{\lambda_2\lambda_1\epsilon}(p_2,p_1,q)b^\dagger_{p_2\lambda_2}d^\dagger_{p_1\lambda_1}a^\dagger_{q\epsilon}|0\rangle \,. \quad (2.26)$$

The wave functions have the forms,

$$\Psi^P_{\lambda_2\lambda_1}(p_2,p_1) = 2(2\pi)^3\delta^3(P-p_2-p_1)\sqrt{p_2^+p_1^+}\ \phi_{\lambda_2\lambda_1}(x_2,\kappa^\perp)\,, \quad (2.27)$$

and

$$\Psi^P_{\lambda_2\lambda_1\epsilon}(p_2,p_1,q) = 2(2\pi)^3\delta^3(P-p_2-p_1-q)\sqrt{p_2^+p_1^+x_0}\ \phi_{\lambda_2\lambda_1\epsilon}(x_2,x_1,\kappa_2^\perp,\kappa_1^\perp)\,. \quad (2.28)$$

Such forms are implied by the kinematical symmetries of the front form of dynamics. Factors of square roots are introduced for convenience. The arguments of the wave functions are defined as follows.

$$x_2 = \frac{p_2^+}{P^+}\,, \quad x_1 = \frac{p_1^+}{P^+}\,, \quad x_0 = \frac{q^+}{P^+}\,. \quad (2.29)$$

In the electron-positron sector we define

$$p_2^\perp = x_2P^\perp + \kappa^\perp\,, \quad (2.33)$$
$$p_1^\perp = x_1P^\perp - \kappa^\perp\,, \quad (2.34)$$

and

$$x_2 + x_1 = 1\,. \quad (2.32)$$

In the electron-positron-photon sector we define

$$p_2^\perp = x_2P^\perp + \kappa_2^\perp\,, \quad (2.33)$$
$$p_1^\perp = x_1P^\perp - \kappa_1^\perp\,, \quad (2.34)$$
$$q^\perp = x_0P^\perp + \kappa_1^\perp - \kappa_2^\perp\,, \quad (2.35)$$

and

$$x_0 = 1 - x_1 - x_2\,. \quad (2.36)$$

These variables resemble variables x and $p^{\perp}$ used in description of hadrons in high energy collisions. The fraction x of the total momentum P^+ resembles the Feynman parton model fractions of the total momentum of a hadron in the IMF.

The bound state equations are obtained by projecting the Hamiltonian eigenvalue equation,

$$H|P\rangle = \frac{P^{\perp 2} + M^2}{P^+}|P\rangle \,, \tag{2.37}$$

on the Fock space $|e^+e^-\rangle \oplus |e^+e^-ph\rangle$ and expressing the resulting eigenvalue problem in terms of coupled integral equations for the two wave functions from Eqs (2.27) and (2.28). The integral equations can be written in an abbreviated fashion as

$$\begin{bmatrix} p_1^- + p_2^- + \sum \int \langle V_{e^+e^-}\rangle \,, & \sum \int \langle H_I\rangle \\ \sum \int \langle H_I\rangle & , \; p_1^- + p_2^- + q^- + \sum \int \langle V_{e^+e^-ph}\rangle \end{bmatrix}$$
$$\times \begin{bmatrix} \Psi^P_{\lambda_2\lambda_1}(e^+e^-) \\ \Psi^P_{\lambda_2\lambda_1 \epsilon}(e^+e^-ph) \end{bmatrix} = \frac{P^{\perp 2} + M^2}{P^+} \begin{bmatrix} \Psi^P_{\lambda_2\lambda_1}(e^+e^-) \\ \Psi^P_{\lambda_2\lambda_1 \epsilon}(e^+e^-ph) \end{bmatrix} \,, \tag{2.38}$$

where we marked symbolically integrations over arguments of the wave functions and sums over spin indices, and indicated matrix elements of operators between the corresponding Fock states by the angular brackets. $\langle V_{e^+e^-}\rangle$ represents interaction kernels which result from matrix elements of operators order e^2 from the QED Hamiltonian in the e^+e^- Fock sector. This term includes the fermion self-interactions due to normal ordering of operators and the LF instantaneous photon exchange interaction which is induced by the last term in Eq. (2.10). $\langle V_{e^+e^-ph}\rangle$ represents instantaneous fermion–photon couplings induced in the electron-positron-photon Fock sector by the fourth term in Eq. (2.10). $\langle H_I\rangle$ represents matrix elements of the first interaction term on the rhs of Eq. (2.10). This term induces emission and absorption of single photons by fermions and leads to transitions between the two Fock sectors. Thus, the positronium is represented by 8 wave functions, four of six variables and four of nine variables, which satisfy a nontrivial set of integral equations.

The total momentum of the bound state, P, can be eliminated from Eqs (2.38). This is one of the unique properties of the LF form of dynamics which we referred to as external relativity in the Introduction. The total momentum components P^+ and $P^{\perp}$ formally drop out from the bound state equations and the mass eigenvalue M for the positronium is independent of the total three momentum of the bound state. In other words, $E(\vec{P}) = \sqrt{\vec{P}^2 + M^2}$ as it should be. Analogous bound state equations in ET dynamics would not lead to such energy-momentum relation for the bound state solutions. In ET equations different choices for $\vec{P}$ would lead to different results for M, which is not acceptable.

Thus, in LF dynamics, one obtains equations for the relative motion wave functions which are independent of the bound state motion as a whole. The relative motion wave functions appear multiplied by three-dimensional delta functions in Eqs (2.27) and (2.28). These are functions of only three and six arguments, respectively. The integral equations for the relative motion wave functions are much simpler than the original equations.

One may observe that the off-diagonal terms in Eq. (2.38) contain a factor e and the interaction $\langle V_{e^+e^-ph}\rangle$ a factor e^2. Therefore, $\langle V_{e^+e^-ph}\rangle$ contributes to the effective dynamics in the e^+e^- sector terms multiplied by e^4 and higher powers of e. Let us assume that all integrals over relative momenta of bare particles are limited by cutoffs on the order of the bare electron mass, m. Then, such interactions are of order $\alpha^2 m$ and can be neglected in comparison to the leading interactions order αm. Once these interactions are neglected one may observe that the wave function of the three particle sector can be algebraically expressed in terms of the wave function of the two particle sector.

We can express the three-body wave function by the two-body wave function and write an equation for the two-body wave function itself. This equation has the familiar structure

$$\left[\frac{m^2+\kappa^2+\Sigma_2}{x_2 M}+\frac{m^2+\kappa^2+\Sigma_1}{x_1 M}-M\right]\phi + \sum\int\left[\langle V_{OPhE}\rangle + \langle V_{e^+e^-}\rangle\right]\phi = 0\,. \qquad (2.39)$$

The wave function ϕ represents a column of four wave functions corresponding to four possible spin configurations of the electron and positron. $\langle V_{OPhE}\rangle$ represents the one photon exchange potential, which depends on the eigenvalue M. In order to explain further procedure, we represent the wave function as a 2×2 matrix,

$$\phi_{2\times 2} = \begin{bmatrix}\phi_{\uparrow\downarrow} & \phi_{\uparrow\uparrow}\\ \phi_{\downarrow\downarrow} & \phi_{\downarrow\uparrow}\end{bmatrix}\,. \qquad (2.40)$$

Then, Eq. (2.39) reads as follows,

$$\left[\frac{m^2+\kappa^2+\Sigma_2}{x_2}+\frac{m^2+\kappa^2+\Sigma_1}{x_1}-M^2\right]\phi_{2\times 2}(x_2,\kappa^\perp)-\frac{4\alpha}{(2\pi)^2}\int dx_2'\int d^2\kappa'^\perp \times\left[\frac{(s_2^i-v^i)\phi_{2\times 2}(x_2',\kappa'^\perp)(s_1^i+v^i)}{\text{denominator}}+\frac{1}{(x_2-x_2')^2}\phi_{2\times 2}(x_2',\kappa'^\perp)\right]=0, \qquad (2.41)$$

where $\alpha = (e^2/4\pi)$, and

$$2s_2^i = \frac{\kappa^\perp\sigma^\perp}{x_2}\sigma^i + \sigma^i\frac{\kappa'^\perp\sigma^\perp}{x_2'} + \sigma^i\sigma^3 m\frac{x_2' - x_2}{x_2 x_2'}\,, \tag{2.42}$$

$$2s_1^i = \frac{\kappa'^\perp\sigma^\perp}{x_1'}\sigma^i + \sigma^i\frac{\kappa^\perp\sigma^\perp}{x_1} + \sigma^i\sigma^3 m\frac{x_2' - x_2}{x_1 x_1'}\,, \tag{2.43}$$

$$v^\perp = \frac{\kappa^\perp - \kappa'^\perp}{x_2 - x_2'}\,, \tag{2.44}$$

and

$$\begin{aligned}
\text{denominator} =& \mu^2 - \tfrac{1}{2}(p_2 - p_2')^2 - \tfrac{1}{2}(p_1 - p_1')^2 \\
&+ \tfrac{1}{2}|x_2 - x_2'|(\mathcal{M}_{12}^2 + \mathcal{M}'^2_{12} - 2M^2) \\
=& \mu^2 + (\kappa^\perp - \kappa'^\perp)^2 + \tfrac{1}{2}(x_2 - x_2') \\
&\times\left(\frac{\kappa^2 + m^2}{x_2} - \frac{\kappa^2 + m^2}{x_1} + \frac{\kappa'^2 + m^2}{x_1'} - \frac{\kappa'^2 + m^2}{x_2'}\right) \\
&+ \tfrac{1}{2}|x_2 - x_2'|\left(\frac{\kappa^2 + m^2}{x_1 x_2} + \frac{\kappa'^2 + m^2}{x_1' x_2'} - 2M^2\right)\,.
\end{aligned} \tag{2.45}$$

We have introduced the mass, μ, which is here put equal zero for photons but will be useful in a later discussion for gluons. We have also defined the on-mass-shell momentum four-vectors p_2 and p_1 so that $p_2^2 = p_1^2 = m^2$, and introduced useful notation $\mathcal{M}_{12}^2 = (p_1 + p_2)^2$ and $\mathcal{M}'^2_{12} = (p_1' + p_2')^2$, where primed symbols refer to the intermediate states under the integrals.

Bound state equations of this type are known for a long time [2, 28]. First numerical solutions for such equations using partial wave analysis in scalar field theory were found by Danielewicz and Namysłowski [29]. Partial wave analysis of similar equations in field theories with spin and examples of numerical solutions were given by Głazek [30, 31].

The fermion self-energies, Σ_1 and Σ_2, are

$$\Sigma_i = -\frac{\alpha}{(2\pi)^2}\int dz \int d^2\kappa \frac{\frac{2}{1-z}\left[m^2(1 - z^{-1})^2 + \kappa^2\left(z^{-2} + 2z^{-1}(1-z)^{-2}\right)\right]}{x_i(\mathcal{M}_{12}^2 - M^2) + \frac{\kappa^2 + m^2}{z} + \frac{\kappa^2 + \mu^2}{1-z} - m^2}\,. \tag{2.46}$$

If we assume that the photon momenta in these integrals are suitably limited in a range on the order of bare electron mass, we can proceed as follows. Σ_i depends on the electron and positron momentum through a characteristic argument $x_i(\mathcal{M}_{12}^2 - M^2)$, which is on the order $(\alpha m)^2$ in positronium (see the discussion below). Therefore, Σ_1 and Σ_2 are predominantly constants order αm^2 plus terms order $\alpha^3 m^2$ and the latter can be neglected. The former can be considered as a renormalization of the bare m^2 term in the first term of the LF QED Hamiltonian density from Eq. (2.10) (but not terms linear in m in the Hamiltonian). Then, one can say that the bare mass squared was chosen precisely to be such that when the leading constant term of Σ is added the result is a physical mass squared of an electron or a positron, denoted again by the same letter m. Thus, in positronium where $\alpha \simeq 1/137$, one can forget Σ's when considering leading interactions order α and small momentum cutoffs.

The first term under the integral in Eq. (2.41) is the one photon exchange interaction and the second term is the instantaneous photon exchange. When the integrals over relative momenta of fermions are suitably cut off at values not exceeding m, one can argue for dramatic simplifications of the interaction kernels as follows.

It is assumed that the relative motion of fermions is selfconsistently described by a wave function which peaks at momenta very small in comparison to the electron mass, order αm. This assumption is verified a posteriori. Then, one can neglect κ and κ' in comparison to m and consider x_1, x_2, x_1' and x_2' to be very close to $1/2$. Thus, $x_2 = 1/2 + y$, $x_1 = 1/2 - y$, $x_2' = 1/2 + y'$ and $x_1' = 1/2 - y'$ and $y \sim y' \sim \alpha$. Therefore, the spin dependent terms from Eqs (2.42) and (2.43) (which are proportional to the fermion velocities $\sim \alpha$) contribute terms order α or higher to the numerator in Eq. (2.41). The special spin independent term from Eq. (2.44) is of order 1 since both numerator and denominator in $v^\perp$ are small. The denominator from Eq. (2.45) is in the leading order in α equal $(\kappa^\perp - \kappa'^\perp)^2 + 4m^2(y-y')^2$. The combined result is that the leading interaction kernel in Eq. (2.41) is

$$-\frac{1}{(y-y')^2}\frac{(\kappa^\perp - \kappa'^\perp)^2}{(\kappa^\perp - \kappa'^\perp)^2 + 4m^2(y-y')^2} + \frac{1}{(y-y')^2}$$
$$= \frac{4m^2}{(\kappa^\perp - \kappa'^\perp)^2 + 4m^2(y-y')^2}\,. \tag{2.47}$$

Next, we change the variables from x and $\kappa^\perp$ to

$$p^\perp = \kappa^\perp, \tag{2.48}$$

and

$$p^3 = y\mathcal{M}_{12} = \left(x_2 - \tfrac{1}{2}\right)\sqrt{\frac{m^2 + \kappa^{\perp 2}}{x_2(1-x_2)}}\,, \tag{2.49}$$

so that we can form a vector, $\vec{p} = (p^\perp, p^3)$, and

$$\mathcal{M}_{12}^2 = 4[(\vec{p})^2 + m^2]\,. \tag{2.50}$$

Similarly for primed variables, so that

$$\mathcal{M}'^2_{12} = 4[(\vec{p}\,')^2 + m^2]\,, \tag{2.51}$$

and

$$dx'_2 = dy' = \frac{1-4y'^2}{\mathcal{M}'_{12}}dp'^3 \simeq \frac{1}{2m}dp'^3\,. \tag{2.52}$$

In this notation, Eq. (2.41) with the leading kernel from Eq. (2.47) reads

$$4[(\vec{p})^2 + m^2]\phi_{2\times 2}(\vec{p}) - \frac{4\alpha}{(2\pi)^2}\int \frac{d^3p'}{2m}\frac{4m^2}{(\vec{p}-\vec{p}\,')^2}\phi_{2\times 2}(\vec{p}\,') = M^2\phi_{2\times 2}(\vec{p})\,. \tag{2.53}$$

Writing the positronium mass as $M = 2m - E_b$, where E_b is the binding energy, and neglecting terms order E_b^2, one can rewrite Eq. (2.53) as

$$\frac{\vec{p}^{\,2}}{m}\phi(\vec{p}) + \int \frac{d^3p'}{(2\pi)^3}\frac{-4\pi\alpha}{(\vec{p}-\vec{p}\,')^2}\phi(\vec{p}\,') = -E_b\phi(\vec{p})\,, \tag{2.54}$$

which is the Shrödinger equation in a center of mass coordinates for two charged particles interacting by the Coulomb potential, independently of their spin. Eq. (2.54) has solutions of the form

$$\phi(\vec{p}) = \frac{N}{(\vec{p}^{\,2} + mE_b)^2}\,, \tag{2.55}$$

where N is a normalization constant and $E_b = {}^1\!/_4\alpha^2 m$. We stress that the wave function from Eq. (2.55) falls off very quickly for momentum $|\vec{p}| \gg \alpha m$ and the integral with Coulomb potential in Eq. (2.54) is insensitive to the upper limit of integration, *i.e.* to the cutoff imposed on bare particle momenta in the original Tamm–Dancoff eigenvalue problem. Thus, we have a selfconsistent approximation for positronium in LF QED. For small cutoffs and a small coupling constant the nonrelativistic Coulomb wave function is a leading order approximation to the positronium state and corrections to the Coulomb wave function due to various neglected terms can be investigated in a systematic way using perturbation theory.

As a side remark we would like to add that the nonrelativistic picture of positronium inspires successful models of heavy quarkonia. There, the Coulomb potential is modified by a model term assumed to represent a string of gluons. Lattice calculations for static quark sources representing a heavy quarkonium at rest, support this idea. Further improvements of the lattice calculations may find the size of corrections to the nonrelativistic potential which are implied by QCD. Nonrelativistic lattice calculations for heavy quarkonia have been recently reported by Thacker and Lepage [32].

Using Eq. (2.55) one may speculate that a good first approximation for the LF positronium wave function has the form

$$\phi(x_2, \kappa^{\perp}) = N \left[\alpha^2 m^2 + \frac{\kappa^{\perp 2} + (x_2 - \frac{1}{2})^2 4m^2}{x_2(1 - x_2)} \right]^{-2} , \tag{2.56}$$

independently of the electron and positron spins. This wave function sharply peaks for $x_2 \simeq x_1 \simeq {}^1/_2$ and $\kappa^{\perp} \sim 0$. Minor corrections to this form can be investigated in perturbation theory. This wave function is strongly convergent for large relative transverse momenta or extreme longitudinal momentum fractions; $\phi \sim x_2^2(1 - x_2)^2 \kappa^{-4}$. The strong convergence, *i.e.* the quick fall off of the wave function in momentum space in the Coulomb potential is the source of success of bound state models in QED and may misleadingly suggest that theoretical problems in relativistic bound state dynamics are of marginal importance. Let us illustrate the scale of problems one has to solve. One can calculate the structure function of positronium in the leading order in α using the wave function from Eq. (2.56). One obtains

$$F_{\text{positronium2}}(x) = 2e^2 x f(x) , \tag{2.57}$$

where the electron–parton distribution function is

$$f(x) = N \frac{[x(1 - x)]^4}{\left[\frac{\alpha^2}{4} x(1 - x) + (x - \frac{1}{2})^2 \right]^3} . \tag{2.58}$$

Since $\int_0^1 dx f(x) = 1$ this function approaches $\delta(x - {}^1/_2)$ when $\alpha \to 0$. In reality, the half-width of the positronium structure function is of the order of 10^{-3}. This should be contrasted with a proton structure function which is a smooth function extending over the whole range of x from 0 to 1. Our knowledge of bound state dynamics in positronium is of little utility in the case of hadrons. In hadronic dynamics a whole range of constituents momenta, at least three to four orders of magnitude larger than in positronium and exceeding constituents' masses many times, must be explored before we will be able to provide a theory of hadronic structure.

Since in positronium one does not allow too much momentum in particle self-energies and vertex corrections, the ambiguities in the theory due to lack of covariance or other limitations are small and hardly visible. Some discussion of retardation effects in a related case of scalar boson exchange in next to leading order terms can be found in Ref. [33]. One should also mention that LF formulae for wave functions analogous to Eq. (2.56), can be used to incorporate knowledge of nonrelativistic wave functions in LF phenomenology of rapidly moving and energetically probed bound states like deuteron. [34–36, 61] This discussion completes our presentation of LF theory of internally nonrelativistic bound states. Precise theoretical understanding of bound states of elementary particles is limited to these simple systems.

When one considers coupling constants which are not as small as in QED and cutoffs are definitely larger than the constituent masses in order to include relativistic motion of constituents then the whole positronium-like picture of bound states goes out the window. Many terms in the Hamiltonian become not negligible and difficulties with combining quantum mechanics and special relativity become numerically significant and conceptually bothering. In the case of strong forces when relativistic effects become important the whole picture of bound states is murky and requires systematic investigation.

We need to understand various divergences in the LF bound state equations and formulate precise definitions of eigenvalue problems for LF Hamiltonians in quantum field theory. Before we proceed to the theoretical discussion of LF Hamiltonians it is instructive to present examples of semi-phenomenological models which demonstrate that relativistic effects in bound state dynamics are not at all negligible. We shall now discuss two examples of physical systems where relativistic effects are relevant to analysis of experimental data. We begin with nuclear physics.

3. Nucleons in nuclei

In the previous Section we have observed that electron-photon interactions in positronium are able to modify electron mass but the effect is extremely small and can be described in perturbation theory. In nuclear physics relativistic effects of altering mass of a nucleon in the bound state dynamics in a large nucleus are much larger. Such effects cannot be described in perturbation theory. Effective nucleon masses appear in model considerations of nuclear matter. Extensive studies have been carried out in the Walecka model [37]. Advanced introduction to relativistic theory of nuclear bound states has been recently published by Celenza and Shakin [38].

Alteration of the nucleon mass in nuclear matter is considered to be one of possible explanations of the observed difference between deep inelastic lepton scattering cross section on free protons and on protons bound in nuclei. However, although there is a great need for relativistic theory of nuclei which could help in understanding nuclear effects in deep inelastic lepton-nucleus scattering and heavy ion collisions, LF theory of many-body systems is virtually nonexistent. Our discussion of the mass shift effects for nucleons in large nuclei is based on the boost invariant theory of nuclear matter developed by Głazek and Shakin [39] and we extensively quote from Ref. [39] in this Section. Average properties of matter inside large nuclei are similar to properties of gedanken infinite nuclear matter. In LF approach nuclear matter itself is considered to be a giant bound state of nucleons, in which surface effects can be neglected. On the basis of the LF nuclear matter model one may attempt to explain some features of the deep inelastic lepton-nucleus scattering.

3.1. Relativistic many-body bound states [39]

We consider a model based on the following Lagrangian density [40]

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\varphi\partial^\mu\varphi - \mu^2\varphi^2) + \bar\psi(i\,\not\partial - m)\psi - g\bar\psi\psi\varphi\,. \tag{3.1}$$

Varying the corresponding action one obtains equations of motion for the scalar field φ, called a meson field, and the fermion field ψ, called a nucleon field,

$$(\partial^2 + \mu^2)\varphi = -g\bar\psi\psi\,, \tag{3.2}$$

$$(i\,\not\partial - m)\psi = g\psi\varphi\,. \tag{3.3}$$

The fermion field equations of motion involve constraints. Following a similar procedure as in QED in Section 2 one can define the field $\tilde\psi$, such that the full fermion field is

$$\begin{aligned}\psi &= \tilde\psi + \frac{1}{i\partial^+}\beta g\varphi\psi_+ \\ &= \left(1 + \frac{\gamma^+}{2i\partial^+}g\varphi\right)\tilde\psi\,,\end{aligned} \tag{3.4}$$

where the latter equality is a consequence of $\gamma^+\gamma^+ = 0$. The LF Hamiltonian reads

$$\begin{aligned}P^- = \tfrac{1}{2}\int dx^- d^2x^\perp \Big[&\varphi(-\partial^{\perp 2} + \mu^2)\varphi + \bar{\tilde\psi}\gamma^+\frac{-\partial^{\perp 2} + m^2}{i\partial^+}\tilde\psi \\ &+ 2g\bar{\tilde\psi}\tilde\psi\varphi + \bar{\tilde\psi}g\varphi\frac{\gamma^+}{2i\partial^+}g\varphi\tilde\psi\Big]\,.\end{aligned} \tag{3.5}$$

The momentum P^+ does not explicitly involve interaction terms

$$P^+ = \tfrac{1}{2}\int dx^- d^2x^\perp \left[\partial^+\varphi\partial^+\varphi + \bar{\tilde\psi}\gamma^+ i\partial^+\tilde\psi\right]\,. \tag{3.6}$$

The fermion field is quantized in order to obtain its particle interpretation and is written as

$$\tilde{\psi}(x) = \int [dp] \sum_{\lambda\tau} b_{p\lambda\tau} u_\tau u_{mp\lambda} e^{-ipx} , \tag{3.7}$$

for $x^+ = 0$. We adopt here the same conventions as in Section 2. The subscript m of the spinor reminds the reader that the fermion mass in this spinor is equal m as for free nucleons, see Eq. (2.20). In addition, we have indicated isospin index τ and isospin spinor u_τ. During further discussion we omit isospin for simplicity but include it back in our final formulae. In Eq. (3.7) we do not include the antinucleon component of the field, since it is not necessary in the mean field or Hartree theory of large nuclei or nuclear matter. It is also not necessary to discuss explicitly quantization of the meson field in the mean field approximation.

The mean field description of nuclear matter is based on the following variational principle. One considers a trial state

$$|P, \text{parameters}\rangle_A \tag{3.8}$$

for a system of A nucleons with total momentum P. The "parameters" are essentially wave functions describing the motion of the nucleons and mesons. One defines a so-called static energy operator

$$\hat{E}_s = \tfrac{1}{2}(\hat{P}_s^- + \hat{P}_s^+) , \tag{3.9}$$

in which the subscript s refers to the static approximation for the meson field. The expectation value of $\hat{E}_s$ is defined in the standard way

$$E(P, \text{parameters}) = \frac{{}_A\langle P, \text{parameters}|\hat{E}_s|P, \text{parameters}\rangle_A}{{}_A\langle P, \text{parameters}|P, \text{parameters}\rangle_A} , \tag{3.10}$$

and called energy. Then we minimize the energy with respect to the parameters. Dirac equation for nucleon separation energies, expression for the nucleon effective mass and an optimal volume condition for a large bound state of nuclear matter with fixed number of nucleons (or, equivalently, the magnitude of the Fermi momentum), result from equations of the form

$$\frac{\delta}{\delta(\text{parameters})} E(P, \text{parameters}) = 0 . \tag{3.11}$$

The mass of nuclear matter which is a giant bound state of nucleons is given by the formula

$$M_A = E(M_A, \text{optimal parameters}) , \tag{3.12}$$

where we considered the nuclear matter at rest and replaced P by M_A.

The relativistic LF Hartree variational ansatz for the ground state of the system of very many nucleons is

$$|P\rangle = \sum_{n=1}^{A} |P\rangle_n \,, \tag{3.13}$$

where states $|P\rangle_n$ represent stationary levels to be filled by A nucleons,

$$|P\rangle_n = \sum_{\lambda\Lambda} \int [dk][dK] f_{P,n}^{\lambda\Lambda}(k,K) b_{k\lambda}^{\dagger} B_{k\Lambda,n}^{\dagger} |0\rangle \,. \tag{3.14}$$

A nucleon of momentum k is accompanied by an effective residual system of $A-1$ nucleons and mesons of total momentum K, created by the operator $B^{\dagger}$. Each orbital is associated with a separate residual system whose parameters are those of the corresponding orbital [41]. The residual systems have masses denoted by M_n. When evaluating the energy or momentum of nuclear matter, we will ignore the noncommutativity of the operators b and $B^{\dagger}$. Therefore, it is legitimate to write the sum over n in Eq. (3.13). The bookkeeping for the residual systems of $A-1$ nucleons is facilitated using either commutation or anticommutation relations according to their spin,

$$\left[B_{K\Lambda,n}, B_{K'\Lambda',n'}^{\dagger}\right]_{\pm} = 2K^{+}(2\pi)^3\delta^3(K-K')\delta_{\Lambda\Lambda'}\delta_{nn'} \,. \tag{3.15}$$

The wave functions $f_{P,n}^{\lambda\Lambda}(k,K)$ have the form

$$f_{P,n}^{\lambda\Lambda}(k,K) = 2P^{+}(2\pi)^3\delta^3(P-k-K) g_n^{\lambda\Lambda}(x,\kappa^{\perp}) \,, \tag{3.16}$$

where

$$x = \frac{k^{+}}{P^{+}} \tag{3.17}$$

and

$$\kappa^{\perp} = k^{\perp} - xP^{\perp} \,. \tag{3.18}$$

This form is familiar from the previous Section describing positronium. The main new element is that the masses of the two particles, the nucleon and the residual system of $A-1$ nucleons, are completely different while in positronium masses of the electron and the positron are equal. The nuclear matter is viewed as a superposition of A pairs of particles. In each pair there is a single nucleon in an orbital around or, in fact, within an infinitely heavier body of the remaining $A-1$ nucleons.

In the static approximation the meson field φ is replaced by the static solution φ_s of the meson field equation of motion, Eq. (3.2),

$$\varphi \to \varphi_s = -\frac{g}{\mu^2}\bar{\psi}\psi\,. \tag{3.19}$$

Using this approximation and integrating by parts in Eq. (3.5) one obtains

$$\hat{P}_s^- = \tfrac{1}{2}\int dx^- d^2x^\perp \left[\bar{\tilde{\psi}}\gamma^+ \frac{-\partial^{\perp 2} + m^2}{i\partial^+}\tilde{\psi} + g\bar{\tilde{\psi}}\tilde{\psi}\varphi_s\right]. \tag{3.20}$$

The remaining three components of the momentum operator become equal to

$$\hat{P}_s^\mu = \tfrac{1}{2}\int dx^- d^2x^\perp\, \bar{\tilde{\psi}}\gamma^+ i\partial^\mu\tilde{\psi}\,, \tag{3.21}$$

since the derivatives of the static meson field are negligible in the big volume of our nuclear matter and the surface effects are neglected. We assume that the system saturates. This will be verified a posteriori. The saturation is a purely relativistic effect and does not occur in an analogous nonrelativistic model.

Now, one can evaluate expectation value of the energy operator from Eq. (3.10), making following steps. Matrix elements of translationally invariant operators, say $\hat{O}$, between the total three-momentum eigenstates are normalized using relation

$${}_A\langle P'|\hat{O}_s|P\rangle_A = 2P^+(2\pi)^3\delta^3(P-P')\,O_s\,. \tag{3.22}$$

Using Eq. (3.4) with φ replaced by φ_s one obtains from Eq. (3.19) that

$$\varphi_s = -\frac{g}{\mu^2}\left[\bar{\tilde{\psi}}\tilde{\psi} + \bar{\tilde{\psi}}g\varphi_s\frac{\gamma^+}{-2i\overleftarrow{\partial}^+}\tilde{\psi} + \bar{\tilde{\psi}}\frac{\gamma^+}{2i\overrightarrow{\partial}^+}g\varphi_s\tilde{\psi}\right]. \tag{3.23}$$

Taking the expectation value of this equation in the nuclear matter trial state and assuming that the expectation values in the nuclear matter are the same as in the residual system with one nucleon less, one obtains the integral equation [39]

$$\begin{aligned}{}_A\langle P'|\varphi_s(0)|P\rangle_A = -\frac{g}{\mu^2}\sum_{n=1}^{A}\sum_{\lambda'\lambda\Lambda}\Bigg[\int_\kappa \frac{\bar{u}_{mk'\lambda'}u_{mk\lambda}}{x}g_n^{\lambda'\Lambda*}\left(x',\bar{\kappa}^\perp\right)g_n^{\lambda\Lambda}(x,\kappa^\perp)\\ + \int_\kappa\int_{\kappa'}\bar{u}_{mk'\lambda'}\left[\frac{\gamma^+}{2(x'P^+ - q^+)} + \frac{\gamma^+}{2(xP^+ + q^+)}\right]u_{mk\lambda}\\ \times g_n^{\lambda'\Lambda*}(x',\kappa'^\perp)\,{}_A\langle K'|\varphi_s(0)|K\rangle_A\, g_n^{\lambda\Lambda}(x,\kappa^\perp)\Bigg],\end{aligned} \tag{3.24}$$

where $\tilde{\kappa}^\perp = \kappa^\perp + (1-x)q^\perp + (x-x')P^\perp$ and we have used the relation

$$\int [dk][dK] = \int [d(k+K)] \int_\kappa \tag{3.25}$$

and the abbreviation

$$\int_\kappa = \int_0^1 \frac{dx}{2x(1-x)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d^2\kappa^\perp}{(2\pi)^3}\,. \tag{3.26}$$

The expectation value of the Hamiltonian in the static approximation involves the vertex described by Eq. (3.24). Namely,

$$P_s^- = \sum_{n=1}^{A} \sum_{\lambda'\lambda\Lambda} \Bigg[\int_\kappa |g_n^{\lambda\Lambda}(x,\kappa^\perp)|^2 \delta_{\lambda\lambda'} \frac{(xP^\perp + \kappa^\perp)^2 + m^2}{xP^+}$$
$$+ \int_\kappa \int_{\kappa'} g_n^{\lambda'\Lambda *}(x',\kappa'^\perp) \frac{g}{2P^+} \bar{u}_{mk'\lambda'} u_{mk\lambda\Lambda} \langle K'|\varphi_s(0)|K\rangle_A g_n^{\lambda\Lambda}(x,\kappa^\perp) \Bigg]\,. \tag{3.27}$$

Other components of the nuclear matter four-momentum are

$$P_s^\mu = \sum_{n=1}^{A} \sum_{\lambda\Lambda} \int_\kappa |g_n^{\lambda\Lambda}(x,\kappa^\perp)|^2 (xP+\kappa)^\mu\,, \tag{3.28}$$

where we use the property of the relative momentum κ,

$$\kappa = (1-x)k - xK\,, \tag{3.29}$$

that $\kappa^+ \equiv 0$.

It is seen that the expectation values of the kinematical momenta contain only contributions from nucleons, while the Hamiltonian contains contributions of the meson field as well. There is no a priori reason for $P_s^+ = P^+$ and $P_s^\perp = P^\perp$ although in each orbital separately

$$P^\mu = k^\mu + K^\mu \tag{3.30}$$

for $\mu = +, \perp$.

Eqs (3.24), (3.27) and (3.28) define the LF description of nuclear matter in the mean field approximation. By minimizing the mass of nuclear matter, while keeping the baryon number in each level fixed to unity, one obtains separate equations for the wave functions $g_n^{\lambda A}(x,\kappa^\perp)$ and masses of the residual systems. Eq. (3.24) provides the selfconsistent scalar field form factor which corresponds to the Hartree potential in nonrelativistic ET quantum mechanics. For finite systems the Hartree theory described above is very complicated. We consider below a simple case of a formal limit of infinitely large nuclear matter. For sufficiently large number of nucleons A one can introduce continuous labeling of states and obtain analytic formulas for the nucleon mass shift.

The limit of continuous level labeling corresponds to so large a volume that the discrete quantum level labels of nucleon states in the giant body of nuclear matter can be replaced by continuous variables,

$$\begin{aligned} g_n^{\lambda A}(x,\kappa^\perp) &\to g^\lambda_{\eta^+\eta^\perp}(x,\kappa^\perp)\delta_{A\bar{\lambda}}\,, \\ &\to g^\lambda_\eta(x,\kappa^\perp)\,, \\ &\to \sqrt{\frac{(2\pi)^3}{V}}\delta(\eta^+ - xM_A)\delta^2(\eta^\perp-\kappa^\perp)s^\lambda_\eta\,, \end{aligned} \tag{3.31}$$

where the discrete labels of states n have been replaced in the continuous labeling by variables $\eta = (\eta^+,\ \eta^\perp)$, so that

$$\sum_{n=1}^{A} \to \frac{V}{(2\pi)^3}\int d\eta^+ d^2\eta^\perp \equiv V\int_\eta\,, \tag{3.32}$$

where V is the volume of the nuclear matter bound state and s^λ_η is a nucleon amplitude for occupying the state labeled by η. In order to simplify notation we include the minimal relativity factors in the occupation amplitudes and use

$$\psi^\lambda_\eta = s^\lambda_\eta[2\eta^+(2\pi)^3]^{-1/2}. \tag{3.33}$$

From Eq. (3.24) in the continuous level density limit one obtains that the meson field formfactor is

$${}_A\langle P'|\varphi_s(0)|P\rangle_A = 2P^+(2\pi)^3\delta^3(P-P')\varphi \tag{3.34}$$

where the constant φ is

$$\begin{aligned} \varphi = -\frac{g}{\mu^2}\int_\eta \sum_{\lambda'\lambda} \psi^{\lambda'*}_\eta \bar{u}_{mk\lambda'}u_{mk\lambda}\psi^\lambda_\eta(2\eta^+)^{-1} \\ \times\left[1+\frac{g^2}{\mu^2}\int_\eta\sum_{\lambda'\lambda}\psi^{\lambda'*}_\eta\bar{u}_{mk\lambda'}\gamma^+u_{mk\lambda}\psi^\lambda_\eta(2\eta^+k^+)^{-1}\right]^{-1}. \end{aligned} \tag{3.35}$$

Therefore, the four-momentum expectation values are

$$P_s^- = V \sum_{\lambda\lambda'} \int_\eta \left[\delta_{\lambda\lambda'} |\psi_\eta^\lambda|^2 \frac{[(\eta^+/M_A)P^\perp + \eta^\perp]^2 + m^2}{(\eta^+/M_A)P^+} + \psi_\eta^{\lambda'*} \bar{u}_{mk\lambda'} u_{mk\lambda} \psi_\eta^\lambda \frac{g\varphi}{(\eta^+/M_A)P^+} \right] \tag{3.36}$$

and

$$P_s^\mu = V \sum_\lambda \int_\eta |\psi_\eta^\lambda|^2 [(\eta^+/M_A)P^\mu + \kappa^\mu] , \tag{3.37}$$

for $\mu = +, \perp$.

The variational parameters in the continuum level density limit are the occupation amplitudes ψ_η^λ. The condition that a level labeled by η contributes a unit of the baryon number is

$$\sum_\lambda |\psi_\eta^\lambda|^2 = 1, \tag{3.38}$$

and we have to introduce a Lagrange multiplier for a corresponding term in the variational principle from Eq. (3.11). The Lagrange multipliers are denoted by $-\omega_\eta^-$ and turn out to determine the separation energies for nucleons occupying levels labeled by η. The variational condition

$$\frac{\delta}{\delta \psi_\eta^{\lambda *}} \left[P_s^- - V \sum_\lambda \int_\eta |\psi_\eta^\lambda|^2 \omega_\eta^- \right] = 0 \tag{3.39}$$

gives

$$(\not{\omega}_\eta - m - g\varphi) \left[1 + \frac{\gamma^+}{2\eta^+} g\varphi \right] \psi_\eta = 0 , \tag{3.40}$$

where we have denoted $\omega_\eta^+ = \eta^+ = k^+$, $\omega_\eta^\perp = \eta^\perp = k^\perp$ and defined the spinor ψ_η as

$$\psi_\eta = \sum_\lambda \psi_\eta^\lambda u_{mk\lambda} . \tag{3.41}$$

Eq. (3.40) is an important result. ψ_η is a Dirac spinor of a nucleon with mass m while $[1 + (\gamma^+/2\eta^+)g\varphi]\psi_\eta$ is a Dirac spinor of mass $m^* = m + g\varphi$. We observe two features. Firstly, an interaction term of the form $(\gamma^+/2i\partial^+)g\varphi$ changes the fermion mass in LF spinors by $g\varphi$. Secondly, the Dirac equation in which the shifted nucleon mass appears, is an equation not for the nucleon wave function but for the occupation amplitude for a state labeled by η. The label η coincides with the nucleon momentum k only in the continuous level density limit.

We can change notation for η by introducing a three-vector label $\vec{\eta} = (\eta^{\perp}, \eta^3)$, where

$$\frac{\eta^+}{M_A} = \frac{e^*_\eta + \eta^3}{e^*_\eta + \sqrt{M^2_\eta + (\vec{\eta})^2}} \tag{3.42}$$

and

$$e^*_\eta = \sqrt{m^{*2} + (\vec{\eta})^2}\,. \tag{3.43}$$

One can solve Eq. (3.42) for the mass M_η of the residual system of $A-1$ nucleons,

$$M_\eta = M_A - e^*_\eta\,. \tag{3.44}$$

This equation is valid when both M_η and M_A are much larger than m^* and k_F, and becomes exact in the infinite nuclear matter limit. In this limit, the rest frame of nuclear matter coincides with the rest frame of a residual system of $A-1$ nucleons.

e^*_η equals a free energy of a nucleon of momentum $\vec{\eta}$ and mass m^*. It can be used to calculate the nucleon separation energy. Although the separation energy appears to be a difference between energies of a free nucleon and a nucleon of the same three-momentum and the effective mass m^*, the nucleon masses have not been physically altered. We have no antinucleons in the trial state nor mesons which locally dress up nucleons, but there is a uniform background meson field which influences energy of nucleons and it appears as if the nucleons had a different mass. The origin and interpretation of the mass shift are confused in ET dynamics by the presence of negative energy solutions to the Dirac equation and the question about how many antinucleons are present in large nuclei. The effective mass is merely a parameter which defines the Lagrange multipliers required by the baryon number conservation.

The notion of Fermi momentum in LF dynamics has been introduced for the first time in this model. One has to specify what states are occupied in the nuclear matter at rest. The trial guess is defined using the parameters

$$\eta^{\perp} = p^{\perp} \tag{3.45}$$

and

$$\eta^+ = \sqrt{m_F^2 + (\vec{p})^2} + p^3\,. \tag{3.46}$$

The Fermi mass m_F is introduced as a variational parameter. The Fermi sphere is formed by declaring that states with $|\vec{p}| \le k_F$ are occupied and states with $|\vec{p}| \ge k_F$ are empty. k_F is the Fermi momentum. Denoting

$$\int_\eta = (2\pi)^{-3} \int_{\text{occupied states}} d\eta^+ d^2\eta^\perp , \tag{3.47}$$

one obtains

$$E_s = 4V \int_\eta \tfrac{1}{2} \left[\frac{\eta^{\perp 2} + m^2 + mg\varphi}{\eta^+} + \eta^+ \right] . \tag{3.48}$$

The factor 4 results from two spins and two isospins for nucleons in a single momentum state. This is the only effect of isospin in the mean field approximation in this model. The constant φ is given by Eq. (3.35) as

$$\varphi = -\frac{g}{\mu^2}\, 4 \int_\eta m(\eta^+)^{-1} \left[1 + 4g^2\mu^{-2} \int_\eta (\eta^+)^{-1} \right]^{-1} . \tag{3.49}$$

The minimum condition

$$\frac{dE_s}{dm_F} = 0 \tag{3.50}$$

implies

$$m_F = m + g\varphi = m^* . \tag{3.51}$$

The Fermi momentum, k_F, the volume of the nuclear matter bound state, V, and the number of nucleons, A, are related. In order to exhibit this relation one can use the vector $\vec{p}$ defined in Eqs (3.45) and (3.46). We have

$$\int \frac{d\eta^+ d^2\eta^\perp}{\eta^+} = \int \frac{d^3p}{e_p^*} , \tag{3.52}$$

where $e_p^* = \sqrt{\vec{p}^{\,2} + m^{*2}}$, and denote

$$\int_F = (2\pi)^{-3} \int d^3p\, \theta(k_F - |\vec{p}|) . \tag{3.53}$$

Then one obtains

$$\varphi = -\frac{g}{\mu^2} 4 \int_F \frac{m^*}{e_p^*} , \tag{3.54}$$

$$E_s = V \left[4 \int_F e_p^* + \tfrac{1}{2}\mu^2\varphi^2 \right] , \tag{3.55}$$

and the relation

$$A = 4V \int_F . \tag{3.56}$$

The last free parameter of our ansatz for the LF nuclear matter bound state, k_F, is fixed by the variational condition

$$0 = \frac{dE_s}{dk_F} = -\frac{3V}{k_F}\left[\tfrac{1}{2}\mu^2\varphi^2 - 4\int_F \frac{\vec{p}^{\,2}}{3e_p^*}\right] . \tag{3.57}$$

Direct inspection of the formulae for P_s^μ shows that in the rest frame of the nuclear matter the equilibrium condition from Eq. (3.57) implies that $E_s = M_A = P_s^+ = P_s^-$. Eq. (3.57) determines the saturation volume for the giant nucleus formed by A nucleons so that there is no pressure on its surface [37].

Eq. (3.57) is our second important result. It demonstrates that the nucleon momentum sum rule used in the nucleon–parton models in phenomenology of deep inelastic lepton-nucleus scattering is a consequence of the dynamical equilibrium in large nuclei. From Eqs (3.32), (3.48) and (3.57) we have

$$\sum_{n=1}^{A} k_n^+ = 4V \int_\eta \eta^+ = P_s^+ , \tag{3.58}$$

which is the same as the nucleon momentum sum rule in the infinite momentum frame,

$$\int_0^1 dy\rho(y)y = 1 . \tag{3.59}$$

Here $\rho(y)$ is *a density of states* for nucleons carrying the fraction y of longitudinal momentum of a whole nucleus in the infinite momentum frame. The LF results are invariant under boosts along the z-axis and provide understanding of the infinite momentum frame results in the rest frame of a nucleus.

The sum rule also says that the average plus momentum of a nucleon in the model nuclear matter, consisting of A nucleons, is A^{-1}. The Fermi mass $m_F = m^*$ is smaller than the free nucleon mass m. The separation energy depends on labels of nucleon states as if nucleons had mass m^*. The meson field contributes to the mass of nuclear matter. Nevertheless, the nucleons carry, on average, as large a fraction of the total plus momentum of the nucleus as they would in a gas of A noninteracting nucleons. These results are useful in phenomenology of deep inelastic lepton–nucelus scattering.

3.2. Phenomenology of the EMC effect

European Muon Collaboration (EMC) has observed that the deep inelastic lepton-nucleus scattering cross section is not a straightforward superposition of cross sections for scattering on individual nucleons [42]. The structure function $F_{2A}(x)$ of a nucleus built from $A/2$ protons and $A/2$ neutrons is a function of the Bjorken scaling variable x which ranges from 0 to 1. In the deep inelastic lepton-nucleus scattering, the leptons from a beam with energy of many GeV per particle emit hard virtual photons and scatter towards a detector which measures their final energy and scattering angles. The hard photons struck individual quarks in the target nuclei. There is a Bjorken variable x which has the interpretation of a fraction of a nucleus longitudinal momentum carried by the struck quark in the IMF. Function $x^{-1}F_2(x)$ for any target has a parton model interpretation of the probability distribution for finding a quark carrying the fraction x of the total momentum of the target in the IMF. EMC has discovered that the ratio of a structure function of iron per nucleon, $A^{-1}F_{2A}(x/A)$ with $A = 56$, to the nucleon structure function $F_2(x)$, does not equal 1. In the range $0.3 \le x \le 0.8$ the ratio is smaller than 1 by a few percent. For $x \ge 0.8$ the ratio raises to infinity due to the Fermi motion of nucleons and below 0.3 the ratio varies from slightly above to below 1 for x approaching 0. The shallow dip of the ratio at the intermediate values of x has raised speculations that nucleons in nuclei may have significantly altered internal quark structure due to some relativistic binding effects. Therefore, the EMC effect has caught attention of many physicists. Models based on the idea that nucleons have different effective mass when bound in nuclei have been considered. Our results from the previous section add to that discussion and demonstrate that the LF Hamiltonian approach to bound state dynamics can be useful in practice.

For example, Akulinichev, Kulagin and Vagradov have proposed that the binding of nucleons leads to corrections in the nucleon energy and this correction may be responsible for the observed EMC ratio [43]. They used ET dynamics in their reasoning.

The struck quark carrying x of a nucleus momentum belongs to a nucleon which carries y momentum of the nucleus, $x \le y$. In order to calculate the structure function of a nucleus one needs to know the distribution of nucleons in the nucleus, $\rho(y)$. The distribution of nucleons is a sharply peaked function centered at $y \simeq A^{-1}$. A small shift from that point towards smaller y by a few percent could explain the EMC data. In ET dynamics the variable y for nucleons can be defined as

$$y = \frac{p^0 + p^3}{M_A} \tag{3.60}$$

and if p^0 can be slightly smaller than the nucleon mass then the typical values of y could be slightly smaller than A^{-1}. Therefore, the effective mass of nucleons in nuclei which is smaller than the mass of free nucleons could explain the effect. Alternatively, one could replace a free nucleon energy by the number p^0 which corresponds to the shell model energy of a nucleon in a stationary nuclear level.

However, the effect is small and other effects of the same order of magnitude have to be taken into account in addition to the mass shift. For example, there is a problem in ET dynamics that the quantity y from Eq. (3.60) can be less than zero or greater than 1 when p^0 represents the off mass shell energy of a nucleon [44]. This effect also leads to corrections to the momentum sum rule for nucleons in relativistic many-body theory [45]. The same problem appears when structure functions are analyzed in a bag model [46].

In LF dynamics the variable y is defined as

$$y = \frac{p^+}{P^+} \tag{3.61}$$

and ranges from 0 to 1 according to the parton model interpretation. In the previous Section we have shown that the nucleon effective mass m^* obtained in the mean field or Hartree approximation for nuclear matter in a Yukawa model does not shift the peak in $\rho(y)$ towards $y < A^{-1}$. The momentum sum rule for nucleons is a consequence of dynamical equilibrium which includes the binding. Additional study of the Walecka model in the mean field approximation including vector mesons may explain if a shift towards smaller y can be obtained in the presence of the vector meson field.

Głazek and Schaden have analyzed data for various nuclei using LF description of nuclear bound states [47]. They assumed that a small part of the total P^+ of a nucleus is carried by mesons, predominantly by pions. They have obtained a fit to all available data for structure functions of nuclei with mass numbers A ranging from 2 for a deuteron to 197 for gold with χ^2 of order 1 using the assumption that the momentum carried by mesons is a few percent and slowly grows for heavier nuclei. The success of the model indicates that one should consider a possibility that mesons do carry a visible part of the nucleus inertia. Then, one would conclude that the mean field approximation is not sufficient for explaining the data, unless the average vector meson field can produce the shift in $\rho(y)$. In order to confirm or exclude that the effective mass shift for nucleons may explain the EMC effect one needs to consider mean field theory with vector mesons and consider Fock sectors which contain bare mesons in addition to the nucleons.

When the meson states are taken into account one encounters effects of retardation in the nucleon–nucleon interactions and also self-interactions of nucleons, similar to those that one encounters in positronium (Section 2).

However, the nuclear forces are much stronger than the electromagnetic ones and approximations valid in QED are not theoretically expected to be reasonably good in nuclear dynamics. A host of problems arise in the relativistic bound state problem in nuclear physics since all terms we could initially neglect in positronium are not negligible in nuclei. Strong form factors in nonlocal nucleon-meson vertices are introduced in nuclear calculations in order to dampen these problems. The EMC effect shows that the lack of precise method of describing relativistic effects in bound state dynamics poses practical problems in explaining experimental data. Even larger relativistic effects appear in dynamics of quarks in nucleons.

4. Quarks in nucleons

Nucleons are relativistic bound states of quarks. The binding of quarks in nucleons is very strong, nonlocal and confining. By strong we mean that quarks move fast in a relatively small volume. The interaction is nonlocal since it binds quarks into a color singlet although the quarks occupy different positions. The structure of nucleons is not yet quantitatively understood in a theory based on first principles. In QCD, in addition to the effects of emission and absorption of vector bosons by fermions, one encounters two problems. Firstly, the bosons strongly interact with themselves (nonabelian gluon coupling). Secondly, the QCD ground state is a complicated sea of condensed quarks and gluons and we do not know precisely how to describe this medium or compute its excitations.

Despite the theoretical problems with solving QCD at hadronic scales, phenomenological knowledge about quarks in nucleons is quite rich. There is a quark model discovered by Gell-Mann and Zweig [48, 49]. There is Feynman parton model [50]. There are many other models designed for description of specific hadrons, processes or aspects of the hadronic structure. Recent progress in the description of hadrons has been achieved by Shifman, Veinshtein and Zakharov in QCD sum rules [51]. The sum rules greatly differ from other models by that they are directly related to QCD itself. The main model assumptions involved in the sum rules are the structure of operators which create or annihilate hadrons and the validity of dispersion relations which connect results obtained in the deep Euclidean asymptotically free region in perturbation theory in QCD to the properties of the hadronic spectrum. Major technique of the sum rules is the short distance operator product expansion (OPE) [52]. OPE is used for products of various local current operators whose vacuum matrix elements are connected to observables for hadrons by the dispersion relations.

In this Section we discuss LF phenomenology of nucleon states which originates in ideas taken from QCD sum rules. Our discussion is based on a model developed by Głazek, Głazek, Namysłowski and Werner [53]. The original derivation of the model is described in Ref. [54].

4.1. Phenomenology of vacuum condensates and running quark masses

The QCD sum rules suggest that it is useful to introduce universal phenomenological parameters called quark and gluon condensates. Namely,

$$\nu^4 = \langle \Omega | \frac{\alpha_s}{\pi} : G^a_{\mu\nu} G^{a\mu\nu} : | \Omega \rangle \tag{4.1}$$

and

$$-\chi^3 = \langle \Omega | \sqrt{\alpha_s} : \bar{\psi}\psi : | \Omega \rangle \,, \tag{4.2}$$

where $G^a_{\mu\nu}$ symbolizes the gluon field operator, ψ the quark field operator, and α_s is the coupling constant of QCD. Commonly accepted values of the condensates are

$$\nu \sim 0.4 \text{ GeV} \tag{4.3}$$

and

$$\chi \sim 0.24 \text{ GeV}\,. \tag{4.4}$$

The normal ordering symbol indicates that the divergence in the product of fields at the same point is subtracted. In practical calculations, the condensates appear as phenomenological constants in the momentum space moments of various Green functions. The moments are obtained by integrating Green functions over regions of soft momenta where perturbation theory is not applicable. One relies on general properties of the Green functions and parametrizes the unknown integrals using the constants like ν or χ. The unexpected result of the QCD sum rules is that the condensates have rather universal values and can be associated with the vacuum properties, independently of what kind of a hadronic sum rule is considered.

Ioffe has estimated the mass of a nucleon in QCD using sum rules [55]. He used three different current operators which have nonvanishing matrix elements between the vacuum state and the nucleon state. The nucleon mass is proportional to the quark condensate. Note that the quark condensate is a signature of spontaneous chiral symmetry breaking in the QCD vacuum.

Politzer has noticed earlier that the quark condensate induces a mass-like term in a quark propagator [56]. This term is a part of the quark self--energy. One considers a virtual quark propagation with a four-momentum p, such that $|p^2| \gg \Lambda^2_{QCD}$. The self-energy diagram contains a mass-like term proportional to $\chi^3 p^{-2}$. This type of a momentum dependent mass-like term is called a running mass. The running mass is small for highly virtual quarks and becomes large for p^2 on the order of masses of nucleons, or constituent quarks. Politzer has suggested that the constituent quark masses may originate from self-energy effects which involve the vacuum structure.

The running mass idea in LF dynamics of quarks was introduced in a model which we briefly present below. The model is useful for the following purposes. It provides a new method of constructing LF quark wave functions for nucleons with all necessary details, and gives realistic examples of such wave functions. It illustrates how fast quarks have to move in order to explain some nucleon properties in the model. It points out that the vacuum condensates may be a useful concept in LF dynamics. The latter is important, since naively the vacuum state in LF dynamics is trivial and there is no vacuum condensates.

4.2. Model wave function for nucleons

The nucleon state is constructed using operators which are similar to the Ioffe currents from the QCD sum rules. The quark masses are functions of quark momenta. The LF formalism provides the machinery which allows calculations of form factors and structure functions.

The nucleon state is represented as three quarks. The only parameters which reflect the complicated structure of the quarks is their running masses. The state is of the form

$$|P\lambda\rangle = \sum_{\lambda_1\lambda_2\lambda_3} \int [dp_1][dp_2][dp_3]\, \psi^{\lambda}_{\lambda_1\lambda_2\lambda_3}(p_1,p_2,p_3) \times \varepsilon^{abc} u^{a\dagger}_{p_1\lambda_1} u^{b\dagger}_{p_2\lambda_2} d^{c\dagger}_{p_3\lambda_3} |\Omega\rangle\,, \tag{4.5}$$

where we use conventions familiar from previous Sections. Indices a, b and c are color indices. $|\Omega\rangle$ is the physical vacuum state. $u^\dagger$ creates the vacuum excitations called quarks up and $d^\dagger$ creates excitations called quarks down. The quarks are assigned three-momenta p_1, p_2 and p_3 and the effective wave function is of the form

$$\psi^{\lambda}_{\lambda_1\lambda_2\lambda_3}(p_1,p_2,p_3) = N\,\delta^3(p_1+p_2+p_3-P)\, f\left(\mathcal{M}^2_{123}\right) \times \sum_{k=1}^{3} a_k I_k(q_i, Q_i, \lambda_1, \lambda_2, \lambda_3)\,, \tag{4.6}$$

where N is a normalization factor which gives

$$\langle P'\lambda'|P\lambda\rangle = 2P^+(2\pi)^3\delta^3(P-P')\delta_{\lambda\lambda'}\,, \tag{4.7}$$

when the creation and annihilation operators for the quark excitations of the vacuum are given standard anticommutation relations, Eq. (2.18), enriched with color and isospin indices. The argument of the function f is a square of a four-vector built from individual momenta of the three quarks, $\mathcal{M}_{123}^2 = (p_1+p_2+p_3)^2$. The quark's momenta have $+$ and $\perp$ components constrained by the δ-function in front of f and the condition that $p_i^+ > 0$ for $i = 1, 2, 3$. The minus components are calculated using the running mass formula [57]

$$p_i^- = \frac{p_i^{\perp 2} + m_i^2}{p_i^+}, \tag{4.8}$$

where m_i is a function of the i-th quark virtuality. The quark virtuality is defined as

$$\hat{p}_i^2 = (P - p_j - p_k)^2 \tag{4.9}$$

for $i \neq j \neq k \neq i$. The nucleon mass is denoted by M. Thus,

$$p_1^\mu + p_2^\mu + p_3^\mu = P^\mu + \tfrac{1}{2}\eta^\mu(p_1^- + p_2^- + p_3^- - P^-), \tag{4.10}$$

where the light-front four-vector η has components

$$(\eta^\mu) = (\eta^- = 2,\ \eta^+ = 0,\ \eta^\perp = 0) \tag{4.11}$$

and

$$\mathcal{M}_{123}^2 = M^2 + P^+(p_1^- + p_2^- + p_3^- - P^-) \tag{4.12}$$

since $\eta^2 = 0$. This illustrates how the difference between the energy of an intermediate state of quarks and the energy of a nucleon, appears in LF dynamics in the invariant mass expression for the quarks. We also have the following expression for the virtuality defined in Eq. (4.9),

$$\hat{p}_i^2 - m_i^2 = x_i(M^2 - \mathcal{M}_{123}^2), \tag{4.13}$$

where the fractions x_i for $i = 1, 2, 3$ are equal p_i^+/P^+ and sum up to 1. The running quark mass is chosen in the form [56, 57, 58]

$$m_i \sim \frac{\chi^3}{m_i^2 - \hat{p}_i^{\,2}}. \tag{4.14}$$

For large negative virtualities the running mass in the denominator is negligible. This running mass formula has interesting properties. If one insists on the consistency of Eqs (4.8), (4.9) and (4.14) with a coefficient of order 1 for all allowed values of the quark momenta p_i^+ and $p_i^\perp$ for $i = 1, 2, 3$ in the nucleon state, one obtains reasonable expressions for the running quark masses. These running masses are close to the constituent quark masses for small quark virtualities, vanish for the large virtualities, and smoothly interpolate between these limiting values for intermediate quark momenta. Numerical consequences of such running quark masses for nucleon observables are discussed in Ref. [58]. The above definitions of virtualities and running masses can be generalized to states with arbitrarily many constituent particles.

The function $f(\mathcal{M}_{123}^2)$ in Eq. (4.6) determines the size of a nucleon state. If it is chosen to be

$$f(\mathcal{M}_{123}^2) = \exp\left(-\frac{\mathcal{M}_{123}^2}{6\alpha^2}\right), \tag{4.15}$$

then the parameter α plays the same role here as the analogous size parameter in a nonrelativistic constituent quark model of Isgur and Karl [59].

The three spin and momentum dependent functions $I_k(q_i, Q_i, \lambda_1, \lambda_2, \lambda_3)$ which appear in the linear combination with coefficients a_k , $k = 1, 2, 3$, in Eq. (4.6), are constructed using the three Ioffe currents from the nucleon sum rules in QCD [55]. The Ioffe spin-$^1/_2$ currents have the structure

$$\eta(x) = u^T \Gamma_1^k C u(x)\, \Gamma_2^k d(x)\,, \tag{4.16}$$

where C is a charge conjugation matrix, $u(x)$ is a quark up field and $d(x)$ is a quark down field, and there are various possibilities to choose Dirac matrices Γ_1 and Γ_2, numbered by k. The fermionic index of the current is the left index of the matrix Γ_2^k. The corresponding functions in Eq. (4.6) are [53, 54]

$$I_k(q_i, Q_i, \lambda_1, \lambda_2, \lambda_3) = \bar{u}_{m_1 p_1 \lambda_1} \Gamma_1^k C \bar{u}_{m_2 p_2 \lambda_2}^T \, \bar{u}_{m_3 p_3 \lambda_3} \Gamma_2^k u_{MP\lambda}\,, \tag{4.17}$$

where the spinors are defined in Eq. (2.20). Each quark spinor involves its own running mass. $u_{MP\lambda}$ is the nucleon spinor which contains the nucleon mass. The coefficients a_1, a_2 and a_3 are not known ab initio. One can check if there is a choice which can fit some nucleon observables. The arguments of I_k are the quark spin projections on z-axis and the relative momenta of quarks, called Jacobi momenta, [17]

$$q_k = (x_j p_i - x_i p_j)(x_i + x_j)^{-1} \tag{4.18}$$

and

$$Q_k = (x_i + x_j) p_k - x_k (p_i + p_j)\,, \tag{4.19}$$

with i, j, k in a cyclic triple. Any choice of a pair (q_i, Q_i) with $i = 1$, $i = 2$ or $i = 3$ in Eq. (4.17), is equally valid since the relative Jacobi momenta from one pair can be evaluated in terms of momenta from another pair. Eq. (4.17) indicates that the functions I_k are independent of the total nucleon momentum P.

The nucleon wave function from Eq. (4.6) obeys all kinematical symmetries of LF dynamics. The wave function describes a nucleon as a superposition of various quark excitations of the vacuum. Various excitations are assumed to differ by their masses. The assignment of masses to the quark excitations is correlated with their relative momenta. The excitations can be the constituent quarks for small relative momenta, or correspond to the valence Fock state of three massless quarks for very large relative momenta. The interpolation over many scales of momentum is made using the simplifying assumption that the whole complexity of the quark and gluon dynamics can be parametrized by the running quark masses only. At very large virtualities the quark masses are negligible. For smaller virtualities the running masses grow in analogy to the self-energy of virtual quarks due to the quark condensate. And for quite small virtualities the selfconsistency of the running mass formula provides expressions for the constituent quark masses which mimic the unknown dressing of quarks with quark-antiquark pairs and gluons.

Numerical analysis of the model nucleon state from Eq. (4.5) leads to the following conclusions [53, 54, 58]. The coefficients of the three Ioffe currents can be chosen together with the radius parameter α from Eq. (4.15) so that all static electroweak properties of protons and neutrons can be reproduced with less than 10% error. One should note that the calculation of a charge radius or a magnetic moment of a nucleon requires inclusion of the recoil effects and the LF scheme is the only one in which this step is under some control for relativistic bound states. The momentum distribution for quarks has to be about 40% broader than in the Isgur–Karl model [59] and the nucleon constituents are highly relativistic. The model of Eq. (4.5) explains also the falling off ratio of the d-quark parton model distribution to the u-quark distribution in the proton for the Bjorken scaling variable $x \to 1$. It is hard to correlate good results for deep inelastic structure of nucleons with good fits to the static nucleon properties in other models.

The LF definition of the nucleon state provides a probabilistic interpretation of the nucleon structure functions in terms of parton distributions. The parton distributions as functions of the longitudinal scaling variable x, are given by the modulus squared of the LF quark wave functions integrated over transverse momenta. For example,

$$\lim_{x\to 1} \frac{d(x)}{u(x)} = 4\left(3 - \frac{a_3}{a_1}\right)^2 \left[\left(\frac{a_3}{a_1}\right)^2 + \left(6 - \frac{a_3}{a_1}\right)^2\right]^{-1} . \tag{4.20}$$

From this result, it is clear that the experimental value which is close to 0 requires a significant presence of the third Ioffe current in our nucleon state. The contribution of the third Ioffe current vanishes in the nonrelativistic limit $c \to \infty$. This fact indicates again that in order to explain the structure of nucleons one needs a relativistic theory of the bound states of quarks.

The analysis described in Ref. [53] was the first three-body LF model description of a whole set of nucleon observables. Further studies could include calculations of form factors and structure functions under additional assumptions about internal structure of the quark excitations of the vacuum which form nucleons in the model. Such studies have not been carried out yet. Building a theory of the nucleon bound states requires precise definitions of various elements of the model which are defined merely intuitively.

A series of investigations of new LF models of hadrons by Dziembowski and Mankiewicz has followed construction of this model [60]. Vast amount of examples of insight into the structure of hadrons is provided by many earlier and more recent LF models. Special summaries are given in Refs [17] and [18], and also in Ref. [61] where various hadronic phenomena in high energy nuclear physics are considered. Relativistic effects in LF models of nucleons have been recently discussed by Chung and Coester [62].

5. Condensates in LF QCD

The confinement of quarks and gluons and formation of hadrons are associated with a nontrivial structure of the QCD vacuum. The sum rules of QCD explicitly involve parameters which have an interpretation of the vacuum condensates. These parameters correlate various hadronic observables. LF models which are based on similar ideas, are capable of describing some properties of nucleons. LF QCD is also useful in describing phenomena dominated by short distance dynamics, like for example large momentum transfer exclusive processes where one may argue for applicability of perturbation theory [16]. LF perturbation theory is rapidly developing [63]. In QCD hadrons are represented as multicomponent states in the LF Fock space. The Fock space wave functions naturally support the parton model picture of hadrons. Special kinematical symmetries of the LF form of dynamics offer a possibility to boost bound states, so that one may correlate the structure of hadronic states in the IMF with their structure at rest. One may hope that the LF formulation of QCD may soon unify the constituent quark model and the parton model in the ultimate theory of hadrons. However, there is a conceptual problem involved in this picture [22]. Namely, it is not clear how the nontrivial vacuum structure manifests itself in LF QCD.

The problem with understanding how a physical vacuum state can be formed in LF QCD can be illustrated by comparison with ET dynamics in which the QCD ground state is very complicated.

In ET dynamics one can expand quantum fields at a given moment of time into their Fourier components. The Fourier components involve operators which satisfy commutation relations characteristic for creation and annihilation operators of bare free particles. Using these operators one can build a whole Fock space of states created by products of the creation operators acting on a vacuum state. Then one can ask how does the vacuum state evolve in time. Unfortunately, the operator which generates the evolution, the ET Hamiltonian, does not lead to a simple answer. A translationally invariant Hamiltonian for quantum fields is written as an integral over a three-dimensional space. The space is defined at the given moment of time. The integrand of this integral is a collection of various terms which contain various products of the field operators. In the case of interacting fields there appear terms which contain products of at least three quantum fields. Such products contain terms which are in turn products of creation and annihilation operators. It is important to observe that some terms contain only creation operators. Integration over the space coordinates produces the condition that a sum of the three-momentum components of all creation operators in the product must be zero. Such terms in the Hamiltonian create particles from the Fock space vacuum. In perturbation theory these terms lead to a spontaneous creation of particles, including disconnected vacuum excitations. The spontaneously created particles have the total three-momentum equal zero in the chosen frame of reference. But they can fly apart unconstrained in all directions. When the Hamiltonian is exponentiated to generate the time evolution of states, then arbitrary high powers of terms involving only creation operators appear, and arbitrary many particles are created from the vacuum state. There is a problem of finding states which are sufficiently stable in time and do not spread to new Fock sectors for ever. In other words, the eigenstates of the ET Hamiltonian cannot be approximated by states with finite numbers of bare particles. So far, this problem prevents computations in the ET Hamiltonian formulation of QCD, since we cannot calculate the necessary infinitely many numbers. We have not guessed yet how to approximate the stable states of the theory so that corrections would be small and could be calculated. In perturbation theory one can eliminate the disconnected diagrams. Problems appear when the coupling constants are not sufficiently small so that one cannot approximate solutions using Feynman diagrams. Acute practical problems appear in bound state equations, where perturbation theory to a finite order is not sufficient even if the coupling constants are not large.

In LF dynamics the situation is surprisingly different. If the reader had an impression that the LF dynamics is merely a sort of a change of variables this is a place where it should become clear to her how complicated this change is. Namely, when one expands the same quantum fields into their Fourier components on a LF then the corresponding Hamiltonian, P^- instead of P^0, does not contain terms which only create particles from the vacuum. Such terms are absent. There is only a possibility that already existing particles emit or absorb other particles, but the vacuum stays untouched. It happens in the following way. Let us use the examples of field expansions from Eqs (2.22) and (2.23). From Eqs (2.15) to (2.17) we see that a product of n fields may contain a term which is a product of n creation operators and no annihilation operator. However, such a product is multiplied by an integral of the type

$$\int_{-\infty}^{+\infty} dx^- \exp \frac{i}{2}(k_1^+ + \cdots + k_n^+)x^- = 4\pi\delta(k_1^+ + \cdots + k_n^+) \tag{5.1}$$

and all individual momenta $k_i^+ > 0$, $i = 1, 2, \ldots, n$. Therefore, if the point $k^+ = 0$ in the Fourier expansions is excluded, there are no terms in the LF Hamiltonian which could spontaneously create particles from the vacuum. The same argument applies in case of all normal-ordered terms.

The limit $k^+ \to 0$ is not simple. On the one hand, it is an infrared limit of plane waves with wavelength going to infinity. This corresponds to the infinite volume limit of a theory defined in a box of a large volume. Therefore, the singularities which appear when $k^+ \to 0$ may introduce complicated effects, typical for the infinite volume limits and known to introduce nonanaliticities [64]. On the other hand, the limit $k^+ \to 0$ can be viewed for massive particles as an ultraviolet limit. For a free particle of mass m we have $k^+ = \sqrt{(\vec{k})^2 + m^2} + k^3 \geq 0$, and the possibility that $k^+ \to 0$ appears only when $(k^3/m) \to -\infty$. In fact, both singularities appear mixed in practical calculations. In gauge theories, a choice of the gauge $A^+ = 0$ leads to additional singularities for $k^+ \to 0$ [65].

So, one might think that the LF formulation of QCD is destined for failure due to appearance of such complicated singularities. However, at the same time many fundamental problems are buried in one place. Namely, on the boundary $k^+ = 0$ of the Fourier expansion of quark and gluon fields. Therefore, the LF formulation offers a unique setup of the theory. There is a large domain of the LF Fock space in which the Hamiltonian and other operators are quite regular and possess helpful symmetries, not available in ET dynamics. The corresponding annihilation and creation operators do not contribute to the vacuum structure. And there is a boundary at $k^+ = 0$ which is a source of new effects.

Since the important singularities come from the region of small k^+ one can do the following. One can split the region of integration over k^+ into two by introducing a splitting point at $k^+ = \delta$, where δ is an arbitrarily small number in comparison to the momenta of incoming or outgoing hadrons. Creation and annihilation operators above δ do not change the vacuum and lead to expressions for various quantities as if the vacuum was trivial. The operators below δ are responsible for the vacuum formation and their matrix elements in the vacuum bring in new effects. In this situation one is offered an opportunity to construct semi-phenomenological approaches to the physics of hadrons. One can replace the Fourier expansion of fields into creation and annihilation operators with $k^+ < \delta$ by some effective model Fourier coefficients which can mimic the vacuum degrees of freedom and supply effects of the missing vacuum structure. For example, one can construct some effective low k^+ Fourier components which reproduce results from the QCD sum rules. Thus, we are led to the new concept that the LF Hamiltonian of QCD contains special terms which reproduce effects associated with the vacuum structure in ET dynamics [22].

5.1. Condensates in old-fashioned LF perturbation theory [22]

The Lagrangian density of chromodynamics is

$$\mathcal{L}_{CD} = -\tfrac{1}{2}\,\mathrm{Tr}\, F^{t\mu\nu}F_{t\mu\nu} + \bar{\psi}_t(i\,\not{D}_t - m)\psi_t\,, \tag{5.2}$$

where the total quark field ψ_t and gluon field A_t are each split into two:

$$\psi_t = \psi + \omega \tag{5.3}$$

and

$$A_t = A + a\,. \tag{5.4}$$

The fields ψ and A are standard LF quark and gluon fields comprising creation and annihilation operators with $k^+ \geq \delta$. The fields ω and a are unknown operators involving arbitrarily small $k^+ < \delta$. These fields are called background fields. In order to be able to solve the constraint equations the background fields are requested to satisfy their own equations of motion:

$$d^\mu f_{\mu\nu} = g\bar{\omega}\gamma_\nu T^a \omega T^a \tag{5.5}$$

and

$$(i\,\not{d} - m)\omega = 0\,, \tag{5.6}$$

where the color matrices are normalized as

$$\mathrm{Tr}\, T^a T^b = \tfrac{1}{2}\delta^{ab} \tag{5.7}$$

and

$$d^\mu = \partial^\mu + iga^\mu\,, \tag{5.8}$$

$$f^{\mu\nu} = (ig)^{-1}[d^\mu, d^\nu]\,. \tag{5.9}$$

Using similar procedure as in electrodynamics in Section 2.1. one can write the LF Hamiltonian for chromodynamics as

$$H = \int dx^- d^2x^\perp \mathcal{H}\,, \tag{5.10}$$

where

$$\begin{aligned}
\mathcal{H} =& \psi_+^\dagger \sigma \frac{1}{i\partial^+} \sigma \psi_+ + \omega_+^\dagger \sigma \frac{1}{i\partial^+} \sigma \psi_+ + (\sigma \frac{1}{i\partial^+} \sigma \psi_+)^\dagger \omega_+ \\
&+ g(\bar{\tilde{\psi}} + \bar{\omega})(\tilde{\not{A}} + \not{a})(\tilde{\psi} + \omega) - g\bar{\omega}\,\not{a}\omega \\
&+ g^2 \Big[(\bar{\tilde{\psi}} + \bar{\omega})\tilde{\not{A}} \frac{\gamma^+}{2i\partial^+} \tilde{\not{A}} (\tilde{\psi} + \omega) + \bar{\tilde{\psi}}\,\not{a} \frac{\gamma^+}{2i\partial^+}\,\not{a}\tilde{\psi} \\
&+ \bar{\tilde{\psi}}\tilde{\not{A}} \frac{\gamma^+}{2i\partial^+}\,\not{a}\tilde{\psi} + \bar{\tilde{\psi}}\,\not{a} \frac{\gamma^+}{2i\partial^+} \tilde{\not{A}}\tilde{\psi} + \bar{\omega}\tilde{\not{A}} \frac{\gamma^+}{2i\partial^+}\,\not{a}\tilde{\psi} + \bar{\tilde{\psi}}\,\not{a} \frac{\gamma^+}{2i\partial^+}\tilde{\not{A}}\omega \Big] \\
&+ g^2\, \mathrm{Tr}\, \Psi \frac{1}{(i\partial^+)^2} \Psi \\
&+ \mathrm{Tr}\, \partial^i \tilde{A}^j \partial^i \tilde{A}^j + \mathrm{Tr}\, \partial^i \tilde{A}^j \partial^i a^j + \mathrm{Tr}\, \partial^i a^j \partial^i \tilde{A}^j \\
&+ 2g\, \mathrm{Tr}\, i\partial^\alpha (\tilde{A}^\beta + a^\beta) \left[\tilde{A}_\alpha + a_\alpha, \tilde{A}_\beta + a_\beta\right] - 2g\, \mathrm{Tr}\, i\partial^\alpha a^\beta \left[a_\alpha, a_\beta\right] \\
&- \tfrac{1}{2} g^2\, \mathrm{Tr} \left[\tilde{A}^\alpha + a^\alpha, \tilde{A}^\beta + a^\beta\right] \left[\tilde{A}_\alpha + a_\alpha, \tilde{A}_\beta + a_\beta\right] \\
&+ \tfrac{1}{2} g^2\, \mathrm{Tr} \left[a^\alpha, a^\beta\right] \left[a_\alpha, a_\beta\right]
\end{aligned} \tag{5.11}$$

and $\Psi = \Psi^a T^a$;

$$\begin{aligned}
\Psi^a =& (\bar{\tilde{\psi}} + \bar{\omega})\gamma^+ T^a (\tilde{\psi} + \omega) - \bar{\omega}\gamma^+ T^a \omega \\
&- \left[i\partial^+ (\tilde{A}^\alpha + a^\alpha), \tilde{A}_\alpha + a_\alpha\right]^a + \left[i\partial^+ a^\alpha, a_\alpha\right]^a .
\end{aligned} \tag{5.12}$$

The Hamiltonian of chromodynamics looks much more complicated than the Hamiltonian of electrodynamics given in Eqs (2.9) and (2.10), due to the nonabelian couplings of gluon fields to quark fields and the gluon field self-couplings. Additional complications are introduced by the presence of the background fields in the quantum chromodynamics.

The quark and gluon fields are quantized by expanding into Fourier components at $x^+ = 0$,

$$\tilde{\psi}(x) = \int_{k^+ > \delta} [dk] \sum_{\lambda c} \left[b^c_{k\lambda} u_{mk\lambda} u_c e^{-ikx} + d^{c\dagger}_{k\lambda} v_{mk\lambda} v^*_c e^{ikx} \right] \tag{5.13}$$

and

$$\tilde{A}^{\mu}(x) = \int\limits_{k^+>\delta} [dk] \sum_{\lambda c} \left[a^c_{k\lambda} \varepsilon^{\mu}_{k\lambda} T^c \mathrm{e}^{-ikx} + a^{c\dagger}_{k\lambda} \varepsilon^{\mu *}_{k\lambda} T^c \mathrm{e}^{ikx} \right] , \tag{5.14}$$

and imposing familiar commutation relations

$$\begin{aligned} \left\{ b^c_{k\lambda}, b^{c'\dagger}_{k'\lambda'} \right\} = \left\{ d^c_{k\lambda}, d^{c'\dagger}_{k'\lambda'} \right\} &= \left[a^c_{k\lambda}, a^{c'\dagger}_{k'\lambda'} \right] \\ &= 16\pi^3 k^+ \delta^3(k - k')\, \delta_{\lambda\lambda'}\, \delta^{cc'} \end{aligned} \tag{5.15}$$

with all other anticommutators and commutators for quark and gluon fields equal zero, respectively. Renormalization effects are not taken into account at this point. c denotes color indices, running from 1 to 3 for quark operators and from 1 to 8 for gluon operators.

The background fields can be written in the LF gauge $A^+ = a^+ = 0$ using the know expansion of long wavelength fields in the Fock-Schwinger gauge [66] and making a gauge transformation to the LF gauge. In the following discussion we need only leading terms in the background field Fourier expansion which read

$$a^{\mu}(x) = \int\limits_{k^+<\delta} d^4k \left\{ \tfrac{1}{2} \left[f_0^{\rho\mu} + g^{\rho}_{+} f_0^{+\mu} + g^{\mu}_{+} f_0^{+\rho} \right] \left[i \frac{\partial}{\partial k^{\rho}} \delta^4(k) \right] + \cdots \right\} \mathrm{e}^{ikx} \tag{5.16}$$

and

$$\omega(x) = \int\limits_{k^+<\delta} d^4k \left\{ \omega_0[\delta^4(k)] + d^{\rho}\omega_0 \left[i \frac{\partial}{\partial k^{\rho}} \delta^4(k) \right] + \cdots \right\} \mathrm{e}^{ikx} . \tag{5.17}$$

Note that the gluon field expansion starts from terms which are linear in x and no constant zero-mode is introduced. A constant zero-mode appears in the background field strength $f^{\mu\nu}$. According to the conjecture of the QCD sum rules we postulate that the vacuum matrix elements of the background fields are

$$\langle \Omega | \bar{\omega}^a_{0\alpha} \omega^b_{0\beta} | \Omega \rangle = \tfrac{1}{3} \delta^{ab} \tfrac{1}{4} \delta_{\alpha\beta} \langle \Omega | \bar{\omega}_0 \omega_0 | \Omega \rangle , \tag{5.18}$$

$$\langle \Omega | \bar{\omega}^a_{0\alpha} d^{\mu} \omega^b_{0\beta} | \Omega \rangle = \tfrac{1}{48i} \delta^{ab} (\gamma^{\mu})_{\beta\alpha} m \langle \Omega | \bar{\omega}_0 \omega_0 | \Omega \rangle , \tag{5.19}$$

and

$$\begin{aligned} &\langle \Omega | g^2 f_0^{\alpha\beta a} f_0^{\gamma\delta b} | \Omega \rangle \\ &\quad = \tfrac{1}{96} \delta^{ab} (g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma}) \langle \Omega | g^2 f^c_{0\mu\nu} f_0^{c\mu\nu} | \Omega \rangle . \end{aligned} \tag{5.20}$$

The matrix elements on the rhs' can be identified with the quark and gluon condensates known from the sum rules. A sketch of the proof from Ref. [22] is following.

The vacuum polarization considered in the charmonium sum rules is

$$\Pi^{\mu\nu}(q) = i \int d^4x e^{iqx} \langle \Omega | T_+ [J_t^\mu(x), J_t^\nu(0)] | \Omega \rangle , \tag{5.21}$$

where

$$J_t^\mu(x) =: \bar{\psi}_t(x) \gamma^\mu \psi_t(x) : \tag{5.22}$$

is the quark current operator which has nonzero matrix elements between the vacuum and the charmonium states. T_+ denotes the x^+ ordering symbol. Assuming that the x^+-evolution of fields is given by an exponential of the Hamiltonian from Eq. (5.10) one can compute $\Pi^{++} = q^+ q^+ \Pi(q^2)$ in the old-fashioned perturbation theory. The background fields commute with the standard creation and annihilation operators with $k^+ > \delta$. The annihilation operators can be commuted through the creation operators until the former act on the vacuum state $|\Omega\rangle$ and annihilate this state as if it was a bare vacuum. The result is of the form of integrals of some functions of three-momenta of quarks times remaining matrix elements of the background fields. Leading terms for large negative q^2 are

$$\Pi(q^2) = -\frac{1}{4\pi^2} \log \frac{-q^2}{\mu^2} + \frac{2m}{q^4} \langle \Omega | \bar{\omega}_0 \omega_0 | \Omega \rangle + \frac{c}{48\pi^2 q^4} \langle \Omega | g^2 f^c_{0\mu\nu} f_0^{c\mu\nu} | \Omega \rangle + \cdots , \tag{5.23}$$

where

$$c = \frac{3(a+1)(a-1)^2}{4a^2} \frac{1}{2\sqrt{a}} \log \frac{\sqrt{a}+1}{\sqrt{a}-1} - \frac{3a^2 - 2a + 3}{4a^2} \tag{5.24}$$

and $a = 1 - 4m^2 q^{-2}$. This result matches the vacuum polarization obtained by Shifman, Vainshtein and Zakharov [51] if we identify the bilinear vacuum matrix elements of the background fields ω and f with the condenstates from the sum rules. In fact, this result is valid in the limit $\delta \ll -q^+ m^2 q^{-2}$. δ is neglected in the final expression. Therefore, Eq. (5.24) cannot be used when $m = 0$.

Thus, we have established that the region of $k^+ < \delta$ is responsible for the formation of condensates in LF QCD. Consequently, prescriptions for avoiding singularities when $k^+ \to 0$ in gauge dependent quantities have to take into account that this region of momenta may be a source of significant effects to the whole theory. At the same time we have achieved a situation in which the vacuum degrees of freedom are well distinguished from the hadronic constituents with $k^+ > \delta$. The vacuum is trivial with respect to action of the constituent annihilation operators. On the other hand, the LF Hamiltonian contains new operators which induce new effects in the constituent dynamics due to the vacuum structure. In particular, one can investigate effects induced by the gluon condensate in transverse dynamics of bare particles in the LF Fock space.

5.2. Bare particles in the gluon condensate

In this Section we briefly present a study of some terms in the Hamiltonian given by Eqs (5.10) and (5.11). One can derive interesting conclusions about potential significance of the gluon condensate effects in LF QCD using a set of simplifying assumptions.

Neglecting the quark background field ω, the gluon field A and the instantaneous quark self-interaction in the QCD Hamiltonian, one obtains the following model Hamiltonian density

$$\mathcal{H}_{\rm model} = \psi_+^\dagger (id^\perp \alpha^\perp + \beta m) \frac{1}{i\partial^+} (id^\perp \alpha^\perp + \beta m) \psi_+ \,, \qquad (5.25)$$

where the background covariant derivative operator is explained in Eq. (5.8) and

$$a^i(x) = \tfrac{1}{2}(x^- f_0^{+i} - x^j f_0^{ji}) \qquad (5.26)$$

in the approximation when one keeps only the first term in expansion from Eq. (5.16) for $x^+ = 0$. The Hamiltonian density in Eq. (5.25) describes quarks interacting only with the background gluon field. The model Hamiltonian we consider is

$$H_{\rm model} = \int dx^- d^2x^\perp \mathcal{H}_{\rm model}(x^+ = 0,\, x^-,\, x^\perp)\,. \qquad (5.27)$$

Using Eq. (5.20) we have

$$\tfrac{1}{3}\,{\rm Tr}_{\rm color}\langle \Omega | g^2 a^\perp(x) a^\perp(y) | \Omega\rangle = \frac{\pi^2}{72} \nu^4 x^\perp y^\perp \sim (0.6\nu)^4 x^\perp y^\perp \qquad (5.28)$$

with ν given in Eqs (4.1) and (4.3). The vacuum expectation value of the background gluon field itself is zero, $\langle \Omega | f_0^{a\mu\nu} | \Omega \rangle = 0$.

One can project $H_{\rm model}$ on the space of quark–antiquark pairs created from $|\Omega\rangle$. The basis states are defined as

$$|p\lambda,\, k\sigma\rangle = \frac{1}{\sqrt{3}} \int dx_2^- d^2x_2^\perp\, dx_1^- d^2x_1^\perp\, e^{-ipx_2} e^{-ikx_1}$$
$$\bar u_+(x_2) u_{mp\lambda} \exp\left[-ig(x_2 - x_1)_\mu a^\mu(z)\right] \bar v_{mk\sigma} v_+(x_1) |\Omega\rangle\,, \qquad (5.29)$$

where $u_+(x)$ and $v_+(x)$ denote the quark and antiquark parts in the independent quark field ψ_+. The exponential gauge factor between the quark fields is introduced in order to preserve gauge invariance of the basis states under a limited class of gauge transformations which are independent of x^- in order to preserve the gauge condition $a^+ = 0$ and are independent of the LF time x^+. We have $z^+ = z^- = 0$ and $z^\perp = (x_2^\perp + x_1^\perp)/2$. The gluon field potential a^- does not enter since the basis is defined at equal LF times $x_1^+ = x_2^+ = 0$. The argument of the exponential function is a result of integrating the background field potential along a straight path in transverse coordinates from $x_1^\perp$ to $x_2^\perp$ for $x^- = 0$.

The spinorial indices of the quark fields are contracted with the spinors written in Eq. (5.29). The color indices of the quark and antiquark fields are contracted with the color indices of the exponential matrix factor, which transports the quark color indices from $x_1^\perp$ to $x_2^\perp$. We also assume that the background gluon field $f_0^{\mu\nu}$ from Eq. (5.26) has a direct tensor product structure in the color and space indices,

$$f^a_{\mu\nu} = f^a \otimes f_{\mu\nu}\,, \tag{5.30}$$

which allows operations with $a^\perp(x)$ as if it was an abelian field. Although colors of quarks and antiquarks in the basis states are contracted in a way which depends on the background gluon field potential, we have simple completeness relations,

$$\langle p'\lambda'\, k'\sigma' | p\lambda\, k\sigma\rangle = 16\pi^3 p^+ \delta^3(p-p')\, 16\pi^3 k^+ \delta^3(k-k')\,, \tag{5.31}$$

as if there was no gauge factor between the quark operators acting on the vacuum $|\Omega\rangle$. The basis states are colorless.

In order to write the eigenvalue problem for the model Hamiltonian projected on the colorless basis of $q\bar{q}$-states, we define a meson-like state as

$$|P\rangle = \sum_{\lambda\sigma} \int [dp][dk] \psi^P_{\lambda\sigma}(p,k) |p\lambda\, k\sigma\rangle\,, \tag{5.32}$$

where

$$\psi^P_{\lambda\sigma}(p,k) = 16\pi^3 \delta^3(P-p-k)\sqrt{p^+k^+}\,\phi_{\lambda\sigma}(x,\kappa^\perp) \tag{5.33}$$

in analogy to Eq. (2.26) for positronium. The meson-like state from Eq. (5.32) differs from positronium-like states in that the basis states contain the color factors which depend on background fields and the background fields have special properties which distinguish them from classical external fields in electrodynamics.

The projected eigenvalue problem for the model Hamiltonian is

$$\langle p\lambda\, k\sigma | H_{\rm model} | P\rangle = P^- \langle p\lambda\, k\sigma | P\rangle\,. \tag{5.34}$$

Evaluation of the matrix element on the rhs of this equation gives the wave function from Eq. (5.33) and evaluation of the matrix element on the lhs produces an operator acting on the same function.

The lhs matrix element involves differentiation in transverse directions, due to the presence of covariant derivatives $d^\perp$ in Eq. (5.25). The derivatives act not only on the wave function but also on the basis states. The argument of the exponential function transporting color indices in transverse direction is

$$ig(x_2 - x_1)^\perp a^\perp(z) = igx_2^\perp a^\perp(x_1) = -igx_1^\perp a^\perp(x_2)\,. \tag{5.35}$$

Therefore, the differentiation with respect to $x_2^\perp$ brings down a factor $a^\perp(x_1)$ and differentiation with respect to $x_1^\perp$ brings down a factor $a^\perp(x_2)$. In the covariant derivatives we have subtracted $a^\perp(x_2)$ and $a^\perp(x_1)$, respectively. One has to be careful about sign changes due to integrations by parts, different signs in the exponents and also the anticommutation relations for fermion creation and annihilation operators. The lhs matrix element depends only on the difference $a^\perp(x_1) - a^\perp(x_2)$. This difference is invariant under translations. Translational invariance is broken by the expansion in Eq. (5.26) which distinguishes the point $x^- = x^\perp = 0$. The exponential color matrix factor in the basis states restores the translational invariance in the Hamiltonian matrix elements. Thus, the gauge invariance restores the translational invariance.

There are terms on the lhs of Eq. (5.34) which are independent of the background field, terms linear in the background field and terms bilinear in the background field. The terms independent of the background field represent familiar LF free energies of quarks. The linear terms would normally lead in electrodynamics to Lorentz forces acting on charged quarks. The expectation values of the potential operator A^μ in QED produce a classical vector potential of external fields. Here the background gluon field operator appears in a vacuum matrix element. There are no classical external gluon fields. The exponential factors in the basis states cancel each other and we are left with the vacuum expectation value of $a^\perp(x_2) - a^\perp(x_1)$, which equals 0 by definition. The vacuum background is not modified by the presence of a quark-antiquark pair. The pair is a local object and cannot interfere with the long wavelength vacuum dynamics. This a qualitatively different situation than in QED. The bilinear terms contribute to the lhs of Eq. (5.34) through the square of $a^\perp(x_1) - a^\perp(x_2) = a^\perp(x_1 - x_2)$ for $x_1^- = x_2^- = 0$. Using Eq. (5.28) we have

$$\tfrac{1}{3}\,\mathrm{Tr}\langle\Omega|g^2[a^\perp(x_1) - a^\perp(x_2)]^2|\Omega\rangle \sim \tilde{\nu}^4(x_1^\perp - x_2^\perp)^2\,, \tag{5.36}$$

where we denoted $\tilde{\nu} = 0.6\nu$. In terms of relative momenta of quarks defined in the same way as for positronium in Eqs (2.29) to (2.32), $x_2 \equiv x$ and

$x_1 \equiv 1 - x$, we have

$$(x_1^\perp - x_2^\perp)^2 = -\frac{\partial}{\partial\kappa^i}\frac{\partial}{\partial\kappa^i} = -(\partial_\kappa^\perp)^2 . \tag{5.37}$$

The center of mass motion can be eliminated by writing the eigenvalue as $P^- = (M^2 + P^{\perp 2})/P^+$. Then, one obtains the following equation for the relative motion wave function $\phi_{\lambda\sigma}(x, \kappa^\perp)$ from Eq. (5.33) and the mass eigenvalue M:

$$\left[\frac{\kappa^{\perp 2} + m^2 - \tilde{\nu}^4(\partial_\kappa^\perp)^2}{x(1-x)} - M^2\right] \phi_{\lambda\sigma}(x, \kappa^\perp) = 0 . \tag{5.38}$$

The spectrum of masses is continuous due to the free motion of quarks in the longitudinal direction. However, the transverse motion is quantized into discrete levels of a two dimensional harmonic oscillator. The wave function is a product of a corresponding delta function of the longitudinal momentum fraction x and a Gaussian function of the transverse relative momentum $\kappa^\perp$ in the ground state, times a polynomial in transverse momentum components for excited states. The Gaussian function is of the form $\exp -\kappa^2/(2\tilde{\nu}^2)$ and $\sqrt{2}\tilde{\nu} \sim 340$ MeV. Experimentally observed width of quark transverse momenta in hadrons is of that order. The high transverse momentum tail of the quark wave function cannot be discussed without including radiation of gluons, which we ignored in order to derive this toy model. Nevertheless, the model illustrates how the infinitesimally small region of $k^+ < \delta$ in LF QCD may induce significant dynamical effects. The transverse confinement of quarks in colorless states which is associated with the vacuum structure in ET dynamics may be supplied by new terms in the LF Hamiltonian. This idea will be further pursued later when we discuss renormalization of LF Hamiltonians.

The free LF energy of quarks is quadratic in their transverse momenta and p^+ in denominator of p^- is analogous to a mass in nonrelativistic kinetic energy expression $(\vec{p})^2/(2m)$ [15]. Led by this observation, Głazek and Schaden have proposed a nonrelativistic constituent quark model in which the confining harmonic forces are induced by the gluon condensate [67]. A constituent quark is connected to an antiquark in a meson state by an exponent of a straight path integral of the background field. Three quarks in a baryon are connected by three exponents of three straight path integrals of the background fields. The three paths form a star with a junction located in a geometrical center of the three quarks. The model naturally leads to the harmonic confining potentials which are familiar from the nonrelativistic constituent quark phenomenology [59]. We refer the reader to Ref. [67] for details.

Following further the analogy with positronium, one may ask what happens in Fock states involving one gluon in addition to the $q\bar{q}$-pair. The basis states are defined as

$$|p\lambda\, q\varepsilon\, k\sigma\rangle = \int dx_2^- d^2x_2^\perp\, dx_0^- d^2x_0^\perp\, dx_1^- d^2x_1^\perp\, \mathrm{e}^{-ipx_2-iqx_0-ikx_1}$$

$$\bar{u}_+(x_2)u_{mp\lambda}\, \exp\left[igx_2^i a^i(x_0^\perp)\right]\left[-i\partial^+\varepsilon^\perp B^{\perp\dagger}(x_0)\right]\bar{v}_{mk\sigma}v_+(x_1)|\Omega\rangle\,, \quad (5.39)$$

where $B^\perp$ is the part of the gluon field potential $A^\perp$ which comprises the gluon annihilation operators. The longitudinal LF coordinates in arguments of the background field are put to zero. The above basis states also satisfy orthogonality relations as free quark-antiquark-gluon states, but the color indices of bare particles are contracted through the exponential factors depending on the gluon background field. One can include the free energy term for gluons and the coupling of gluons to the background field in the model Hamiltonian, still neglecting all couplings of gluons to quarks and gluons to themselves. One can carry out the procedure of projecting the eigenvalue equation for such extended model Hamiltonian on the $qg\bar{q}$-states. One obtains the following result for the energy of such system:

$$\frac{p^{\perp 2}+m^2+\tilde{\nu}^4(x_2^\perp-x_0^\perp)^2}{p^+}+\frac{k^{\perp 2}+m^2+\tilde{\nu}^4(x_1^\perp-x_0^\perp)^2}{k^+}$$
$$+\frac{q^{\perp 2}+\tilde{\nu}^4[(x_2^\perp-x_1^\perp)^2-(C_A/C_F)(x_0^\perp-x_1^\perp)(x_0^\perp-x_2^\perp)]}{q^+}\,, \quad (5.40)$$

where $x_2^\perp$, $x_0^\perp$ and $x_1^\perp$ denote the transverse space coordinates of the quark, gluon and antiquark fields, respectively. For SU(3) the ratio of the Casimir invariants is $^9/_4$. We see that the quarks are attracted to the gluon by harmonic forces in transverse direction. The gluon interaction with quarks through the background field provides a position dependent mass-like term for the gluon. This can be rewritten as

$$\tfrac{7}{16}(x_2^\perp-x_1^\perp)^2+\tfrac{9}{4}\left[x_0^\perp-\frac{x_1^\perp+x_2^\perp}{2}\right]^2 \quad (5.41)$$

times $\tilde{\nu}^4$. The formula explains that the gluon energy grows when quarks move apart and the gluon is strongly attracted to the center between quarks in the transverse plane. One may speculate that when more gluons are allowed a string of them is formed between quarks and leads to a linear potential for large separation of the quarks.

One can study the QCD Hamiltonian given by Eqs (5.10), (5.11) and (5.12) in application to the quarkonium eigenvalue problem in analogy to the positronium case described in Section 2. One can project the Hamiltonian eigenvalue problem on the space spanned by the new basis states for the $q\bar{q}$ and $qg\bar{q}$-states in the gluon background fields. Complications quickly grow with the number of included terms from the QCD Hamiltonian. Our experience with the positronium case from Section 2 becomes insufficient. Many terms which are negligible in positronium are of unknown size in the quarkonium case. One cannot eliminate the three-body sector algebraically due to complicated interactions of the three constituents with the background fields, potentially significant seagull interactions of quarks with gluons and instantaneous potentials between quarks and a gluon in the three-body sector. Despite extensive efforts little progress has been made beyond merely writing down the equations themselves and studying their partial wave structure. Preliminary attempts to understand these equations were reported by Głazek in Refs [68] and [69] but they contained a mistake in analysis of the transverse divergence in the instantaneous gluon exchange between quarks. That mistake invalidated some of the conclusions. The mistake was discovered by Franke and Prokhvatilov [70]. Further developments were blocked by the fact that the LF QCD Hamiltonian is strongly divergent. The divergences appear when one tries to remove the cutoff δ limiting k^+ from below and the cutoff Λ which limits the transverse momenta from above. There also appear infrared singularities for massless particles when $k^\perp \to 0$, and complications due to the gauge choice $A^+ = 0$.

Naive attempts to diagonalize a divergent Hamiltonian by brute force without understanding divergences in its matrix elements fail. The LF divergences pose difficult problems and are different from typical divergences in ET Hamiltonians. The QCD Hamiltonian contains many structures which need understanding. We shall have to study these and similar structures for a long time before it will become clear how much of the quark and gluon physics can be described by the Hamiltonian. The remaining Sections of this paper are devoted to such studies and aim at finding new methods for solving renormalization problems in LF Hamiltonians of quantum field theories.

6. Local interactions and divergences

The matrix elements of the QCD Hamiltonian depend on the cutoffs which limit the momentum range in the Fourier expansion of fields ψ_+ and $A^\perp$ in Eqs (5.13) and (5.14). For example, we describe below cutoff effects which appear in the eigenvalue problem of the LF QCD Hamiltonian. We consider a simplified model situation for definiteness. The eigenvalue equation for the QCD Hamiltonian given in Eqs (5.10) and (5.11) is projected on the Fock space spanned by the colorless basis of $q\bar{q}$-states, Eq. (5.29), and $qg\bar{q}$-states, Eq. (5.39) [71]. Then, in order to simplify presentation of the cutoff dependence we throw out all interactions of the quarks and gluons with the background fields. This way we obtain a simple model picture of the quarkonium wave functions as if we considered a canonical QCD Hamiltonian, expanded fields into creation and annihilation operators and projected the eigenvalue problem on the space of $q\bar{q}$-states and $qg\bar{q}$-states which are created by the creation operators from the bare vacuum state $|0\rangle$, *i.e.* as if not from the fully interacting ground state $|\Omega\rangle$. The bare eigenvalue problem which we describe is a small part of the full eigenvalue problem in QCD. However, this problem is already so strongly cutoff dependent that we need to understand this cutoff dependence before we can estimate any size of finite terms in the Hamiltonian and proceed to more complicated problems.

The projected $q\bar{q}$ eigenvalue problem is derived by analogy to the positronium case from Section 2. Many terms can be neglected and verified a posteriori to be small in the case of a very small coupling constant. A long range instantaneous Coulomb potential is a dominant interaction in positronium and the bound state is selfconsistently described in the nonrelativistic approximation. The momentum cutoffs force interaction energies to be always small in comparison to the bare particle masses. In QCD we cannot achieve such simple situation for light quarks. Therefore, we do not omit here various terms which were neglected in positronium.

The quarkonium state is constructed as

$$|P\rangle = \sum_{\lambda\sigma} \int [dp][dk] \Psi^P_{\lambda\sigma}(p,k) |\, p\lambda\, k\sigma\rangle + \sum_{\lambda\epsilon\sigma} \int [dp][dq][dk] \Psi^P_{\lambda\epsilon\sigma}(p,q,k) |\, p\lambda\, q\varepsilon\, k\sigma\rangle \,. \quad (6.1)$$

In order to evaluate matrix elements of the Hamiltonian one needs to limit momenta in the Fourier expansions of the independent fields ψ_+ and $A^\perp$. Various limits can be imposed [72]. We come back to this issue in next Sections devoted to the theory of renormalization. Here we assume that cutoffs are imposed independently on the transverse and longitudinal momenta in the Fourier expansions of both quark and gluon fields equally. Namely, $|p^\perp| \le \Lambda$ and $\Delta \ge p^+ \ge \delta$. This choice regulates most of the singularities arising in the procedure of normal ordering the Hamiltonian and renders finite expressions which depend on Λ, Δ and δ. One has to introduce additional regulators which damp the fields at spatial infinities on the LF, in order to validate integrations by parts without complicated boundary terms. This is achieved by introducing factors $\exp[-\epsilon(|x^1| + |x^2| + |x^-|)]$, which multiply the quark and gluon fields, $\delta \gg \epsilon \to 0$. We shall not exhibit such details unless they appear in our results. Introducing cutoffs in the quark and gluon field expansions is not sufficient to remove all singularities from the Hamiltonian matrix elements. The instantaneous seagull terms require additional specification of bounds on arguments of functions of momenta transferred in the instantaneous vertices.

The subtlety involved in the instantaneous seagull terms appears in evaluating inverse powers of ∂^+. These are defined to be

$$[(i\partial^+)^{-1}f](x) = \frac{1}{4i}\int_{-\infty}^{+\infty} dy^-\,\epsilon(x^- - y^-)f(y^-, x^\perp), \tag{6.2}$$

and

$$[(i\partial^+)^{-2}f](x) = -\tfrac{1}{8}\int_{-\infty}^{+\infty} dy^-\,|x^- - y^-|f(y^-, x^\perp), \tag{6.3}$$

for functions which vanish when $y^- \to \pm\infty$. This condition is insured by the exponentially damping factors introduced above. For example, the instantaneous gluon exchange involves the following type of integrals

$$\tfrac{1}{8}\int_{-\infty}^{+\infty} dx^- \int_{-\infty}^{+\infty} dy^-\, e^{(i/2)q_1^+ x^- - 2\epsilon|x^-|}|x^- - y^-|e^{-(i/2)q_2^+ y^- - 2\epsilon|y^-|}. \tag{6.4}$$

The exponent $-2\epsilon(|x^-| + |y^-|)$ leads to a complicated algebra of oscillating functions which can be avoided when $|x^- + y^-| + |y^- - x^-|$ is put instead of $|x^-| + |y^-|$. The latter also merely damps fields at infinity. It makes no difference which exponent is used for finite distances $|x^-| \ll \epsilon^{-1}$ and $|y^-| \ll \epsilon^{-1}$. The latter leads to simple expressions in momentum space. For example, the expression (6.4) becomes equal after this modification to

$$4\pi\delta(q_1^+ - q_2^+)\frac{q_1^{+2} - 16\epsilon^2}{[q_1^{+2} + 16\epsilon^2]^2}. \tag{6.5}$$

q_1^+ or q_2^+ have the interpretation of the longitudinal momentum transferred by an instantaneous gluon exchange. If we require $|q_1^+| > \delta \gg \epsilon$ the subtlety of choosing the damping factors becomes irrelevant and the instantaneous gluon exchange gives a factor

$$4\pi\delta(q_1^+ - q_2^+)\frac{\theta(|q_1^+| - \delta)}{q_1^{+2}}. \tag{6.6}$$

Both sectors in Eq. (6.1) contain a $q\bar{q}$-pair. Only those terms of the QCD Hamiltonian contribute to the eigenvalue problem projected on the space $|q\bar{q}\rangle \oplus |qg\bar{q}\rangle$ which preserve the numbers of quarks and antiquarks and change the number of gluons by at most one. Using field expansions from Eqs (5.13), (5.14) and (5.16), it is a straightforward but tedious calculation to evaluate matrix elements of the Hamiltonian in the space $|q\bar{q}\rangle \oplus |qg\bar{q}\rangle$. One derives two coupled bound state equations for the multicomponent spin wave functions

$$\Psi^P_{\lambda\sigma}(p,k) = 2(2\pi)^3\delta^3(P-p-k)\sqrt{p^+k^+}\,\phi_{\lambda\sigma}(x_2,\kappa^\perp)\,, \tag{6.7}$$

and

$$\Psi^P_{\lambda\epsilon\sigma}(p,q,k) = 2(2\pi)^3\delta^3(P-p-q-k)\sqrt{p^+x_0k^+}\,\phi_{\lambda\epsilon\sigma}(x_2,x_1,\kappa^\perp,\rho^\perp) \tag{6.8}$$

with the same interpretation in terms of quark and gluon momenta as we had for electrons and photons in the case of positronium in Eqs (2.27) and (2.28), *cf*. Eqs (6.11) to (6.13). Since we keep all interactions induced by the QCD Hamiltonian in the three-body sector we cannot express the three-body wave function in terms of the two-body wave function algebraically. The coupled integral equations are

$$\begin{aligned}
&\left[\frac{p^{\perp 2}+m_2^2+\sigma_2+G_{21}}{x_2}+\frac{k^{\perp 2}+m_1^2+\sigma_1+G_{12}}{x_1}\right]\phi_{\lambda\sigma}(x_2,\kappa^\perp)\\
&-4C_F\alpha(2\pi)^{-2}\int dx_2'd^2\kappa'^\perp\frac{R}{(x_2-x_2')^2}\phi_{\lambda\sigma}(x_2',\kappa'^\perp)\\
&-\frac{\sqrt{3}}{2}\frac{C_Fg}{(2\pi)^3}\sum_{\lambda'\epsilon'}\int dx_2'd^2\kappa'^\perp\, d^2\rho'^\perp\delta^2(k'^\perp-k^\perp)\\
&\times\frac{1}{\sqrt{x_0'}}u^\dagger_\lambda[2](\epsilon')u_{\lambda'}\phi_{\lambda'\epsilon'\sigma}(x_2',x_1,\kappa'^\perp,\rho'^\perp)\\
&-\frac{\sqrt{3}}{2}\frac{C_Fg}{(2\pi)^3}\sum_{\epsilon'\sigma'}\int dx_1'd^2\kappa'^\perp d^2\rho'^\perp\delta^2(p'^\perp-p^\perp)\\
&\times\frac{1}{\sqrt{x_0'}}v^\dagger_{\sigma'}[1](\epsilon')v_\sigma\phi_{\lambda\epsilon'\sigma'}(x_2,x_1',\kappa'^\perp,\rho'^\perp)\\
&=(M^2+P^{\perp 2})\phi_{\lambda\sigma}(x_2,\kappa^\perp)
\end{aligned} \tag{6.9}$$

and

$$
\begin{aligned}
&\left[\frac{p^{\perp 2}+m_2^2+\sigma_2+G_{20}}{x_2}+\frac{q^{\perp 2}+\sigma_0+G_{201}}{x_0}\right. \\
&\left.+\frac{k^{\perp 2}+m_1^2+\sigma_1+G_{01}}{x_1}\right]\phi_{\lambda\varepsilon\sigma}(x_2,x_1,\kappa^\perp,\rho^\perp) \\
&-4\tilde{C}_F\alpha(2\pi)^{-2}\int dx_2'\,dx_1'\,d^2\kappa'^\perp\delta(x_1+x_2-x_1'-x_2') \\
&\times\frac{R}{(x_2-x_2')^2}\phi_{\lambda\varepsilon\sigma}(x_2',x_1',\kappa'^\perp,\rho^\perp) \\
&+\frac{8\alpha}{(x_1+x_2)^2(2\pi)^2}u_\lambda^\dagger v_\sigma\sum_{\lambda'\sigma'}\int dx_2'\,dx_1'\,d^2\kappa'^\perp \\
&\times\delta(x_1+x_2-x_1'-x_2')v_{\sigma'}^\dagger u_{\lambda'}\phi_{\lambda'\varepsilon\sigma'}(x_2',x_1',\kappa'^\perp,\rho^\perp) \\
&-\frac{C_A\alpha}{(2\pi)^2}\int dx_2'\,d^2\kappa'^\perp\,d^2\rho'^\perp \\
&\times\frac{x_0+x_0'}{\sqrt{x_0x_0'}}\frac{R}{(x_2-x_2')^2}\delta^2(k^\perp-k'^\perp)\phi_{\lambda\varepsilon\sigma}(x_2',x_1,\kappa'^\perp,\rho'^\perp) \\
&-\frac{C_A\alpha}{(2\pi)^2}\int dx_1'\,d^2\kappa'^\perp\,d^2\rho'^\perp \\
&\times\frac{x_0+x_0'}{\sqrt{x_0x_0'}}\frac{R}{(x_1-x_1')^2}\delta^2(p^\perp-p'^\perp)\phi_{\lambda\varepsilon\sigma}(x_2,x_1',\kappa'^\perp,\rho'^\perp) \\
&-\frac{\alpha}{2\pi^2}\sum_{\lambda'\varepsilon'\sigma'}\int dx_2'\,d^2\kappa'^\perp\,d^2\rho'^\perp \\
&\times\frac{1}{\sqrt{x_0x_0'}}u_\lambda^\dagger\left[\tilde{C}_F\frac{\not\varepsilon_0'\,\not\varepsilon_0^*}{x_2-x_0'}+C_F\frac{\not\varepsilon_0^*\,\not\varepsilon_0'}{x_2+x_0}\right]u_{\lambda'} \\
&\times\delta^2(k^\perp-k'^\perp)\delta^{\sigma\sigma'}\phi_{\lambda'\varepsilon'\sigma'}(x_2',x_1,\kappa'^\perp,\rho'^\perp) \\
&-\frac{\alpha}{2\pi^2}\sum_{\lambda'\varepsilon'\sigma'}\int dx_1'\,d^2\kappa'^\perp\,d^2\rho'^\perp \\
&\times\frac{1}{\sqrt{x_0x_0'}}v_{\sigma'}^\dagger\left[C_F\frac{\not\varepsilon_0'\,\not\varepsilon_0^*}{x_1+x_0}+\tilde{C}_F\frac{\not\varepsilon_0^*\,\not\varepsilon_0'}{x_1-x_0'}\right]v_\sigma \\
&\times\delta^2(p^\perp-p'^\perp)\delta^{\lambda\lambda'}\phi_{\lambda'\varepsilon'\sigma'}(x_2,x_1',\kappa'^\perp,\rho'^\perp) \\
&-\sqrt{3}C_Fg\sum_{\lambda'\sigma'}\int dx_2'\,d^2\kappa'^\perp\frac{1}{\sqrt{x_0}}u_\lambda^\dagger[2](\varepsilon^*)u_{\lambda'} \\
&\times\delta(x_1-x_1')\delta^2(k^\perp-k'^\perp)\delta^{\sigma\sigma'}\phi_{\lambda'\sigma'}(x_2',\kappa'^\perp)
\end{aligned}
$$

$$
\begin{aligned}
&- \sqrt{3} C_F g \sum_{\lambda'\sigma'} \int dx_2'\, d^2\kappa'^{\perp} \frac{1}{\sqrt{x_0}} v^\dagger_{\sigma'}[1](\varepsilon^*) v_\sigma \\
&\times \delta(x_2 - x_2')\delta^2(p^\perp - p'^\perp)\delta^{\lambda\lambda'}\phi_{\lambda'\sigma'}(x_2', \kappa'^\perp) \\
&= (M^2 + P^{\perp 2})\phi_{\lambda\varepsilon\sigma}(x_2, x_1, \kappa^\perp, \rho^\perp)\,. \qquad (6.10)
\end{aligned}
$$

We have

$$x_2 = \frac{p^+}{P^+}\,,\ x_0 = \frac{q^+}{P^+}\,,\ x_1 = \frac{k^+}{P^+}\,, \qquad (6.11)$$

$$\kappa^\perp = (x_1 p^\perp - x_2 k^\perp)(x_1 + x_2)^{-1}\,, \qquad (6.12)$$

$$\rho^\perp = (x_1 + x_2) q^\perp - x_0(p^\perp + k^\perp)\,, \qquad (6.13)$$

$$u_\lambda = \Lambda_+ \begin{bmatrix} \chi_\lambda \\ 0 \end{bmatrix}, \qquad (6.14)$$

$$v_\sigma = \Lambda_+ \begin{bmatrix} 0 \\ \sigma\chi_{-\sigma} \end{bmatrix}, \qquad (6.15)$$

and χ denotes a two component Pauli spinor. Similar notation is adopted for primed variables.

We do not indicate explicitly the complicated integration limits for relative momentum variables. The total momentum of the bound state cannot be completely removed since it appears also in the ratio $\delta' = \delta/P^+$ limiting integrations over various momentum fractions x, and the total transverse momentum $P^\perp$ cannot be quite removed since it modifies limits of integration over transverse momenta around Λ. We could have no such cutoff dependence if and only if the eigenvalue problem would be insensitive to the cutoffs, and this is not guaranteed. In positronium we have chosen a special approximation in which the cutoff effects were very small. In the quarkonium case we cannot argue the same way since the coupling constant is allowed to be significantly larger and we assume the quarks to have momenta comparable with their masses. We can still consider $P^\perp = 0$ and $P^+ = M$, where M is the bound state mass. We shall have to come back to the issue of cutoff dependence and restoration of symmetries after we find the form of divergences in renormalization theory.

The symbol R in Eq. (6.10) indicates regularized expressions given in Eqs (6.5) or (6.6). Also,

$$[2](\varepsilon') = \frac{p^\perp\alpha^\perp + \beta m_2}{x_2}\alpha^\perp\varepsilon'^\perp + \alpha^\perp\varepsilon'^\perp\frac{p'^\perp\alpha^\perp + \beta m_2}{x_2'} - \frac{2\varepsilon'^\perp q'^\perp}{x_0'} \qquad (6.16)$$

and

$$[1](\varepsilon') = \frac{-k'^\perp\alpha^\perp + \beta m_1}{x_1'}\alpha^\perp\varepsilon'^\perp + \alpha^\perp\varepsilon'^\perp\frac{-k^\perp\alpha^\perp + \beta m_1}{x_1} + \frac{2\varepsilon'^\perp q'^\perp}{x_0'}\,. \qquad (6.17)$$

The self energies σ_i, $i = 2, 0, 1$, result from normal ordering of various seagull terms in the Hamiltonian. They are divergent functions of the cutoffs and depend on the constituents longitudinal momenta. Therefore, they cannot be removed by introducing constant counterterms. We have

$$\sigma_2 = 2C_F\alpha\frac{\Lambda^2}{4\pi}\left[\log\frac{p^+ - \delta}{\delta} - 2\frac{p^{+2}}{p^{+2} - \delta^2}\right], \tag{6.18}$$

$$\sigma_0 = C_A\alpha\frac{\Lambda^2}{4\pi}\left[\log\frac{\Delta^2}{\delta^2} - \frac{2N_f}{C_A}\log\frac{\Delta^2}{q^{+2} - \delta^2} + \log\frac{q^+ + \delta}{\delta} - 2\frac{q^{+2}}{q^{+2} - \delta^2}\right], \tag{6.19}$$

$$\sigma_1 = 2C_F\alpha\frac{\Lambda^2}{4\pi}\left[\log\frac{k^+ + \delta}{\delta} - 2\frac{k^{+2}}{k^{+2} - \delta^2}\right], \tag{6.20}$$

where the quadratic divergence originates from

$$\frac{\Lambda^2}{4\pi} = \int\frac{d^2k^\perp}{(2\pi)^2}\Theta(\Lambda^2 - k^{\perp 2}). \tag{6.21}$$

N_f is the number of the quark flavours. For SU(3), $C_A = 3$, $C_F = {}^4/_3$ and $\tilde{C}_F = -{}^1/_6$.

The interactions induced by the gluon background fields are indicated in Eqs (6.9) and (6.10) by letters G with various subscripts. We have considered these terms in Section 5, *cf.* Eqs (5.38) and (5.40). The subscript 2 refers to a quark, 0 to a gluon and 1 to an antiquark. These terms do not matter in our discussion of the cutoff dependence below.

The second term in Eq. (6.9) represents an instantaneous gluon exchange between quarks. The third term represents absorption of a gluon by a quark in a transition from the three-body to the two-body sector, and the forth term represents the gluon absorption by an antiquark.

The second term in Eq. (6.10) is the instantaneous gluon exchange between quarks in the three-body sector. The third term is an instantaneous $q\bar{q}$ annihilation channel interaction in a color octet state, accompanied by a gluon in the three-body state. The fourth and fifth terms represent instantaneous gluon exchanges between a quark and a gluon, and between an antiquark and a gluon, respectively, in the three-body sector. The sixth term describes instantaneous absorption and reemission of a gluon by a quark, and the seventh term the same by an antiquark, in the three-body channel. The terms eighth and ninth represent emission of a gluon from a quark and from an antiquark, respectively, in a transition from the two-body sector to the three-body sector.

Eqs (6.9) and (6.10) are strongly cutoff dependent and singular. It is visible that many large matrix elements emerge from interaction terms in the LF QCD Hamiltonian when the bare coupling constant g is not very small and cutoffs become extreme. Local interactions would require $\Lambda = \infty$ and $\delta = 0$. It is clear that one cannot discuss the relativistic bound state structure in local QCD without formulating a new renormalization theory which is capable of explaining how to treat the singular cutoff dependence. In the baryon-like states of three quarks exchanging gluons, similar singularities appear in more complicated combinations of various terms.

6.1. Divergences in quark self-energy

The quark self-energy σ_2 from Eq. (6.18) originates from normal ordering of two terms in the Hamiltonian. The logarithm results from a gluon emitted and absorbed in a fermion seagull term. The other term originates from a fermion emitted and absorbed in the instantaneous gluon exchange term. (Useful tables of matrix elements of various terms in the LF QCD Hamiltonian are given by Brodsky and Pauli in Ref. [18], where one can also find references to earlier literature.) Several features of the divergent term are visible. The self-energy appears in a form similar to a mass squared which is quadratically divergent as a function of the transverse momentum cutoff. However, it is also a diverging function of the longitudinal momentum of a quark. This divergence cannot be removed by introducing a constant bare mass squared term in the free energy of quarks. This indicates a qualitatively new feature of LF dynamics which is distinct from ET dynamics. Removal of divergences in LF Hamiltonians may require counterterms which are functions of quark momenta, while in ET dynamics counterterms involve only constants. From Eqs (6.18) and (6.20) it follows that the antiquark self-energy σ_1 as a function of the antiquark longitudinal momentum k^+, and the quark self-energy σ_2 as a function of the quark longitudinal momentum p^+, are different functions. The notion of a mass squared for a quark or an antiquark is obscured by the cutoff dependent effects and we cannot decide how large are the quark masses without understanding renormalization theory for LF Hamiltonians.

Suppose we had no divergent fermion self-energies of the above type. Assume we could neglect other troublesome terms in the three-body sector. Suppose we could express the three-body wave function by the two-body wave function as in the case of positronium in Eq. (2.38). Then, we would have obtained another self-energy term for the quark, and still another one for the antiquark. They would be equal to the self-energies from Eq. (2.46), with a coefficient C_F in front due to the color algebra and α denoting $g^2/(4\pi)$ instead of $\sim 1/137$. The limits of the integration would depend on the total momenta P^+ and $P^\perp$, individual particle momenta and the cutoffs, in a complicated way. However, it is clear that the quark self-interaction due to emission and absorption of gluons, is quadratically divergent as a function of the transverse cutoff Λ and linearly as a function of the inverse of the longitudinal cutoff δ. A different choice of cutoffs would lead to a different divergence structure.

Thus, we see that the quark mass squared term is significantly altered by the interactions. Separating a finite part of the self-energy and calling it a quark mass squared requires precise explanation of what happens with the cutoff dependent pieces which are potentially much larger than any finite pieces. For example, even if the coupling constant would vanish for large cutoffs as inverse powers of logarithms of the cutoffs, the quadratically or linearly divergent terms would still diverge to infinity. The most singular pieces have to be subtracted first.

If we were considering perturbation theory to a finite order, we could consider interplay of various terms, require special cancellations due to special choices of some parameters and eventually get rid of the cutoff dependence. For example, one often finds that the terms resulting from normal ordering automatically cancel most divergent parts of the self-energies induced by radiation and absorption of transverse gluons. The terms resulting from normal ordering are sometimes called self-induced inertia and some discussions of this type of terms can be traced, for example, in references quoted by Pauli and Brodsky in Ref. [18].

However, we are facing a difficulty that the divergences appear in a Hamiltonian eigenvalue problem. For example, the self-energy from Eq. (2.46) depends on the eigenvalue M^2 through the denominator of the integrand. The quadratic divergence can be subtracted. But after this subtraction the remaining term still contains logarithmic divergences and the diverging logarithms are multiplied by the eigenvalue. The eigenvalue is unknown before we solve the secular equation. In order to remove the divergence by a counterterm we would have to know the eigenvalue solution before we know the Hamiltonian counterterm. This is not possible without imposing complicated selfconsistency requirements on the Hamiltonian. Fulfillment of such requirements is hardly conceivable when more interactions and more Fock sectors are included. In order to understand what to do with the eigenvalue dependent divergences one needs to develop a renormalization scheme which can consistently deal with all divergences; quadratic, linear and logarithmic, and can tell us how to avoid the problem of divergences proportional to eigenvalues. The self-induced inertia are not helpful until one can show how they enter in such a procedure. We discuss the renormalization theory which solves the problem of the eigenvalue dependent divergences in Section 9.

Numerical studies and phenomenology may be helpful in finding useful solutions to problems with diverging self-interactions. We have already seen in Sections 3, 4 and 5 that phenomenologically interesting dynamical effects in many bound states can be described as corrections to a mass formula for the constituents. One can even use variational technics to find optimal values of the mass parameters, which involve subtractions of the cutoff dependent diverging terms [72]. Unfortunately, the variational approach introduced in Ref. [72] is not applicable to theories like QCD where fields have spin degrees of freedom. Straightforward variational estimates are also not useful if they are strongly cutoff dependent.

The important guide in defining the quark self-energies and masses can be attributed to chiral symmetry and its breaking. However, we cannot preserve symmetries when introducing cutoffs. For example, chiral symmetry and gauge symmetry are not quite compatible with each other [73]. The spontaneous chiral symmetry breaking is related to the formation of the quark condensate. We have seen in Section 5 that in LF dynamics the condenstates are formed from the modes of arbitrarily small k^+. Therefore, the modes which might lead to the spontaneous chiral symmetry breaking are cut off by introducing the longitudinal cutoff δ. Thus, renormalization and removal of this cutoff dependence should be able to contribute to our understanding of the mechanism of chiral symmetry breaking in QCD [22,74].

6.2. Gluon self-energy and related divergences

We consider gluon self-interactions and consequences of their appearance. Important constraints on the gluon self-energy may be expected due to the gauge invariance.

In QED one is formally used to have massless photons. Vanishing of the photon mass in perturbation theory is inevitably connected with gauge invariance, current conservation and renormalizability. Still, some calculations may require introduction of an infinitesimally small photon mass parameter for regulating behavior of some intermediate gauge dependent results which are singular when photons are massless. The limit of the photon mass going to zero is taken only in the complete gauge invariant result which is well defined in this limit.

In LF Hamiltonians of QED or QCD gauge invariance is not explicit. Firstly, one has to choose a gauge to define a Hamiltonian. For example, one requires $A^+ = 0$ in order to solve the constraint equations in LF gauge theories. Secondly, one has to limit the number of gluons and their momenta in order to define a finite domain of a Hamiltonian. Thirdly, in the bound state problem, experience from perturbation theory for free incoming and outgoing particles is of little use since binding causes significant deviations from the free particle spectrum. Gauge invariance is by no means clear in the cutoff Hamiltonian eigenvalue problems in QCD. These issues will have to be discussed in more detail later.

Eq. (6.19) illustrates some difficulties with divergent gluon self-interactions. There is one constant term which diverges when any single cutoff becomes extreme. This is the first term on the rhs of Eq. (6.19). It originates from normal ordering of the usual quartic gluon vertex in which one gluon is emitted and reabsorbed, the last line in Eq. (5.11) with a^μ put zero, and also from normal ordering of the four-gluon seagull interaction due to the instantaneous gluon exchange, a pure gluon part of the fifth line in Eq. (5.11). There is also a number of singular functions of the gluon longitudinal momentum in Eq. (6.19). The second term depends on the number of quarks in the theory. It results from emission and reabsorption of fermions by a gluon, due to the fermion seagull interaction, third line in Eq. (5.11) with a^μ and ω put zero. The terms third and fourth result from normal ordering the quartic seagull gluon interaction, which is again the purely gluon part of the fifth line of Eq. (5.11).

If we considered corrections to the gluon state due to creation and annihilation of a $q\bar{q}$-pair (vacuum polarization), we would also find that the corresponding gluon self-energy contains a term resembling a mass squared.

In QCD gluons couple to gluons and low energy behavior of gluon self-energy is not known. Phenomenology does not indicate low mass excitations which could correspond to light constituent gluons. It seems reasonable to consider that the gluon mass which results from cancellations of huge divergent numbers, is not zero and quite sizable. Strictly speaking, in renormalization theory one is forced to consider what effects are caused by the gluon self-energy. One could even turn the whole reasoning around and consider massless gluons as quite unrealistic idea in comparison to massive gluons [74,75].

Here we illustrate what may happen when gluons acquire a self-energy term. For simplicity, the self-energy will be represented as a mass squared term in the gluon energy and denoted by μ^2; $q^- = (q^{\perp 2} + \mu^2)/q^+$. The number of gluon degrees of freedom is 2×8 with 2 for two transverse polarizations and 8 due to the SU(3) color group. The number of the gluon degrees of freedom is not changed by the presence of the self-energy.

When in Eq. (6.10) we neglect the background fields, drop the σ_1 and σ_2 in quark self-energies assuming that m_1^2 and m_2^2 are some effective mass squared parameters and ignore all interactions but the last two terms on the lhs, then we can make the same steps in the case of Eqs (6.9) and (6.10) which we have already discussed in case of Eqs (2.38) to (2.45) for positronium. We have purposely indicated there the photon mass squared by the same symbol μ^2 and can use here the formulae from Section 2. The gluon mass squared does not contribute to the numerator of the fermion self-energy term in Eq. (2.46). But it appears in the denominator, where it removes a part of the infrared singularity when $\kappa^2 \to 0$ and $z \to 0$. The fermion self-energy will not concern us here any more.

In Eq. (2.47) in the limit $y \to y'$ when the gluon mass is not zero one has

$$-\frac{1}{(y-y')^2}\frac{(\kappa^\perp - \kappa'^\perp)^2}{\mu^2 + (\kappa^\perp - \kappa'^\perp)^2} + \frac{1}{(y-y')^2} = \frac{1}{(y-y')^2}\frac{\mu^2}{\mu^2 + (\kappa^\perp - \kappa'^\perp)^2}\,. \tag{6.22}$$

This is a singular interaction which would normally disappear when the gluon mass squared is not allowed. However, we know that forcing the gluon mass squared to zero requires control over various large cutoff dependent terms. Such control is still lacking. Instead of discarding the possibility that $\mu^2 \neq 0$ for no apparent reason, we can inspect the singular $q\bar{q}$ interaction in Eq. (6.22) and observe that the cutoff dependence problem may have an interesting alternative solution [74]. Namely, one can limit the momentum transferred by the gluons to be $|y - y'| \geq \delta' = \delta/P^+$, *cf.* Eq. (6.6), and consider very small δ. Integration over y' produces a factor $2/\delta'$. This interaction is very large for small δ' and strongly depends on this infrared cutoff. The counterterm to this cutoff dependence has an intriguing structure,

$$-\frac{2}{\delta'}\delta(y)\frac{\mu^2}{\mu^2 + (\kappa^\perp - \kappa'^\perp)^2}\,. \tag{6.23}$$

This term is a $q\bar{q}$-potential acting in the transverse direction, it is strong for small δ' and involves $\delta(y)$. This last feature is particularly interesting. Wilson suggested that this feature may be the key to constructing a LF theory of quarks and gluons [74]. He observed that counterterms which contain such δ-functions require operators of the form

$$\int dx^- \, \psi_+^\dagger \psi_+\,, \tag{6.24}$$

in order to induce no transfer of the longitudinal momentum. Such operators have the unique LF property discussed already in Section 5. They can neither create nor annihilate particles. The remarkable feature of the counterterms built from such operators is that they obey Zweig's rule: they preserve the number of quarks, and gluons too when the full color density operator is considered. Therefore, the counterterms to the longitudinal singularities lead to a conjecture that the renormalized Hamiltonian of LF QCD may contain strong potentials which preserve the number of bare particles.

Such possibility is welcome, since it provides a hope that one may be able to incorporate confinement of quarks in the LF QCD Hamiltonian using potentials. Such potentials should have structure dictated by the appropriate renormalization theory. We shall study methods of constructing such theory in next Sections. This idea adds a new element to the original Dirac proposal of having relativistic LF dynamics with nontrivial interactions. Dirac had only considered most general covariance conditions imposed by special relativity in quantum mechanics. Renormalization theory helps in finding more details of the interactions using local quantum field theory.

After subtracting the leading infrared divergence in the one gluon exchange one is left with a logarithmic infrared divergence which is proportional to the eigenvalue [76]. Again, as in the case of self-interaction of fermions, we face the problem that construction of a counterterm seems to require knowledge of the eigenvalue which is not known until we know the counterterm. In Ref. [77] in the case of positronium the problem of infrared divergences has been avoided by inventing modifications of the one photon exchange so that the divergence disappeared. Liu and Soper proposed to adapt the Leibbrandt–Mandelstam prescription for removing similar singularities to the case of exchange of massless photons in some bound state equations [78]. In Section 9 we present an introduction to a renormalization theory for LF Hamiltonians which is capable of avoiding eigenvalue dependent divergences. We also describe there the corresponding counterterm for second order one gluon exchange interaction between quarks since the method applies to massive as well as massless bosons.

6.3. Wave function divergence

An example of a wave function divergence appears when one considers an exchange of bosons with large transverse momentum between two fermions. Making ad hoc simplifications and eliminating the three-body sector from Eqs (6.9) and (6.10) one obtains an integral equation for the $q\bar{q}$ wave function, as discussed in Section 6.2. One can drop the divergent quark and antiquark self-interactions due to emission and reabsorption of transverse gluons. One can also cut off the infrared singularity in the one gluon exchange interaction between the quark and the antiquark. The resulting equation is still divergent [30, 31]. The remaining divergence is called the wave function divergence and arises in the following way.

The one boson exchange behaves for large transverse momentum transfers as a constant. Therefore, the wave function falls off as the inverse of the squared relative transverse momentum of fermions. The inverse of the squared momentum originates from the inverse of the two fermions free energy. Such a slow falloff of the wave function produces a logarithm of the upper limit of integration in the integral equation. Another way to see the divergence is to iterate the bound state equation and consider a "box" diagram which contains two one boson exchanges one after another. The box diagram is logarithmically divergent. The result of integration in the box loop depends on the cutoff on the transverse fermion momenta.

It may look like that the above problems are peculiar to the model where the number of constituents is arbitrarily restricted to two. This is not the case. Such divergences are generic to LF Hamiltonians for relativistic fermions. We postpone further discussion of this point to Section 7.

6.4. Vertex corrections and asymptotic freedom

In the projected eigenvalue problem used for illustration in this Section some features of QCD are not visible and were not discussed so far. One such feature requires mentioning. Namely, the coupling of quarks and gluons, as well as other couplings in QCD, exhibit asymptotic freedom. It is calculable in perturbation theory in vertex corrections. Incorporation of asymptotic freedom in Hamiltonian studies of QCD is essential. Unfortunately, little progress has been made in this direction in the Hamiltonian eigenvalue problems.

However, there already exist results obtained by Perry and Wilson in the old-fashioned LF Hamiltonian perturbation theory and using renormalization group techniques [79]. One can find references to some earlier studies of asymptotic freedom in Lagrangian and Hamiltonian QCD in Refs [79]. The important observation discussed by Perry is that the most divergent cutoff dependent terms in the vertex corrections can be subtracted by counterterms and the remaining divergence is only the familiar logarithmic ultraviolet divergence which leads to the asymptotic freedom. All light-front infrared divergences cancel out in the ultraviolet divergent vertex corrections. But they do not cancel out in ultraviolet finite terms. Perry and Wilson, and Perry initiated extensive studies of renormalization group differential equations in application to LF Hamiltonians [79].

7. Cutoffs and renormalization

In the previous Sections it is shown that LF Hamiltonians of quantum field theories are divergent in the following sense. Suppose we have a Hamiltonian

$$H = H_0 + V \,, \tag{7.1}$$

where the part H_0 has a known spectrum of suitably normalized eigenstates

$$H_0|i\rangle = E_i|i|\rangle \,. \tag{7.2}$$

The number of the eigenstates $|i\rangle$ is infinite and the energies (P^- in the LF case) are ranging from 0 to ∞. In order to solve the eigenvalue problem for H we have to define the domain and image of H. We are forced to choose a subset of eigenstates of H_0 in order to be able to define a finite space in which our Hamiltonian is to act. Then we can pose a finite eigenvalue problem which we may attempt to solve. The boundary of the chosen set of basis states is called a cutoff, and denoted symbolically Λ. The space spanned by the chosen set is called Λ-space. The Hamiltonian which acts in the Λ-space is denoted as H^Λ. The first natural guess is that

$$H^\Lambda_{\text{guess}} = P_\Lambda H P_\Lambda \,, \tag{7.3}$$

where P_Λ is a projection operator on the Λ-space. In principle, one can solve the eigenvalue problem for H^Λ_{geuss} from Eq. (7.3) because it is reduced to a finite problem by introducing the cutoff. Divergences arise when we attempt to let $\Lambda \to \infty$ in order to remove the artificial cutoff and obtain the spectrum of the Hamiltonian H itself. H is divergent if a finite limit of the spectrum of H^Λ_{guess} when $\Lambda \to \infty$ does not exist. In such cases we face a fundamental problem of how should we define H^Λ so that it can give us interesting solutions to the eigenvalue problem for H. This problem is called a renormalization problem. We need to define renormalized Hamiltonians in order to be able to calculate observables. LF Hamiltonians for local quantum field theories pose particularly complicated divergence problems. This is the price to pay for taking advantage of the unique properties of LF dynamics in constructing a relativistic bound state theory.

7.1. Theory of fixed sources [80]

Surprisingly little is known about renormalization of LF Hamiltonians for quantum fields. In ET dynamics the Hamiltonian renormalization theory has been developed by Wilson [24]. Wilson considered a model of a fixed source interacting with a scalar field. A demonstration of how the original Wilson model can be formulated in LF dynamics has been given by Głazek and Perry [80]. Although the model is unrealistic from the phenomenological point of view, it is instructive to see the full analysis and to appreciate the procedure required to find a renormalized Hamiltonian that can be used to calculate physical observables. We describe the model as a preliminary step in the development of the renormalization theory for LF Hamiltonians. We extensively quote Ref. [80] below in order to provide the reader with a selfcontained material. The reader may consult Ref. [80] for more detailed discussion.

The canonical Hamiltonian for Yukawa theory has been already considered for quite different purpose in Section 3, where we assumed to ignore renormalization problems. Here we focus our attention on the singularities which appear in the limit where the fermion mass becomes infinitely large. A single heavy fermion will play the role of the fixed source for a scalar boson field.

The canonical LF Hamiltonian for Yukawa theory can be written as

$$\begin{aligned} H_c = &\sum_{\lambda} \int [dp] \frac{m^2 + p^{\perp 2}}{p^+} b^\dagger_{p\lambda} b_{p\lambda} + \int [dq] \frac{\mu^2 + q^{\perp 2}}{p^+} a^\dagger_q a_q \\ &+ g \sum_{\lambda_1} \int [dp_1] \int [dq] \sum_{\lambda_2} \int [dp_2] 2(2\pi)^3 \delta^3 (P_{\text{created}} - P_{\text{annihilated}}) \\ &\times \bar{u}_{mp_2\lambda_2} \Gamma u_{mp_1\lambda_1} b^\dagger_{p_2\lambda_2} (a^\dagger_q + a_q) b_{p_1\lambda_1} \\ &+ [\text{seagull terms with operators } b^\dagger,\ b,\ a^\dagger \text{ and } a] \\ &+ [\text{terms that change fermion number or involve antifermions}]\,. \end{aligned} \tag{7.4}$$

The subscripts λ are spin/isospin indices and Γ is a spin/isospin matrix. One may suppress isospin indices in the initial discussion. It is sufficient here to focus on the case $\Gamma = 1$. The seagull terms and the terms involving antifermions are not important in the discussion of a fixed fermionic source and we skip them.

The momentum integrals in the canonical Hamiltonian extend to infinity and must be replaced by limits of integrals over select finite ranges of momenta in order to define the heavy fermion limit. The simplest way to consider the infinite fermion mass limit is to specify that the ratio of the fermion mass to the range of the momentum integrals becomes infinite. The fermion mass then becomes the dominant scale in the Hamiltonian. However, strictly speaking, restrictions on the particle momenta would require us to limit the total momentum of any system considered. In the ET analysis one is forced to simultaneously limit both total and relative momenta in order to obtain the fixed source Hamiltonian. In the LF scheme it is possible to exactly separate the total momentum of the dressed fermion source from its internal dynamics. Therefore, we require only relative momenta to be negligible in comparison to the fermion mass, and can allow arbitrary motion of the source.

We consider the effective Hamiltonian in the one fermion sector. Eigenstates of different fermion number are widely separated in the spectrum because of the large fermion mass. The one fermion eigenstates of the Hamiltonian have the following general form implied by the LF symmetries

$$|P\lambda\rangle = \sum_{n=0}^{\infty}\sum_{\sigma}\int[dp]\int[dq_1]\dots\int[dq_n]$$

$$2(2\pi)^3P^+\delta^3(P-p-q_1-\dots-q_n)\phi^{(n)}_{\lambda\sigma}(y_1,\kappa_1^\perp,\dots,y_n,\kappa_n^\perp)|p\sigma q_1\dots q_n\rangle\,, \quad (7.5)$$

where

$$|p\sigma q_1...q_n\rangle = b^\dagger_{p\sigma}a^\dagger_{q_1}...a^\dagger_{q_n}|0\rangle\,. \quad (7.6)$$

P^+ and $P^\perp$ are components of the total momentum of the fermion eigenstate. The arguments of the Fock space wave functions $\phi^{(n)}$ are

$$y_i = \frac{q_i^+}{P^+}\,, \quad (7.7)$$

$$\kappa_i^\perp = q_i^\perp - y_iP^\perp\,. \quad (7.8)$$

It follows that the bare fermion momentum in the n-th sector is

$$p_n^+ = x_nP^+\,, \quad (7.9)$$

$$p_n^\perp = x_nP^\perp - \kappa_1^\perp - \dots - \kappa_n^\perp\,, \quad (7.10)$$

where,

$$x_n = 1 - y_1 - \dots - y_n\,. \quad (7.11)$$

The effective Hamiltonian for the fixed fermionic source is obtained by projecting the equation

$$H|P\lambda\rangle = \frac{P^{\perp 2}+M^2}{P^+}|P\lambda\rangle \quad (7.12)$$

on the one fermion Fock space sectors,

$$\langle p\sigma q_1...q_n|H|P\lambda\rangle =$$

$$2(2\pi)^3P^+\delta^3(P-p-q_1-\dots-q_n)\frac{P^{\perp 2}+M^2}{P^+}\phi^{(n)}_{\lambda\sigma}(y_1,\kappa_1^\perp,\cdots,y_n,\kappa_n^\perp)\,. \quad (7.13)$$

One evaluates the fermionic part of the Hamiltonian matrix elements for large m and leaves the bosonic part untouched. M is the physical fermion mass. One obtains

$$\frac{P^{\perp 2}+M^2}{P^+}\phi^{(n)}_{\lambda\sigma}(y_1,\kappa_1^\perp,\dots,y_n,\kappa_n^\perp) =$$

$$\sum_{l=0}^{\infty}\int[d^3k'_1]\dots\int[d^3k'_l]\sum_{\sigma'}\phi^{(l)}_{\lambda\sigma'}(y'_1,\kappa'^{\perp}_1,\dots,y'_l,\kappa'^{\perp}_l)$$
$$\times\langle q_1\dots q_n|\left[\frac{(p^{\perp}_n)^2+m^2}{p^+_n}\delta^{\sigma\sigma'}+P^-_{0,\mathrm{bosons}}\delta^{\sigma\sigma'}\right.$$
$$\left.+g\int[d^3q][\frac{\bar{u}_{p_n\sigma}u_{p_n+q,\sigma'}}{p^+_n+q^+}a^{\dagger}_q+\frac{\bar{u}_{p_n\sigma}u_{p_n-q,\sigma'}}{p^+_n-q^+}a_q]\right]|q'_1\cdots q'_l\rangle\,. \quad (7.14)$$

Only three terms on the right hand side of this equation contribute to the infinite fermion mass limit; the free energy term with $\phi^{(n)}$ and two interaction terms, one with $\phi^{(n+1)}$ and one with $\phi^{(n-1)}$.

In LF dynamics we are not forced to assume that P^+ differs little from M and that $P^{\perp}/M$ is negligible, in contrast to the ET analysis. Here, we can treat P^+ and $P^{\perp}$ exactly and separate the center of mass motion of the dressed fermionic source from its internal dynamics. P^+ and $P^{\perp}$ drop out of the eigenvalue equation for the physical fermion mass M^2 as usually in LF eigenvalue problems. As a consequence, one can obtain the boost invariant results by considering only the special case $P^+=M$ and $P^{\perp}=0$.

Boost invariance is usually violated by the introduction of momentum cutoffs. Demanding that individual momenta of all particles be much smaller than the bare fermion mass prevents the fermion from moving with large velocity and implies that there cannot be boost invariance in the Hamiltonian spectrum. Fortunately, in the LF form of dynamics we can impose cutoffs on relative momenta. It is these that are small in comparison to the fermion mass. We can only maintain explicit boost invariance while limiting the relative momentum variables. Thus, a "fixed source" in the LF dynamics can move with arbitrary velocity. Evaluating the spinor matrix elements and working in the rest frame of the physical fermion we obtain

$$M\phi^{(n)}_{\lambda\sigma}(1,\dots,n)=\sum_{n'=n-1}^{n'=n+1}\int[1']\dots[n']\phi^{(n')}_{\lambda\sigma}(1',\dots,n')$$
$$\times\langle 1,\dots,n|\left[\frac{m^2+(\sum_{i=1}^{n}\kappa^{\perp}_i)^2}{(1-\sum_{i=1}^{n}y_i)M}+\sum_{i=1}^{n}\frac{\mu^2+\kappa^{\perp 2}_i}{y_iM}\right.$$
$$\left.+2g\int[dq](a^{\dagger}_q+a_q)\right]|1',\dots,n'\rangle\,, \quad (7.15)$$

where we use an abbreviated notation for momentum variables, exhibiting only their subscripts. We consider eigenvalues of the form

$$M=m+E\,, \quad (7.16)$$

for $(E/m) \ll 1$ and $\sum_{i=1}^{n} y_i \ll 1$, both of which are easily shown to hold for low-lying eigenstates a posteriori. Eq. (7.15) becomes

$$E\phi^{(n)}(1,\ldots,n) = \sum_{n'=n-1}^{n'=n+1} \int [1'] \ldots [n'] \phi^{(n')}(1',\ldots,n')$$
$$\times \langle 1,\ldots,n| \left[\sum_{i=1}^{n} \tfrac{1}{2} \left(y_i m + \frac{\mu^2 + \kappa_i^{\perp 2}}{y_i m} \right) + g \int [dq](a_q^\dagger + a_q) \right] |1',\ldots,n'\rangle . \tag{7.17}$$

Therefore, the effective LF Hamiltonian for the fermionic source internal dynamics is

$$H_{\rm eff} = \int [dq] \left[\tfrac{1}{2} \left(q^+ + \frac{\mu^2 + q^{\perp 2}}{q^+} \right) a_q^\dagger a_q + g(a_q^\dagger + a_q) \right] . \tag{7.18}$$

The integral extends over momenta that are negligible in comparison to the fermion mass. Therefore, for the infinite fermion mass we obtain the effective Hamiltonian of Eq. (7.18) in which no restrictions on the boson momenta appear. Note that when the fermion mass is large the LF energy q^- is naturally accompanied by q^+ to form the analog of the ET energy, $(q^+ + q^-)/2$. Similar feature appeared in Section 3 where we considered nuclear matter. In the rest frame of the dressed source the transverse meson momentum $q^\perp$ coincides with the relative momentum $\kappa^\perp$ and the longitudinal momentum q^+ coincides with ym, because the difference between m and $P^+ = M$ can be neglected in the product yP^+.

The eigenvalue equation for the Hamiltonian of Eq. (7.18) leads to divergent results. The Hamiltonian is divergent in the sense defined at the beginning of Section 7. Therefore, it requires renormalization. Wilson found a way to define a class of renormalized Hamiltonians that corresponds to the fixed source Hamiltonian in ET dynamics, and discovered the renormalization group theory for quantum field Hamiltonians. Wilson's work on the fixed source model in ET dynamics produced the first nonperturbative renormalization group analysis of coupling constant renormalization. We refer the reader to the two remarkable articles by Wilson [24] where he formulated foundations of the renormalized Hamiltonian dynamics. It is recommended to read the two articles in Ref. [24] before attacking the more complicated theory of renormalization for LF Hamiltonians.

Now we describe the situation in LF dynamics. In order to define Wilson's model on the LF we need to introduce momentum ranges analogous to his ET energy shells [24]. Since we have only one unrestricted momentum integration to sample we can use our previous experience from the model in Section 3, Ref. [39], and from Ref. [72], where a mass parameter has been introduced to define limits on the Fourier expansions of scalar fields. We introduce the following regions of momenta:

$$\mu < \tfrac{1}{2}\left(q^+ + \frac{\mu^2 + q^{\perp 2}}{q^+}\right) < E_0\,, \tag{7.19}$$

$$\sqrt{\left(\tfrac{1}{2}\Lambda^n k_0\right)^2 + \mu^2} < \tfrac{1}{2}\left(q^+ + \frac{\mu^2 + q^{\perp 2}}{q^+}\right) < \sqrt{(\Lambda^n k_0)^2 + \mu^2}, \quad n \geq 1\,, \tag{7.20}$$

where

$$E_0 = \sqrt{\mu^2 + k_0^2}\,. \tag{7.21}$$

The LF eigenvalue problem for the resulting Hamiltonian is isomorphic to the ET problem considered by Wilson. We change variables from q^+ and $q^\perp$ to k^3 and $k^\perp = (k^1, k^2)$ which form the three-vector $\vec{k}$;

$$q^\perp = k^\perp\,, \tag{7.22}$$

$$q^+ = \omega_\mu(\vec{k}) + k^3\,, \tag{7.23}$$

$$\omega_\mu(\vec{k}) = \sqrt{\mu^2 + (\vec{k})^2}\,, \tag{7.24}$$

$$[dq] = \frac{d^3k}{2\omega_\mu(\vec{k})(2\pi)^3} := [dk]\,, \tag{7.25}$$

The Hamiltonian from Eq. (7.18) in terms of these variables reads

$$H = \int_{\text{shells of } k} [dk][\omega_\mu(\vec{k}) a_k^\dagger a_k + g(a_k^\dagger + a_k)]\,. \tag{7.26}$$

One can now step back to the initial theory with fermions and bosons carrying isospin. We recall that there are two kinds of bosons in the Hamiltonian (operators a and b) and isospin matrices τ^+ and τ^-. We change the normalization of the creation and annihilation operators to match Wilson's convention and let the coupling g be Wilson's bare coupling constant g_0. After these steps the LF fixed source Hamiltonian is identical with Wilson's model described in Ref. [24]. The whole discussion from Ref. [24] follows.

This last step is a major one. Following Wilson's work in ET formulation one obtains the LF version of Wilson's model of the coupling constant renormalization in quantum field theory. To our knowledge this allows one to complete the first example of a nonperturbative renormalization group analysis in LF dynamics.

This example provides a starting point for studies of significantly more complicated renormalization problems which we already met in previous Sections and will have to confront in future in LF quantum field theories of bound states of elementary particles.

This model implies several conclusions concerning the renormalization of relativistic LF Hamiltonians. The LF construction of the model of coupling constant renormalization is not merely a change of variables in the ET model. It is actually both nontrivial and instructive. In the ET analysis one employs momenta canonically conjugate to the ET spatial variables. One places the fixed source at the origin of the coordinate system, and all boson coordinates are given relative to this position. The LF analysis is necessarily more complicated, because a source fixed in LF space would move at the speed of light in ET coordinates and has an infinite energy. In the LF analysis a different approach is needed. One considers a fixed source as an arbitrarily moving dressed fermion bound state, which contains infinitely many bosons piled up into a hierarchy of layers of structure. The Poincaré generators of boosts do not contain interactions and one is able to completely separate the total momentum of the bound state from the problem. In an ET analysis the boost operators contain interactions and the separation of the total momentum from the problem is not possible. In ET dynamics one is forced to assume that the fermion momentum is small in comparison to the fermion mass in order to complete the analysis. In the LF model one automatically obtains a theory of heavy sources that can move with arbitrary velocity. The momentum $\vec{k}$, (or q^+ and $q^\perp$), is considered as a relative momentum of a boson and the center of mass of the *dressed* source. The motion of the dressed source is separated from its internal dynamics. This is a second and distinct example of highly nontrivial bound state dynamics of relativistic bound states of many elementary particles. The first example is described in Section 3.

For infinitely heavy fermions the distinction between relative and absolute spatial coordinates is not significant. However, for quarks (baryons) coupled to gluons (mesons) there is no reason to believe that the difference does not matter. In order to understand relativistic bound state dynamics in QCD or nuclear physics, one must understand the relative motion of constituents over many scales of momenta at once. It is not possible to employ nonrelativistic ideas of considering only rest frame of reference or fixed sources everywhere. One must expect several important new features to appear in renormalization of relativistic Hamiltonians.

In Wilson's ET model the only interactions allowed are between a non-relativistic source and relativistic bosons. In the analogous LF model one can see that it is the negligible recoil of the source that selects the momentum scales in Eqs (7.19) and (7.20). These scales imply a relationship between the scaling of longitudinal and transverse momenta. However, we know no reason for this relationship to hold for light sources. We consider it an outstanding problem to discover the principle that will allow one to establish a set of scales appropriate for the study of relativistic sources analogous to those employed by Wilson for the study of a fixed source. We shall come back to this problem in Section 10.

In the case of infinite fermion mass the key role is played by the meson mass parameter, μ, which is used in sampling longitudinal and transverse momenta in the LF renormalization group analysis. The sampling is patterned after ET sampling of a free boson energy because the ET free boson energy is naturally selected in the LF Hamiltonian for the internal dynamics of a heavy source, as displayed in Eq. (7.18). When fermion and boson masses are comparable, or when the bosons have self-interactions, the ET free boson energy does not naturally arise in a LF analysis and a unique momentum sampling does not exist. Both fermion and boson masses need renormalization, as we have clearly seen in previous Sections. It becomes clear that one is forced to formulate an analysis that samples longitudinal and transverse relative momenta in a new way.

One is also forced to make restrictions on the number of particles in Fock space as a practical limitation on the analysis [81]. A considerable effort over a period of time will have to be devoted to the study of how the renormalization group transformation depends on the Fock space sectors considered. In Wilson's model the number of bosons in a single quantum state has been artificially limited to one and the number of fermions has necessarily been fixed. In order to be able to firmly connect renormalized LF Hamiltonians for quantum chromodynamics at hadronic scales with Feynman's parton model and perturbative QCD one has to go a long way. An example which illustrates that LF Hamiltonians need more general renormalization counterterms than just a coupling constant renormalization is given in the next Section.

7.2. Overlapping logarithmic divergences [82]

Głazek and Wilson have constructed a new example of a renormalization procedure for LF Hamiltonians [82]. Their model is specially designed to resemble some phenomenological models which we have discussed in previous Sections. In order to be able to understand basic features of renormalization theory, drastic simplifications have to be made in comparison to what one might imagine to be required when approaching canonical Hamiltonians for local quantum field theories. Our aim is to understand the rules one can use in building H^{Λ} from a known bare H. The particular structure or limitations of H used for the exposition are of secondary importance. In this Section we reproduce essential parts of Ref. [82]. We present the main elements of the model but many details of the original work are omitted. Previous Sections provide a lot of examples of the notation used here.

The construction starts from a Hamiltonian which acts in a space spanned by free states of two fermions and states of two fermions and one scalar boson. The starting Hamiltonian resembles a canonical Yukawa Hamiltonian projected on the model space. It is regularized by chopping factors in the interaction vertices. Then, an effective Hamiltonian acting in the space of two fermions is derived in analogy to the case of positronium in Section 2 and to the case of $q\bar{q}$-states in Section 6. At that point ad hoc simplifications in the effective two-body Hamiltonian are made to produce a model that one can analyse. Namely, fermion self-interactions are dropped and the eigenvalue in the effective one boson exchange term is replaced by a constant. The resulting model Hamiltonian involves logarithmic ultraviolet transverse divergences which we have already mentioned in Section 6.3. These divergences are analogous to overlapping divergences in perturbative Lagrangian S-matrix calculations. We introduce a small cutoff $\lambda \ll \Lambda$ and explain how one can derive the renormalized interaction V_λ from the knowledge of the bare and divergent interaction V. We explain the construction of a Hamiltonian counterterm that removes divergences from V_λ to all orders in the Hamiltonian perturbation theory. The renormalization group transformation for V_λ, is determined by a nonlinear integro-differential equation of the form $dV_\lambda/d\lambda = -V_\lambda K_\lambda V_\lambda$, where K_λ is a known kernel.

The model counterterm turns out to be local in the transverse direction and contains an arbitrary function of the longitudinal momenta of fermions. A suitable choice of the arbitrary function may partly remove the violation of rotational invariance in the spectrum of the model Hamiltonian. Initial results of numerical analysis concerning this issue are described in the next Section.

In Section 6, a set of perturbative power counting rules discovered by Wilson is mentioned. Wilson used his power counting to classify operators which may appear in the LF QCD Hamiltonian [74]. Here we consider a much simpler model situation. The power counting in the model suggests which counterterms need to be present in the Hamiltonian to all orders in perturbation theory. We briefly describe the general idea of power counting and then return to our simple model.

Transverse and longitudinal dimensions count differently. The light-front fermion fields have dimension $(x^\perp\sqrt{x^-})^{-1}$ while the LF Hamiltonian density has dimension $x^{\perp-4}$, since the light-front Hamiltonian has dimension $x^-/x^{\perp 2}$ and the density is integrated over the front $x^+ = 0$ to obtain the Hamiltonian. It is straightforward to analyze dimensions of fields in Hamiltonian densities from Eqs (2.10), (3.5) or (5.11). The notion of a size of a quantum field is introduced by expanding it into a complete set of orthgonalized wave packets called wavelets. Creation and annihilation operators for the wave packet states are of order one. An excellent introduction to such analysis in ET dynamics is given in Ref. [24]. LF version requires a distinction between the transverse and longitudinal dimensions in construction of the wavelets. Roughly speaking, Heisenberg's uncertainty principle tells us how large the wavelet coefficient must be in front of a single particle creation or annihilation operator, if the corresponding wave packet state (wavelet) has some dimension $x^\perp$ in transverse and x^- in longitudinal direction. Power counting amounts essentially to counting orders of magnitude of terms which appear in old-fashioned Hamiltonian perturbation theory. The expansion of quantum fields into wavelets times corresponding creation and annihilation operators must be limited, in the sense that a finite number of wavelets can be used in practice. One needs to introduce cutoffs on the space of wavelets. Results of the perturbation theory depend on the cutoffs and diverge for their extreme values. Divergences in perturbation theory arise due to the large energies of single particle wavelets, large sizes of individual wavelets and large numbers of wavelets coupled together by the interactions. Power counting allows estimates of the cutoff dependence and provides clues about the structure of necessary counterterms. This means, it tells us crudely how the Hamiltonian H^Λ must differ from $H^\Lambda_{\mathrm{guess}}$. For example, power counting tells us that there can be a LF Hamiltonian counterterm which removes transverse ultraviolet divergences. Such counterterm involves four fermion fields. It must be local in the transverse dimension, should not involve either derivatives in the transverse direction or mass parameters, and must involve a function of longitudinal fermion field coordinates which has dimension $(x^-)^2$. We describe, following Ref. [82], how such counterterms appear in a simple model.

The ultraviolet transverse momentum cutoff in the model is analogous to the transverse momentum cutoff Λ of Section 6. It is different from the infrared cutoff δ which is responsible for cutting off the vacuum problem. However, understanding the renormalization procedure in the model is a prerequisite to tackling the renormalization problems with severe longitudinal singularities in QCD where the δ-dependence of the spectrum has to be removed by different counterterms.

We proceed to the description of the model. The initial Hamiltonian acts in a space spanned by Fock states containing two fermions and states containing two fermions and a boson. After elimination of the three-body wave function, the two-body wave function satisfies a familiar effective two-body equation

$$\left[\frac{m^2+\kappa^2+\delta m^2_{\Lambda\varepsilon}+\Sigma_2}{x_2 M}+\frac{m^2+\kappa^2+\delta m^2_{\Lambda\varepsilon}+\Sigma_1}{x_1 M}-M\right]\phi + \sum\int[\langle V_{OBE}\rangle+\langle V_\Lambda\rangle]\phi = 0\,. \quad (7.27)$$

The new features of this integral equation are the special integration limits and the presence of a counterterm $\langle V_\Lambda\rangle$. The counterterm is the difference between H^Λ and H^Λ_{guess} in the case of this model. Without the counterterm the model contains a transverse divergence of the type discussed in Section 6.3. Principles of construction of the counterterm $\langle V_\Lambda\rangle$ in Eq. (7.27) are explained below after some simplifying assumptions are made. The wave function ϕ represents a column of four wave functions corresponding to four possible spin configurations of two fermions. $\langle V_{OBE}\rangle$ represents the one scalar boson exchange potential, which depends on the eigenvalue M. The vertices of a fermion–boson coupling are given by products of fermion spinors as in Yukawa theory. Detailed derivation of this equation is given in Ref. [82]. The simplified model is obtained by making two ad hoc changes in Eq. (7.27). First, $\delta m^2_{\Lambda\varepsilon}$, Σ_2 and Σ_1 are completely neglected. Second, the one boson exchange term, $\langle V_{OBE}\rangle$, contains the eigenvalue M in the energy denominator. We replace the eigenvalue in the denominator by a constant, M_0, which is smaller than $2m$.

The resulting bound state equation can be viewed as an eigenvalue equation for a simplified Hamiltonian which acts in the space of two fermions only. The model Hamiltonian commutes with the kinematical operator J_z, generator of rotations about the z-axis. The total z-component of the angular momentum can be decomposed into the orbital angular momentum of the bound state relative to an arbitrary axis parallel to the z-axis, L_z, and the total internal angular momentum of the bound state, $j_z = l_z + s_z$. The model Hamiltonian commutes also with j_z. Divergences appear when $|j_z| \le 1$. Let us consider here the case $j_z = 0$.

Using notation $\phi^{l_z}_{\lambda_2\lambda_1}$ one obtains the following $j_z = 0$ part of the model Hamiltonian eigenvalue equation

$$(\mathcal{M}^2_{12}-M^2)\begin{bmatrix}\phi^0_{\uparrow\downarrow}\\ \phi^0_{\downarrow\uparrow}\\ \phi^{-1}_{\uparrow\uparrow}\\ \phi^{+1}_{\downarrow\downarrow}\end{bmatrix} - \frac{\alpha_g}{4\pi}\int dx'\int d\kappa'^2 C(x\kappa, x'\kappa')$$

$$\times\left\{\begin{bmatrix}\alpha & \beta & \gamma & \delta\\ \beta & \alpha & -\delta & -\gamma\\ \varepsilon & -\varphi & \omega & \rho\\ \varphi & -\varepsilon & \rho & \omega\end{bmatrix} + M\langle V_\Lambda^0\rangle\right\}\begin{bmatrix}\phi^0_{\uparrow\downarrow}\\ \phi^0_{\downarrow\uparrow}\\ \phi^{-1}_{\uparrow\uparrow}\\ \phi^{+1}_{\downarrow\downarrow}\end{bmatrix} = 0\,. \tag{7.28}$$

The integration limits in this equation are defined by the factor

$$\begin{aligned} C(x\kappa, x'\kappa') = \ & \Theta(x - x' - \varepsilon x)\Theta(x - x' - \varepsilon(1-x)) \\ & \times \Theta(x-\varepsilon)\Theta(1-x-\varepsilon)\Theta(\Lambda^2 - \kappa^{\perp 2}) \\ & \times \Theta(x'-\varepsilon)\Theta(1-x'-\varepsilon)\Theta(\Lambda^2 - \kappa'^{\perp 2}) + (x \leftrightarrow x')\,. \end{aligned} \tag{7.29}$$

Thus, the transverse relative momenta of fermions are limited in modulus by Λ. Longitudinal momentum fractions carried by fermions are limited from below by the longitudinal cutoff $\varepsilon = \delta/M$.

From Eq. (7.29) we see also that the longitudinal momentum integrals are chopped off in the region where $x \simeq x'$, *i.e.* the momentum fraction carried by the boson in the intermediate states is not allowed to vanish. In QCD, $x \simeq x'$ is a region where a strong longitudinal singularity appears and a renormalization group analysis of the singularity is essential. Therefore, it is assumed in the model that ε is finite and we do not need to worry about the limit $\varepsilon \to 0$. We study divergences that appear in the transverse direction for finite ε which is sufficiently large.

V_Λ^0 is proportional to the $j_z = 0$ projection of the counterterm. The counterterm must commute with j_z because the initial Hamiltonian does. V_Λ^0 is present to make solutions to Eq. (7.28) independent of the cutoff Λ. The coupling constant $\alpha_g = g^2/4\pi$, where g is a coupling constant analogous to g in front of $\bar\psi\psi\phi$ in the Yukawa theory. For completeness we quote from Ref. [82] various functions denoted by Greek letters in Eq. (7.28). They are

$$\alpha = m^2\left(\frac{1}{x}+\frac{1}{x'}\right)\left(\frac{1}{1-x}+\frac{1}{1-x'}\right)A^0\,, \tag{7.30}$$

$$\begin{aligned}\beta = & \left(\frac{\kappa'^2}{x'(1-x')}+\frac{\kappa^2}{x(1-x)}\right)A^0 \\ & - \kappa\kappa'\left(\frac{1}{x'(1-x)}-\frac{1}{x(1-x')}\right)A^1\,,\end{aligned} \tag{7.31}$$

$$\gamma = m\left(\frac{1}{x}+\frac{1}{x'}\right)\left(\frac{\kappa}{1-x}A^1 - \frac{\kappa'}{1-x'}A^0\right)\,, \tag{7.32}$$

$$\delta = m\left(\frac{1}{1-x}+\frac{1}{1-x'}\right)\left(\frac{\kappa}{x}A^1-\frac{\kappa'}{x'}A^0\right), \tag{7.33}$$

$$\varepsilon = m\left(\frac{1}{x}+\frac{1}{x'}\right)\left(\frac{\kappa'}{1-x'}A^1-\frac{\kappa}{1-x}A^0\right), \tag{7.34}$$

$$\varphi = m\left(\frac{1}{1-x}+\frac{1}{1-x'}\right)\left(\frac{\kappa'}{x'}A^1-\frac{\kappa}{x}A^0\right), \tag{7.35}$$

$$\omega = m^2\left(\frac{1}{x}+\frac{1}{x'}\right)\left(\frac{1}{1-x}+\frac{1}{1-x'}\right)A^1, \tag{7.36}$$

$$\rho = -\left(\frac{\kappa'^2}{x'(1-x')}+\frac{\kappa^2}{x(1-x)}\right)A^1 + \kappa\kappa'\left(\frac{1}{x'(1-x)}-\frac{1}{x(1-x')}\right)A^0, \tag{7.37}$$

where

$$A^l = \frac{1}{\sqrt{a^2-b^2}}\left[\frac{\sqrt{a^2-b^2}-a}{b}\right]^{|l|}, \tag{7.38}$$

$$a = \mu^2+\kappa^2+\kappa'^2 + \tfrac{1}{2}(x'-x)\left(\frac{\kappa^2+m^2}{x}-\frac{\kappa^2+m^2}{1-x}+\frac{\kappa'^2+m^2}{1-x'}-\frac{\kappa'^2+m^2}{x'}\right) + \tfrac{1}{2}|x'-x|\left(\frac{\kappa^2+m^2}{x(1-x)}+\frac{\kappa'^2+m^2}{x'(1-x')}-2M_0^2\right), \tag{7.39}$$

and

$$b = -2\kappa\kappa'. \tag{7.40}$$

Asymptotic behavior of the wave functions $\phi^{l_z}_{\lambda_2\lambda_1}(\kappa)$ when the relative transverse momentum becomes large is given by the inverse of the invariant mass squared of the two fermions, $\mathcal{M}_{12}^{-2}$, multiplying the potential term. From Eqs (7.30) to (7.37) we see that almost all entries in the one boson exchange potential in Eq. (7.28) behave for large transverse momentum as at least one inverse power of the transverse momentum. These terms lead to asymptotic behavior of the wave functions which is at least as convergent as κ^{-3} and produce no large cutoff dependence in the integral equations. However, the first term of β in Eq. (7.31) behaves like a constant as a function of the transverse momentum for large κ. The asymptotic tail of β is a function of x and x' only. Therefore, the wave functions with $l_z = 0$ obtain contributions which fall off at large κ like κ^{-2}. Substituting such a function under the integral with a constant potential, we see that asymptotic behavior of the wave function in the transverse direction leads to a logarithmic divergence in the one boson exchange potential. This is a common feature of all one boson exchange interactions between relativistic LF fermions.

The logarithmically divergent integral in the potential term generates Λ-dependence of the eigenvalue problem. The spectrum is cutoff dependent. Numerical estimates of the cutoff dependence will be discussed in the next Section. We cannot simply let Λ tend to ∞. The Hamiltonian H^{Λ} must contain the counterterm indicated in Eq. (7.28).

The model Hamiltonian with the cutoff Λ is now denoted by H and its free part by H_0. E denotes the eigenvalue and gV the potential term, where g symbolizes a coupling constant which is equal to α_g in our model. Thus, $H = H_0 + gV$. One introduces projection operators P_λ and $Q_\lambda = 1 - P_\lambda$. P_λ projects on the space of states of two fermions with relative transverse momentum squared smaller or equal to $\lambda \leq \Lambda^2$. The corresponding space of two fermion states is called λ-space. The free Hamiltonian H_0 commutes with the projection operators. Note that the dimension of the cutoff λ is transverse momentum squared. The effective Hamiltonian in the λ-space, denoted by H_λ, contains the effective interaction, V_λ, which is given by the formula [83]

$$V_\lambda = P_\lambda g V P_\lambda + P_\lambda g V Q_\lambda \frac{1}{E - H_0 - Q_\lambda g V Q_\lambda} Q_\lambda g V P_\lambda \,. \tag{7.41}$$

If the limit of V_λ when $\Lambda \to \infty$ existed, then H_λ would have a cutoff independent spectrum and the counterterm would not be necessary. Since the limit does not exist one can attempt to add counterterms to H so that the dependence of V_λ on Λ disappears for $\Lambda^2 \gg \lambda$. In other words, one is seeking a method to solve a divergent Hamiltonian eigenvalue problem and find a special new operator which removes the divergent parts from all eigenvalues in a well defined procedure. H_λ which results from H including counterterms has a finite limit when Λ formally tends to infinity. The new finite limiting H_λ is called the renormalized Hamiltonian, $H_{R\lambda}$.

$$H_{R\lambda} = \lim_{\Lambda \to \infty} H_\lambda \,. \tag{7.42}$$

Since the cutoff λ is chosen arbitrarily, the spectrum of common eigenstates of the renormalized Hamiltonians $H_{R\lambda}$ must be independent of the cutoff λ. Renormalized Hamiltonians at various cutoffs λ are related by a renormalization group transformation. The objective of the renormalization procedure is to find the necessary counterterms to be added to the Hamiltonian H so that the formal limit of the new H when Λ is sent to infinity exists but it is not obvious how to construct the Hamiltonian counterterms.

Analysis of divergences in the effective Hamiltonians H_λ when $\Lambda \to \infty$, indicates which counterterms should be added.

There are two steps to make. The first step is to find the general structure of the counterterms. One can use power counting in perturbation theory to isolate divergences in the effective Hamiltonian and discover the structure of the necessary counterterms.

The second step involves identifying special conditions, such as, for example, Poincaré invariance of the spectrum, and using these to constrain finite parts of the counterterms. This means one fits the free finite parameters in the Hamiltonian to observable physical features. That includes also symmetries like Poincaré symmetry. The second step may require a numerical procedure involving diagonalization of the full Hamiltonian with counterterms containing the adjustable finite parts.

The first step can be done using differential equations for the effective interactions as functions of the cutoff λ. By definition

$$dV_\lambda = P_\lambda V_{\lambda+d\lambda} P_\lambda - V_\lambda \,. \tag{7.43}$$

Multiplication of matrices in spin space and integration over longitudinal momentum fractions are represented as multiplication. Integration over transverse momenta is denoted by integration over variable z for κ^2, z' for κ'^2 *etc.*. In this convention, the kernels of the projection operators are represented by

$$P_\lambda(z,z') = \Theta(\lambda - z)\Theta(z)\delta(z - z')\,, \tag{7.44}$$
$$Q_\lambda(z,z') = \Theta(\Lambda^2 - z)\Theta(z - \lambda)\delta(z - z')\,, \tag{7.45}$$

and an infinitesimal projection operator is introduced

$$dQ_\lambda(z,z') = -d\lambda\delta(z - \lambda)\delta(z - z')\,. \tag{7.46}$$

The corresponding space of states is called $d\lambda$-space. Using the identity

$$\frac{1}{E - H_0 - Q_\lambda gVQ_\lambda} = \frac{1}{1 - \frac{Q_\lambda}{E-H_0} gVQ_\lambda} \frac{1}{E - H_0} \tag{7.47}$$

one can rewrite Eq. (7.41) as

$$\begin{aligned} V_\lambda =& P_\lambda gVP_\lambda \\ &+ P_\lambda gV[\frac{Q_\lambda}{E-H_0} + \frac{Q_\lambda}{E-H_0} gV \frac{Q_\lambda}{E-H_0} \\ &+ \frac{Q_\lambda}{E-H_0} gV \frac{Q_\lambda}{E-H_0} gV \frac{Q_\lambda}{E-H_0} + \ldots]gVP_\lambda \,. \end{aligned} \tag{7.48}$$

Then,

$$dV_\lambda = P_\lambda gV[\frac{dQ_\lambda}{E-H_0} + \frac{dQ_\lambda}{E-H_0}gV\frac{Q_\lambda}{E-H_0} + \frac{Q_\lambda}{E-H_0}gV\frac{dQ_\lambda}{E-H_0}$$
$$+ \frac{dQ_\lambda}{E-H_0}gV\frac{Q_\lambda}{E-H_0}gV\frac{Q_\lambda}{E-H_0}$$
$$+ \frac{Q_\lambda}{E-H_0}gV\frac{dQ_\lambda}{E-H_0}gV\frac{Q_\lambda}{E-H_0}$$
$$+ \frac{Q_\lambda}{E-H_0}gV\frac{Q_\lambda}{E-H_0}gV\frac{dQ_\lambda}{E-H_0} + \ldots]gVP_\lambda\,, \qquad (7.49)$$

so that

$$dV_\lambda = P_\lambda V_\lambda \frac{dQ_\lambda}{E-H_0}V_\lambda P_\lambda\,. \qquad (7.50)$$

Due to the simplicity of the model Hamiltonian the projection operator dQ_λ is infinitesimally small and action of the potential term takes states out of the $d\lambda$-space. Therefore, dQ_λ appears at most once in each term of the sum in Eq. (7.49) and the resulting Eq. (7.50) takes a simple form. In terms of potential kernels the latter equation reads

$$\frac{d}{d\lambda}V_\lambda(z,z') = V_\lambda(z,\lambda)\frac{1}{H_0(\lambda)-E}V_\lambda(\lambda,z')\,. \qquad (7.51)$$

Divergences in the effective potential V_λ when $\Lambda \to \infty$ and no counterterms are included, are only logarithmic, and one may consider such cutoffs λ that eigenvalues E are much smaller than $H_0(\lambda)$. Therefore, one may neglect the eigenvalue in the denominator on the right hand side of Eq. (7.51). It is convenient to include a factor P^+ in the denominator, and consider interaction term which includes another factor of P^+ and a minus sign. These conventions allow easier notation when we focus on the transverse divergence. So, denoting the inverse of the product $P^+H_0(\lambda)$ by K_λ, one obtains the following differential equation,

$$\frac{d}{d\lambda}V_\lambda(z,z') = -V_\lambda(z,\lambda)K_\lambda V_\lambda(\lambda,z')\,. \qquad (7.52)$$

The integral form of this equation is

$$V_\lambda(z,z') = V_{\Lambda^2}(z,z') + \int_\lambda^{\Lambda^2} ds V_s(z,s)K_s V_s(s,z')\,. \qquad (7.53)$$

The initial condition at $s = \Lambda^2 \to \infty$ is

$$V_{\Lambda^2}(z,z') = gV(z,z') + c_\Lambda(z,z')\,, \qquad (7.54)$$

where $c_\Lambda(z,z')$ denotes a counterterm. The problem is to find the structure of the counterterm in the limit $\Lambda \to \infty$. The difficulty is that one has to solve a first order differential equation with two boundary conditions. Namely, one needs to specify the potential at $s = \Lambda^2$ and request that the effective potential is independent of Λ at $s = \lambda \ll \Lambda^2$.

The generally valid structure of the counterterm can be defined by recursive relations among terms involving different powers of g in the appropriate expansion. One writes the counterterm as a series in the coupling constant. The series for the counterterm starts with a term proportional to g^2. The family of kernels $V_s(z,z')$ for $\lambda_0 \le s \le \Lambda^2$ is also written as a series in the coupling constant g. The family is viewed as a single function of three arguments z, z', and s. The first term in the series expansion of the kernel is proportional to g. It is equal to the original kernel $gV(z,z')$ (including the factor P^+). The term of order g is independent of s. λ_0 is much larger than M^2 but otherwise it is arbitrary. Order by order in the series expansion one can show that the divergences due to $\Lambda \to \infty$ appear in the effective potential at small cutoffs only in $V_{\lambda_0}(0,0)$. Therefore, it is sufficient to subtract the divergent part of $V_{\lambda_0}(0,0)$ to obtain cutoff independence of the effective Hamiltonian H_λ. The recursion and proof to all orders in g are given in Ref. [82].

Once it is established that divergences appear only in $V_{\lambda_0}(0,0)$ the renormalized Hamiltonians can be found in the following iterative procedure.

We assume that the counterterm is independent of z and z', *i.e.* $c_\Lambda(z,z') = c_\Lambda$. Eq. (7.53) takes the form

$$V_\lambda(z,z') = gV(z,z') + c_\Lambda + \int_\lambda^{\Lambda^2} ds V_s(z,s) K_s V_s(s,z') \,. \tag{7.55}$$

The first approximation to the effective kernel is the bare potential itself,

$$V_{1\lambda}(z,z') = gV(z,z') \,, \tag{7.56}$$

and

$$c_{1\Lambda} = 0 \,. \tag{7.57}$$

The second approximation is given by

$$V_{2\lambda}(z,z') = gV(z,z') + c_{2\Lambda} + \int_\lambda^{\Lambda^2} ds gV(z,s) K_s gV(s,z') \,, \tag{7.58}$$

and

$$c_{2\Lambda} = f_2 - \int_{\lambda_0}^{\Lambda^2} dsgV(0,s)K_sgV(s,0)\,, \tag{7.59}$$

where f_2 is allowed to be a finite spin matrix which is a function of the longitudinal momenta of the fermions. Looking at Eqs (7.30) to (7.37) and K_λ we see that the counterterm contains two factors of the longitudinal momentum fractions in denominator. This implies that the counterterm is, in fact, containing a function of dimension $(x^-)^2$. In the second approximation it is $(\partial^+)^{-1}$ acting twice on various fermion fields in the four-fermion term suggested by the power counting.

The procedure can be iterated according to the recursion

$$V_{(n+1)\lambda}(z,z') = gV(z,z') + f_{n+1} - \int_{\lambda_0}^{\Lambda^2} dsV_{ns}(0,s)K_sV_{ns}(s,0) + \int_{\lambda}^{\Lambda^2} dsV_{ns}(z,s)K_sV_{ns}(s,z')\,. \tag{7.60}$$

At the $n+1$ iteration, one has to choose the arbitrary finite function f_{n+1} so that some symmetry requirements are satisfied. It is a matter of case by case study to find how can one implement such constraints.

If the limit of the iteration procedure for $n \to \infty$ exists, the solution to the renormalization problem is given by the potential which satisfies the following equation

$$V_\lambda(z,z') = gV(z,z') + f - \int_{\lambda_0}^{\Lambda^2} dsV_s(0,s)K_sV_s(s,0) + \int_{\lambda}^{\Lambda^2} dsV_s(z,s)K_sV_s(s,z')\,, \tag{7.61}$$

when $\Lambda^2 \to \infty$. $f = o(g^2)$ is the finite part of the counterterm that is defined using the cutoff λ_0 as the lower limit of integration. Therefore, f and λ_0 are related.

In practical calculations, one considers V_{Λ^2} which is a sum of $gV(z,z')$ and certain unknown function of the longitudinal momentum fractions of the fermions. Equation (7.61) can be solved numerically. One has to fit the Λ-dependence of the counterterm so that no Λ-dependence appears in observables. In simple models, one may also study analytic examples that illustrate what kind of solutions are expected. The renormalized interactions V_λ may develop singularities for some values of the cutoff λ.

We quote below three instructive examples from Ref. [82]. If the kernel is independent of transverse momenta then the renormalized interaction is too,

$$c_\lambda(z, z') = c_\lambda \,. \tag{7.62}$$

Eq. (7.61) simplifies to

$$c_\lambda = f - \int_{\lambda_0}^{\Lambda^2} ds[g + c_s]K_s[g + c_s] + \int_{\lambda}^{\Lambda^2} ds[g + c_s]K_s[g + c_s] \,, \tag{7.63}$$

which is equivalent to

$$c_\lambda = f - \int_{\lambda_0}^{\lambda} ds[g + c_s]K_s[g + c_s] \,. \tag{7.64}$$

The solution is

$$c_\lambda = [1 + (f + g) \int_{\lambda_0}^{\lambda} dsK_s]^{-1}[f - (f + g) \int_{\lambda_0}^{\lambda} dsK_s g] \,, \tag{7.65}$$

so that the renormalized interaction is

$$V_\lambda = [1 + (f + g) \int_{\lambda_0}^{\lambda} dsK_s]^{-1}(f + g) \,. \tag{7.66}$$

In this example the renormalized interaction vanishes when the cutoff λ tends to infinity, which is analogous to asymptotic freedom in QCD. When the bare interaction is of the form $gV(z, z') = ig\theta(z - z') - ig\theta(z' - z)$, the integration of Eq. (7.61) produces a singular behavior of the renormalized interaction at certain values of the cutoff λ. The example is motivated by analysis of the partial wave potentials for $|j_z| = 1$ in the Yukawa model, where antisymmetric imaginary kernels appear. One considers

$$V_s(z, z') = ig\theta(z - z') - ig\theta(z' - z) + c_s \,, \tag{7.67}$$

where c_s is independent of the transverse momentum as in the example above. Here it is assumed that g and c_s are numbers independent of the longitudinal momenta and spins of the fermions. Only then can a simple analytic form for c_s be easily derived. K_s is put equal to s^{-1} for simplicity. With the above simplifications direct integration of Eq. (7.61) leads to the solution

$$c_\lambda = g\frac{f - g\tan\left(g\log\frac{\lambda}{\lambda_0}\right)}{g + f\tan\left(g\log\frac{\lambda}{\lambda_0}\right)}. \tag{7.68}$$

Note that the renormalized coupling strength tends to infinity for certain values of the cutoff λ. Near such values of the cutoff perturbation theory becomes invalid. In order to avoid such features it is desirable to incorporate asymptotic freedom in model Hamiltonians for quarks and gluons. The advantage of asymptotic freedom is that it forces the renormalized couplings to approach zero so they cannot go to infinity instead. If the renormalized kernel V_λ can be approximated by a separable form,

$$V_\lambda(z, z') = \sum_{ij} h_i^*(z)c_{ij}(\lambda)h_j(z'), \tag{7.69}$$

or, equivalently, in abbreviated notation,

$$V_\lambda(z, z') = h^\dagger(z)c_\lambda h(z'), \tag{7.70}$$

and satisfies Eq. (7.52), then the matrix c_λ may be written explicitly as

$$c_\lambda = [1 + c_{\lambda_0}\int_{\lambda_0}^{\lambda} ds h(s)K_s h^\dagger(s)]^{-1}c_{\lambda_0}. \tag{7.71}$$

This example implies that using basis functions instead of grids in momentum space helps in constructing numerical renormalization procedures.

7.3. Numerical studies in the Yukawa model

First numerical attempts to investigate renormalization effects in two fermion bound states in the LF Yukawa theory have been reported by Głazek, Harindranath, Pinsky, Shigemitsu and Wilson [84]. These authors studied a Yukawa model of Eq. (7.27) in full detail. When the self-energies of fermions are included one has to subtract their divergent parts. The fermion self-energy in the Yukawa model, analogous to Σ from Eq. (2.46) in QED and a similar expression for quarks from Section 6, is

$$\Sigma_i = -\frac{1}{4}\frac{\alpha_g}{(2\pi)^2}\int dz \int d^2\kappa \frac{\frac{1}{1-z}\left[m^2(1+z^{-1})^2+\kappa^2 z^{-2}\right]}{x_i(\mathcal{M}_{12}^2 - M^2) + \frac{\kappa^2+m^2}{z} + \frac{\kappa^2+\mu^2}{1-z} - m^2}. \tag{7.72}$$

This is divergent and needs subtraction. The self-energy can be written as

$$\Sigma_i = \Sigma[x_i(\mathcal{M}_{12}^2 - M^2)]. \tag{7.73}$$

The function $\Sigma(a)$ can be written in the form

$$\Sigma(a) = \Sigma(0) + a\Sigma'(0) + \text{rest}. \tag{7.74}$$

The first term is strongly cutoff dependent and can be subtracted by introducing appropriate δm^2 for fermions in the two-fermion Fock sector. The second term is logarithmically divergent and, combined with the free energies of fermions in the two-fermion sector, leads to the term

$$[1+\Sigma'(0)](\mathcal{M}_{12}^2 - M^2) \tag{7.75}$$

in the once subtracted two-body equation. $\Sigma'(0)$ is logarithmically divergent and positive. In order to remove the divergence one may divide the effective two-body bound state equation by the bare coupling constant α_g. Then, one can see that the logarithmic divergence can be removed by requesting

$$\frac{1+\Sigma'(0)}{\alpha_g} = \frac{1}{\alpha}, \tag{7.76}$$

where α is a "physical", *i.e.* cutoff independent, coupling constant. This is an ad hoc procedure. It leads to the problem that the bare coupling constant is related to the physical coupling by

$$\alpha_g = \frac{\alpha}{1 - \alpha\alpha_g^{-1}\Sigma'(0)}. \tag{7.77}$$

Since $\Sigma'(0)$ is positive and grows when the cutoff becomes more extreme we have the feature that g becomes imaginary for sufficiently large cutoffs if α is kept fixed. This is called a triviality problem, since when one attempts to send the cutoff to infinity the "physical" coupling goes to zero and no binding or nontrivial interaction occurs. Thus, for a fixed value of the coupling constant α we have certain maximal allowed value of the cutoff. The numerical studies in Ref. [84] were carried out within the bounds set by the triviality. The coupling α should be sufficiently strong in order to create bound states; for massive bosons the fermion–boson coupling constant is required to exceed certain minimal value for that the binding is strong enough and bound states exist. On the other hand the coupling should not be too large in order to allow sufficiently large cutoffs. Large cutoffs are needed to make the renormalization procedure, based on the assumption that the cutoff is large, work. The compromise has stiff limits. For each value of the cutoff there is a critical value of the coupling, α_c which is the maximal possible coupling for that cutoff. In the range of the transverse momentum cutoff $10m < \Lambda < 100m$ the approximate result is that

$$\alpha_c \sim 3.2 - 0.5 \log \frac{\Lambda}{m} . \tag{7.78}$$

For such couplings and cutoffs the binding is weak. The bound state mass squared, M^2 is only a few % smaller than $4m^2$. Still, the difference $4m^2 - M^2$ may vary in the above range of the cutoff Λ by about 7 to even 13% if no counterterms are included. Therefore, although the cutoff dependence is not dramatic, it is clearly visible. By introducing counterterms of only fourth order in g (second order in α) one can reduce this cutoff dependence by an order of magnitude, roughly.

The important point is that the spectrum of bound states of the bare Hamiltonian does not exhibit degeneracy which is required by rotational invariance. The counterterms not only remove the cutoff dependence but also provide a possibility to make the spectrum become degenerate as required. The finite parts of the counterterms are fitted to obtain the degeneracy. Then, one can calculate the bound state wave functions and consider their implications for the bound state observables. Ref. [84] provides examples of numerical solutions for the wave functions.

One should keep in mind that the ad hoc coupling constant renormalization used in Ref. [84] is not a general solution to the acute problem with the logarithmic divergences in self-energies which are proportional to an eigenvalue. It is the simplicity of the Yukawa model which lets us get away with the coupling constant renormalization as a cure for the eigenvalue dependent divergences. This problem has already been discussed in Section 6.

7.4. Covariance conditions

In the previous Section it was observed that the finite parts of the counterterms provide an opportunity to obtain a physically acceptable spectrum of bound states which exhibits the mass degeneracy as required by rotational invariance. Without having the freedom to fit the finite parts of the counterterms one could not achieve this result. Wilson suggested that matrix elements of current operators should provide additional constraints [75].

Below, we explain how a covariance condition may enforce constraints on a theory.

Let us consider a hypothetical matrix element

$$v^\mu = \langle 0|J^\mu(0)|\text{bound state}\rangle\,, \tag{7.79}$$

where $J^\mu(x)$ is called a current operator and the bound state is assumed to be

$$|\text{bound state}\rangle = \sum_{\lambda\sigma}\int dx d^2\kappa^\perp \phi^{\lambda\sigma}(x,\kappa^\perp) b^\dagger_{xP+\kappa,\lambda} d^\dagger_{(1-x)P-\kappa,\sigma}|0\rangle\,. \tag{7.80}$$

Suppose the current operator is

$$J^\mu(0) = \bar{\tilde{\psi}}(0)\gamma^\mu\tilde{\psi}(0)\,. \tag{7.81}$$

Then

$$v^\mu = \sum_{\lambda\sigma}\int dx d^2\kappa^\perp \bar{v}_{m,(1-x)P-\kappa,\sigma}\gamma^\mu u_{m,xP+\kappa,\lambda}\phi^{\lambda\sigma}(x,\kappa^\perp)\,. \tag{7.82}$$

Let us assume that

$$\phi^{\lambda\sigma}(x,\kappa^\perp) = \delta^{\lambda,-\sigma} f^\lambda(x,|\kappa^\perp|)\,. \tag{7.83}$$

Evaluating the spinorial matrix elements one obtains the result

$$v^\mu = c_1 P^\mu + c_2 g^{+\mu}\,, \tag{7.84}$$

where

$$c_1 = -\sum_\lambda \int dx d^2\kappa^\perp f^\lambda(x,|\kappa^\perp|)2\sqrt{x(1-x)} \tag{7.85}$$

and

$$c_2 = \frac{1}{P^+}\sum_\lambda\int dx d^2\kappa^\perp f^\lambda(x,|\kappa^\perp|)\sqrt{x(1-x)}\left[M^2 + \frac{\kappa^2+m^2}{x(1-x)}\right]\,. \tag{7.86}$$

We see that the matrix element has a covariant structure in v^+ and $v^\perp$ independently of details of the wave function but contains a noncovariant part in v^-. Moreover, if the function $f(x,|\kappa^\perp|)$ is real and positive the coefficient c_2 cannot vanish. A function which changes sign is required. One could also introduce nondiagonal spin structure and angle dependent wave functions in Eq. (7.83). More complicated discussion is required for more complicated wave functions. However, this example already illustrates some important features.

The first feature is that in LF dynamics under some general assumptions the structure of certain phenomenologically relevant matrix elements may automatically have the structure required by special relativity, independently of details of a wave function. This result follows from the LF symmetries which determined the structure of the state and the spinors in the current.

Second feature is that in order to obtain acceptable structures of some bound state matrix elements one has to consider Hamiltonians acting in a bigger space than the space sufficient for the description of the bound state itself. Namely, some matrix elements, here v^-, are not obviously relevant to physics until one formulates a more complete theory in which such matrix elements may appear. For example, the problem with v^- does not appear in the case when a vector field component A^+ which is able to couple to v^-, is put equal to zero.

The third feature is that there may be required considerable variations in the bound state Hamiltonian parameters in order to obtain desired structure of some bound state matrix elements. One has also to consider the possibility that the current operators which may give physically relevant transition amplitudes will have to have more complicated structure than in free field theory or in perturbation theory. For example, in the elementary example of this Section one could pose a question how should the operator J^μ be defined so that $c_2 = 0$ in v^- and the same c_1 is obtained from calculating v^- as from calculating the other matrix elements with $\mu = +$ or $\mu = \perp$.

In more complicated matrix elements the issues of rotational invariance and angular momentum (spin) become complicated. It is not possible to obtain rotationally invariant results without introducing new terms in a Hamiltonian. The new terms should allow wave functions and operators to combine into the rotationally invariant answers for physical observables. The importance of such considerations in numerical calculations is stressed in Refs [36] and [85]. See also Ref. [34] which gives an explicit evaluation of current matrix elements in a model spin-1 bound state. In the next Section, we discuss another model example which further illustrates how Poincaré covariance constrains LF Hamiltonians.

8. Model study

In this section we present a model which illustrates that a Hamiltonian acting in a small space restricted by strong cutoffs may lead to nontrivial relativistic results. We have seen in LF QED that imposing a restriction on the number of bare particles to at most three and considering only small relative momenta leads to phenomenologically valid approximation for positronium. The small coupling constant made this approximation selfconsistent in QED. However, if the coupling was considerably larger one would encounter problems discussed in Section 6. The present model provides an example of a LF Hamiltonian theory which can be considered for coupling constants of order one, or even much larger in some cases, and the solution for the fermion–boson scattering amplitude is covariant. This is achieved by introducing counterterms. It is important to have this example since otherwise one might question the possibility to obtain relativistic results from quantum theories constructed in limited Hilbert spaces. The model we discuss has been originally formulated by Głazek and Perry [86] and we reproduce parts of that formulation in Section 8.1. The Hamiltonian of the model acts in a space of two Fock sectors. Namely, a sector with one fermion and a sector with one fermion and one boson. This feature makes the model resemble a sector of the Lee model [87]. The major difference is that the Lee model and its relativistic generalizations were constructed using ET form of dynamics while the new model is constructed using LF dynamics in a way which is motivated by renormalization theory for Hamiltonians. Fuda [88] has considered a model resembling this one but he does not discuss the problems of regularization and renormalization of LF Hamiltonian eigenvalue problems which is the central issue of this paper. Without special terms in the Hamiltonian one could not obtain rotational invariance in the model. There are two kinds of such terms. There are seagull terms which correspond to the fermion–antifermion pair creation in the ET dynamics and there are counterterms which remove divergences. A side effect of the renormalization is that the new model exhibits triviality, a feature found in a two-fermion system from the previous Section and known in the Lee model. In this Section we discuss three physical properties of the new model: fermion–boson scattering, physical fermion form factors and the fermion structure functions.

8.1. Fermion–boson binding and scattering [86]

To restrict the number of necessary counterterms in the model Hamiltonian, we want to maintain as many symmetries as possible. We must violate the dynamical rotational symmetry, but it is still possible to maintain the kinematic symmetries. For example, these symmetries are maintained if we choose cutoffs that are functions of longitudinal momentum fractions and relative transverse momenta. In order to present the model as an example of a more general LF Hamiltonian renormalization program we give a slightly broader discussion than this limited example truly requires.

The Hamiltonian must be constructed using only the dynamical degrees of freedom on the LF and it is restricted by simple dimensional arguments. In canonical LF Hamiltonian densities one has field operators, derivatives and masses. There appear inverse powers of ∂^+ but no inverse powers of $\partial^\perp$ or masses. As we already explained in Section 7, one must consider longitudinal and transverse dimensions separately, so there are two length scales, x^- and $x^\perp$ [74]. Dimensions of fields are easily determined from a free field theory. Two components of a Dirac field are constrained, leaving two dynamical degrees of freedom, $\psi_+(x)$. The scalar field is $\phi(x)$. The engineering dimension of each of these variables is

$$[\partial^+] = \left[\frac{1}{x^-}\right], \quad [\partial^\perp] = \left[\frac{1}{x^\perp}\right], \quad [m] = \left[\frac{1}{x^\perp}\right],$$
$$[\psi_+(x)] = \left[\frac{1}{x^\perp\sqrt{x^-}}\right], \quad [\phi(x)] = \left[\frac{1}{x^\perp}\right]. \tag{8.1}$$

The Hamiltonian $H = \int dx^- d^2x^\perp \mathcal{H}$ is the integral of the Hamiltonian density, so that

$$[H] = \left[\frac{x^-}{x^{\perp 2}}\right], \quad [\mathcal{H}] = \left[\frac{1}{x^{\perp 4}}\right]. \tag{8.2}$$

One should mention that the association of dimensions with fields is not equivalent to estimating their size in a sense of some metric. The latter requires introduction of wave packets (wavelets). We will not be concerned with that issue here, *cf.* Section 7.2. and Ref. [24]. Before we can list the allowed terms in the Hamiltonian we need to mention that operators that require inverse powers of a mass are typically discarded as non-renormalizable. For example, ϕ^5/m leads to a ϕ^6/m^2 counterterm in perturbation theory, which in turn leads to higher powers of ϕ. In perturbation theory one must introduce all powers of ϕ if one includes ϕ^5/m, and delicately balance their couplings to obtain finite results. This problem does not occur in the model Hamiltonian because high powers of a field operator cannot act in a truncated Fock space.

With all of these simplifications we can list the remaining terms allowed by such dimensional analysis. The terms that involve no fermion operators are

$$m^3\phi, \; m^2\phi^2, \; \partial^{\perp 2}\phi^2, \; m\phi^3, \; \phi^4. \tag{8.3}$$

All of these terms should be understood to be normal-ordered. We do not specify how the derivatives are to be distributed. The term linear in ϕ contributes only if we allow a zero longitudinal momentum mode. Since the free energy of any mode with zero longitudinal momentum is infinite, we will discard this mode when we regulate the theory; therefore, we can ignore the linear term. We will not encounter infrared features which would force us to reconsider this step in the limited model. Also, neither the ϕ^3 term nor the ϕ^4 term can contribute because we never allow more than one boson. Including fermion fields one obtains

$$\frac{1}{\partial^+}\psi_+^\dagger \Gamma \Big\{ m^2 \,, \partial^{\perp 2} \,, m\gamma^\perp \partial^\perp \,, \phi\gamma^\perp\partial^\perp \,, m\phi \,, \phi^2 \Big\} \psi_+ \,, \tag{8.4}$$

$$\left(\frac{1}{\partial^+}\right)^2 \psi_+^\dagger \Gamma_1 \psi_+ \psi_+^\dagger \Gamma_2 \psi_+ \,. \tag{8.5}$$

Again we do not specify how derivatives, including inverse powers of derivatives, are to be distributed. Moreover, we do not list all possible gamma matrices that can occur in these terms. For example, in equation (8.4) one can have $\Gamma = 1 \,, \gamma_5 \,, \sigma_{12}$, *etc*. These cannot be restricted, except to insist that all transverse indices be contracted, until one brings in other considerations, such as parity.

At this point we have drastically limited the class of Hamiltonians we consider, and as a final limitation we assume that the tensor structure of the interactions is the same as found in the canonical scalar Yukawa Hamiltonian. This leads to the Hamiltonian

$$H = H_0 + H_1 + H_2 + H_3 \,, \tag{8.6}$$

$$\begin{aligned} H_0 &= \tfrac{1}{2} \int dx^- d^2x^\perp \Big[\phi(x) \Big(-\partial^{\perp 2} + \mu^2 \Big) \phi(x) \\ &\quad + 2\psi_+^\dagger(x) \Big(-\partial^{\perp 2} + m_0^2 \Big) \frac{1}{i\partial^+} \psi_+(x) \Big] \,, \end{aligned} \tag{8.7}$$

$$H_1 = g_1 \int dx^- d^2x^\perp \, \psi_+^\dagger(x) \phi(x) \frac{1}{i\partial^+} \Big(i\gamma^\perp \cdot \partial^\perp \Big) \psi_+(x) + \text{h.c.} \,, \tag{8.8}$$

$$H_2 = g_2 m_2 \int dx^- d^2x^\perp \, \psi_+^\dagger(x) \phi(x) \frac{1}{i\partial^+} \psi_+(x) + \text{h.c.} \,, \tag{8.9}$$

$$H_3 = g_3^2 \int dx^- d^2x^\perp \, \psi_+^\dagger(x) \phi(x) \frac{1}{i\partial^+} \phi(x) \psi_+(x) \,. \tag{8.10}$$

All of these terms occur in the canonical Hamiltonian; however, in the canonical Hamiltonian $m_1 = m_2 = m$ and $g_1 = g_2 = g_3 = g$. If we were to consider more particles (*eg*., two fermions and one boson), we would need terms that are not found in the canonical Hamiltonian to obtain finite covariant results.

This Hamiltonian requires one further modification before we can obtain covariant results. We must allow the parameters to depend on the Fock space sectors in or between which they act [89, 81]. To accomplish this we can introduce Fock space projection operators

$$P_f = \sum_\lambda \int [dp]\, |p\lambda\rangle\langle p\lambda| \,, \tag{8.11}$$

$$P_{fb} = \sum_\lambda \int [dp][dq]\, |p\lambda, q\rangle\langle p\lambda, q| \,, \tag{8.12}$$

where

$$|p\lambda\rangle = b^\dagger(p,\lambda)|0\rangle \tag{8.13}$$

and

$$|p\lambda, q\rangle = b^\dagger(p,\lambda)a^\dagger(q)|0\rangle \,. \tag{8.14}$$

We need to apply proper combinations of these projection operators to each term in the Hamiltonian, and introduce separate coefficients for each resultant term. Since the projection operators are simple only in momentum space, it is easiest to work entirely in momentum space. One uses the familiar expansions of fields into creation and annihilation operators. We also rewrite various terms in the Hamiltonian using operators which involve the Dirac spinors from Eq. (2.20), $u_{mp\lambda}$, instead of using only the dynamically independent components ψ_+. This simplifies the task of deriving a manifestly covariant T-matrix, because the fermion field corresponds to a physical particle and covariant matrix elements are most easily written in terms of $u_{mp\lambda}$ in this case. These spinors involve the mass parameter m, as indicated by the subscript. Moving to momentum space, recombining H_1 and H_2 so that $u_{mp\lambda}$ appears, and redefining some of the mass parameters, we have

$$H = H_0^f + H_0^{fb} + H_1 + H_2 + H_3 \,, \tag{8.15}$$

$$H_0^f = \sum_\lambda \int [dp]\, |p\lambda\rangle \left(\frac{p^{\perp 2} + m_1^2 + g^2\omega^2}{p^+} \right) \langle p\lambda| \,, \tag{8.16}$$

$$H_0^{fb} = \sum_\lambda \int [dp]\,[dq]\, |p\lambda, q\rangle \left(\frac{p^{\perp 2} + m^2}{p^+} + \frac{q^{\perp 2} + \mu^2}{q^+} \right) \langle p\lambda, q| \,, \tag{8.17}$$

$$
\begin{aligned}
H_1 + H_2 &= g \sum_{\lambda\sigma} \int [dp]\,[dk]\,[dq]\,\theta\big(\Lambda^2 - \kappa^2\big)\, 2(2\pi)^3 \delta^3(p+q-k) \\
&\times \left[|p\lambda, q\rangle\, \overline{u}_{mp\lambda} \left(1 + \frac{\delta m}{2k^+}\gamma^+\right) u_{mk\sigma}\, \langle k\sigma| \right. \\
&\left. + |k\sigma\rangle\, \overline{u}_{mk\sigma} \left(1 + \frac{\delta m}{2k^+}\gamma^+\right) u_{mp\lambda}\, \langle p\lambda, q| \right], \qquad (8.18) \\
H_3 &= g^2 \sum_{\lambda\sigma} \int [dp_1][dp_2][dq_1][dq_2] \\
&\times \theta(\Lambda^2 - \kappa_1^2)\theta(\Lambda^2 - \kappa_2^2) 2(2\pi)^3 \delta(p_2 + q_2 - p_1 - q_1) \\
&\times |p_2\lambda, q_2\rangle\, \overline{u}_{mp_2\lambda} \frac{\gamma^+}{2(p_1^+ + q_1^+)} u_{mp_1\sigma}\, \langle p_1\sigma, q_1| \,. \qquad (8.19)
\end{aligned}
$$

The relative momentum variables are

$$
\kappa^\perp = \frac{q^+ p^\perp - p^+ q^\perp}{p^+ + q^+}\,, \quad \kappa_i^\perp = \frac{q^+ p_i^\perp - p_i^+ q^\perp}{p_i^+ + q^+}\,, \quad (i = 1,2)\,. \qquad (8.20)
$$

We have dropped a part of H_3 in going from Eq. (8.10) to Eq. (8.19), a part that is identical in form to the expression shown in Eq. (8.19) but with $p_1^+ - q_2^+$ replacing $p_1^+ + q_1^+$. As mentioned above, when we make the Fock space truncation, we must project each interaction and break it into pieces, allowing each piece to vary in strength to restore covariance. H_3 provides a simple example. When one goes beyond this model such functions of longitudinal momentum must be further generalized.

In this Hamiltonian the transverse momentum cutoffs appear as vertex regulators that limit only the momentum transfer. The total transverse momentum is not limited. This will regulate all ultraviolet divergences that are caused by states with large relative transverse momentum, but will not regulate divergences caused by states with small longitudinal momentum. To regulate these one imposes an additional cutoff on the longitudinal momentum integrals everywhere in the Hamiltonian,

$$
p^+, k^+ > f^+\,, \quad q^+ > b^+\,. \qquad (8.21)
$$

In other words, all fermion longitudinal momenta are greater than f^+ and all boson longitudinal momenta are greater than b^+. Since this cutoff employs the longitudinal momenta themselves instead of the momentum fractions, it breaks longitudinal boost invariance and we have to carefully choose the ratio f^+/b^+ to restore this symmetry. It is possible to use other cutoffs.

In general one should allow separate couplings to appear in $H_1 + H_2$ and in H_3, but in this special example covariance requires that the same coupling appears. H_0^f gives the free energy of a fermion before interactions are turned on, and the bare mass is divided into two terms, $m_1^2 + g^2\omega^2$, for later algebraic convenience. H_0^{fb} gives the free energy of the states with one fermion and one boson. Because of the Fock space truncation, a state in which a fermion and a boson propagate at large separation experiences no interactions. We have not allowed direct long-range interactions, so at large separation particles must exchange additional particles to interact, and the Fock space truncation excludes this possibility in this sector. As a result, the masses in H_0^{fb} must be the physical masses. H_1 and H_2 are grouped together, but the possibility that they differ in strength is maintained by the terms proportional to δm.

The operator $\left[1 + (\delta m\gamma^+)/(2k^+)\right]$ acts on the spinor $u_{mk\lambda}$ and converts it into a spinor which satisfies a Dirac equation with mass $m + \delta m$ [39]. If we use a subscript for the fermion mass appropriate to a given spinor, we find that [*cf*. Eq. (3.40) in many-body nuclear bound states]

$$\overline{u}_{mp\lambda}u_{mk\sigma} + \overline{u}_{mp\lambda}\frac{\gamma^+}{2k^+}(m_1 - m)u_{mk\sigma} = \overline{u}_{mp\lambda}u_{m_1k\sigma}\,. \tag{8.22}$$

Thus, $H_1 + H_2$ causes transitions from a fermion of mass m_1 to a fermion of mass m and a boson, and vice versa. The occurrence of such an operator should not be surprising, because fermions in different sectors of Fock space can become dressed to a different extent. In addition to requiring different masses to occur in H_0 when projected on different sectors, this requires different spinors to appear in transition matrix elements that occur in the interactions that connect different sectors. This completes the definition of the model as described in Ref. [86].

The calculation of all fermion–boson scattering observables begins with the division of the Hamiltonian into a free and interacting part. The free part governs the propagation of particles when they are separated by a sufficient distance that interactions can be neglected. We write $H = H_0 + H_I$. H_0 is given by $H_0^f + H_0^{fb}$, with $g^2\omega^2$ set equal to zero in H_0^f. The remainder of the Hamiltonian gives H_I.

Following Gell-Mann and Goldberger [90], the S-matrix elements can be calculated from

$$S_{fi} = \langle\phi_f|S|\phi_i\rangle = \delta_{fi} - 2\pi i\delta(E_f - E_i)\langle\phi_f|T(E_i)|\phi_i\rangle\,, \tag{8.23}$$

where $T(E)$ satisfies the equation,

$$T(E) = H_I + H_I\frac{1}{E - H_0 + i\epsilon}T(E)\,. \tag{8.24}$$

$|\phi_i\rangle$ is an initial state with a fermion of mass m, momentum p and polarization λ, and a boson of mass μ and momentum k, while $|\phi_f\rangle$ is a final state with a fermion of mass m, momentum p' and polarization λ', and a boson of mass μ and momentum k'. Eq. (8.24) for the T-matrix can be solved analytically in the present case because H_I has a sufficiently simple algebraic structure. Solving for T includes defining how ω, m_1, δm, f^+ and b^+ are chosen when Λ is large in comparison to m and μ. The choice which renders a covariant scattering amplitude is described in detail in Ref. [86]. Here we quote the result

$$\langle\phi_f|T(E)|\phi_i\rangle = 2(2\pi)^3\delta^3(P_f - P_i)\bar{u}_{mp'\lambda'}\frac{g^2}{\not{P}_i - m_1 - g^2\Sigma(P_i) + i\epsilon}u_{mp\lambda}\,. \tag{8.25}$$

The self-energy $\Sigma(P_i)$ has the covariant form

$$\Sigma(P_i) = \alpha(M_i)\,\not{P}_i + \beta(M_i)m\,, \tag{8.26}$$

where

$$\alpha(M) = -\frac{1}{16\pi^2}\int d\kappa^2\Theta(\Lambda^2-\kappa^2) \times\int_0^1 dx\frac{x}{\kappa^2+\mu^2x+m^2(1-x)-M^2x(1-x)-i\epsilon} \tag{8.27}$$

and

$$\beta(M) = -\frac{1}{16\pi^2}\int d\kappa^2\Theta(\Lambda^2-\kappa^2) \times\int_0^1 dx\frac{1}{\kappa^2+\mu^2x+m^2(1-x)-M^2x(1-x)-i\epsilon}\,. \tag{8.28}$$

The functions $\alpha(M)$ and $\beta(M)$ depend on the cutoff Λ and diverge when $\Lambda\to\infty$. Recently, Burkardt and Langnau studied similar fermion self-interactions in perturbation theory [91]. The scattering matrix T has a pole at the invariant mass M_i of the scattering state equal to the physical fermion mass. The physical fermion states of various velocities form a continuum of eigenstates of our model Hamiltonian with a single discrete bound state mass eigenvalue m. The bare mass is defined to be

$$m_1 = \left\{1 - g^2\left[\alpha(m)+\beta(m)\right]\right\}m\,. \tag{8.29}$$

This choice follows from solving the Hamiltonian eigenvalue problem for a physical fermion state. Due to this definition the divergence in $\beta(M_i)$ in the fermion self-energy in the T matrix is cancelled. Note that the physical fermion bound state eigenvalue equation is not derivable in perturbation theory. The remaining divergence due to $\alpha(M_i)$ can be cancelled by introducing the renormalized coupling constant $\tilde{g}(M)$,

$$\tilde{g}^2(M) = \frac{g^2}{1 - g^2\alpha(M)} = \frac{\tilde{g}^2(m)}{1 - \tilde{g}^2(m)\left[\alpha(M) - \alpha(m)\right]} . \tag{8.30}$$

On the other hand, Eq. (8.30) and the requirements that the bare coupling constant is real and finite, impose triviality limits on the maximal value of the renormalized coupling constant at a given cutoff Λ,

$$\alpha_{\text{max}} \equiv \frac{\tilde{g}^2_{\text{max}}(m)}{4\pi} = -\frac{1}{4\pi\alpha(m)} . \tag{8.31}$$

(The longitudinal cutoffs f^+ and b^+ are sent to zero at an appropriate ratio [86].) α_{max} behaves like $(\log(\Lambda/\mu))^{-1}$ for large Λ. This means that for larger boson masses stronger renormalized coupling constants are allowed.

Substituting the renormalized expressions into Eq. (8.25) one obtains the scattering amplitude A_{fi} which equals T_{fi} without factors $2(2\pi)^3\delta^3(P_f - P_i)$. Namely

$$A_{fi} = \bar{u}_{mp'\lambda'} \frac{\tilde{g}^2(M_i)\theta(\Lambda^2 - \kappa'^2)\theta(\Lambda^2 - \kappa^2)}{\not{P}_i - \tilde{m}(M_i)} u_{mp\lambda} . \tag{8.32}$$

The running mass $\tilde{m}(M_i)$ is

$$\tilde{m}(M_i) = \left\{1 + \tilde{g}^2(M_i)\left[\alpha(M_i) - \alpha(m) + \beta(M_i) - \beta(m)\right]\right\} m \tag{8.33}$$

and equals m for $M_i = m$ at the bound state pole. The total unpolarized fermion–boson scattering cross section is

$$\sigma_{fi} = \frac{1}{64\pi^2 M_i^2} \int d\Omega \sum_{\text{pol}} |A_{fi}|^2 \tag{8.34}$$

and using Eq. (8.32) one obtains [92]

$$\sigma = \frac{1}{8\pi M^2}\theta(\Lambda^2 - \kappa^2)\theta(\Lambda^2 - \kappa'^2)\frac{|\tilde{g}(M)|^4}{|M^2 - \tilde{m}^2(M)|^2} \times \left\{(2m^2 + p^2)(M^2 + |\tilde{m}(M)|^2) + 4mM\sqrt{m^2 + p^2}\, Re\ \tilde{m}(M)\right\} , \tag{8.35}$$

where p^2 is the center of mass momentum;

$$M = M_i = M_f = \sqrt{m^2 + p^2} + \sqrt{\mu^2 + p^2}\,. \tag{8.36}$$

A comment is necessary about the seemingly noncovariant factors of θ-functions in Eqs (8.32) and (8.35). κ and κ' are invariant under kinematic LF transformations (*eg.*, boosts and rotations about the z-axis), but they change under rotations about one of the transverse axes. Therefore, covariance is violated if two observers disagree on whether either of the step functions is zero or one. The invariant mass for scattering states is given by

$$M^2 = \frac{\kappa^2 + m^2}{x} + \frac{\kappa^2 + \mu^2}{1-x} = \frac{\kappa'^2 + m^2}{x'} + \frac{\kappa'^2 + \mu^2}{1-x'}\,. \tag{8.37}$$

It follows that

$$\kappa^2 = \left(M^2 - \frac{m^2}{x} - \frac{\mu^2}{1-x}\right)x(1-x) < \Lambda^2\,. \tag{8.38}$$

Since $x(1-x) \leq 1/4$, we cannot find a frame of reference in which $\kappa^2 > \Lambda^2$ if

$$M^2 < 4\Lambda^2\,. \tag{8.39}$$

In other words, all scattering observables are perfectly covariant for states whose invariant mass satisfies $M^2 < 4\Lambda^2$. This is a remarkable result in a Hamiltonian approach to quantum field theory. To our knowledge this is the first time such precise maintenance of Lorentz covariance has been achieved using a field theory truncated to a few particles. In fact, even for cutoffs smaller than $m+\mu$ one can obtain covariant results. All interactions required for this result are found in the canonical Hamiltonian for the Yukawa theory. However, one cannot use the canonical Hamiltonian without modifications. The strength of various parameters in the Hamiltonian must be allowed to depend on the Fock space sectors within which or between which they act. In this special example one does not encounter the need to introduce renormalization functions as it was necessary in the model from Section 7.2. This is because this model resembles closely perturbation theory and does not involve full complexity of the relativistic Hamiltonian eigenvalue problem.

Numerical studies of the cross section from Eq. (8.35) are presented in Ref. [92]. Practically no residual cutoff dependence appears in the whole range of cutoffs allowed by the triviality bounds. We proceed to the discussion of the physical fermion form factors.

8.2. Form factors [92]

The triviality bounds force us to keep the ultraviolet cutoff finite and we cannot make the cutoff go to infinity, as it is possible in the case of asymptotically free theories. A critical question arises how strong is the dependence of observables on the large but finite cutoff. The answer determines whether it is useful to consider such models with finite cutoffs and without asymptotic freedom.

We consider fermions to carry a charge $e = 1$ and bosons to be uncharged. The physical fermion state is a superposition of a bare fermion state and a fermion–boson state

$$|P\lambda\rangle = N\left[b^{\dagger}_{P\lambda} + \sum_{\sigma}\int [dp][dk]2P^{+}(2\pi)^3\delta^3(P-p-k)f^{\lambda}_{\sigma}b^{\dagger}_{p\sigma}a^{\dagger}_{k}\right]|0\rangle\,, \tag{8.40}$$

where

$$f^{\lambda}_{\sigma} = g\theta(\Lambda^2-\kappa^2)\left[m^2 - \frac{\kappa^2+m^2}{x} - \frac{\kappa^2+\mu^2}{1-x}\right]^{-1}\bar{u}_{mp\sigma}u_{mP\lambda} \tag{8.41}$$

and N is a normalization constant which also gives $F_1(0) = 1$.

We extract the form factors from matrix elements of the current $j^{\mu} = \bar{\psi}\gamma^{\mu}\psi$ for $\mu = +$, between states with the same P^{+} so that the momentum transfer $q^{+} = 0$. In this way one can avoid contribution from the pair creation by the current. The standard expression for a fermion current matrix element is

$$\langle P'\lambda'|j^{+}(0)|P\lambda\rangle = \bar{u}_{mP'\lambda'}\left[F_1(q^2)\gamma^{+} + \frac{F_2(q^2)}{2m}q_{\nu}i\sigma^{\nu+}\right]u_{mP\lambda} \tag{8.42}$$

and evaluation of the matrix element in the model gives

$$\begin{aligned}&\langle P'\lambda'|j^{+}(0)|P\lambda\rangle = \bar{u}_{mP'\lambda'}\\ &\times eN^2\left[\gamma^{+} + \frac{g^2}{(2\pi)^3}\int\frac{dx}{x^2(1-x)}d^2\kappa^{\perp}\frac{\theta'\theta}{D'D}\left(\not{p}'+m\right)\gamma^{+}\left(\not{p}+m\right)\right]u_{mP\lambda}\,,\end{aligned} \tag{8.43}$$

where $\theta = \theta(\Lambda^2-\kappa^2)$, $\theta' = \theta(\Lambda^2-\kappa'^2)$, and D and D' are the denominators of the initial and final state wave functions of Eq. (8.41) with $\kappa'^{\perp} = \kappa^{\perp} + (1-x)q^{\perp}$. Using properties of the Dirac spinors and substituting $\kappa^{\perp} = u^{\perp} - {}^1\!/_2(1-x)q^{\perp}$ and dropping terms linear in $u^{\perp}$ which contribute zero when integrated one can replace the product $(\not{p}'+m)\gamma^{+}(\not{p}+m)$ under the integral by a simpler expression and obtain the matrix element in the form of Eq. (8.42). Thus, the form factors are equal

$$F_1(q^2) = N^2[1 + g^2 I(q^2)] \tag{8.44}$$

and

$$F_2(q^2) = N^2 g^2 J(Q), \tag{8.45}$$

where $N^{-2} = 1 + g^2 I(0)$,

$$I(q^2) = \frac{1}{(2\pi)^3}\int \frac{dx d^2\kappa^\perp}{x^2(1-x)}\frac{\theta'\theta}{D'D}\left[(1+x)^2 m^2 + \frac{1}{4}(1-x)^2 q^2 + u^{\perp 2}\right] \tag{8.46a}$$

and

$$J(q^2) = \frac{1}{(2\pi)^3}\int \frac{dx d^2\kappa^\perp}{x^2(1-x)}\frac{\theta'\theta}{D'D} 2m^2(1-x^2)\,. \tag{8.46b}$$

When the bare coupling constant is expressed by the renormalized coupling $\tilde{g} = \tilde{g}(m)$ one obtains

$$F_1(q^2) = \frac{1 + \tilde{g}^2[I(q^2) + \alpha(m)]}{1 + \tilde{g}^2[I(0) + \alpha(m)]} \tag{8.47}$$

and

$$F_2(q^2) = \frac{\tilde{g}^2 J(q^2)}{1 + \tilde{g}^2[I(0) + \alpha(m)]}\,. \tag{8.48}$$

One finds by inspection that the divergent parts of the integrals in $I(q^2)$ or $I(0)$ are cancelled against the divergence in $\alpha(m)$. $J(q^2)$ is a convergent integral. Thus one could obtain fully cutoff independent predictions for the form factors in the model if not the triviality bounds which prevent letting $\Lambda \to \infty$. Numerically, the residual cutoff dependence of the form factors within the triviality bounds is negligibly small, on the order of a few percent or less for light bosons [92]. The cutoff dependence originates from the momentum dependent fermion spin effects. For light bosons, the bare fermions do not move much within the physical fermion bound state and the form factors are not sensitive to the cutoff. For heavy mesons, effects of the fermion motion are growing up to the scale of the meson mass and the two-body sector contribution to the Dirac form factor becomes considerably sensitive to the cutoff. However, when the boson mass becomes much larger than the fermion mass the contribution of the fermion–boson sector to the physical fermion eigenstate is strongly suppressed and the physical fermion approaches a pointlike bare fermion structure rapidly. The study of Ref. [92] suggests that renormalized Hamiltonian models with triviality may be useful in phenomenology. This is encouraging for relativistic nuclear physics where asymptotic freedom does not appear and cutoffs cannot be made arbitrarily large.

It is not enough to calculate only j^+ matrix elements for $q^+ = 0$, since other matrix elements in configurations rotated around transverse axes may lead to different results and they actually do. A model cannot aspire to an explanation of the composite particle structure if it is not consistently describing the full scattering amplitude. One is forced to consider corrections to the current operator and to the bound state. Model corrections in perturbation theory involve fermion–antifermion pairs [85]. The fermion–antifermion pairs may alter results for matrix elements of j^+ through a change in the normalization of states and through their own contributions even for $q^+ = 0$ if the fermion and antifermion contributions do not cancel each other.

It has been stressed by Karmanov [36] that few-body models usually do not automatically give results which conform to requirements of special relativity, charge conservation, etc. One can obtain different results for form factors choosing different components of the currents in different frames of reference [93]. For example, for a spin one bound state of two fermions one can build models which lead to acceptable structure of matrix elements for $+$ or $\perp$ components of the current and two longitudinal polarizations when $q^+ = 0$, but not otherwise [34]. In order to obtain covariant answers for the scattering amplitudes one is forced to consider the bound state dynamics in the presence of projectiles which are introduced to measure the bound state structure and construct a complete Hamiltonian for such system. This is not meant to contradict the principle that the nature of the projectile may not alter the bound state structure. However, one needs a consistent description of the bound state currents and couplings to the fields or particles mediating the projectile-target interaction. That includes a theory of virtual intermediate states involving the projectiles. So far, no complete bound state theory of that kind, or even a practical suggestion how to formulate it has been known to the present author. The more general need for corrections to the current operators and Hamiltonians in renormalization theory has already been discussed in Section 7. We proceed to a limited discussion of the deep inelastic structure functions in the model.

8.3. Structure functions

Our interest in the structure functions follows from the unusual property of the model that one can calculate these structure functions using fully interacting final states as described by the model Hamiltonian. This is another feature of this model which deserves attention as a source of understanding and intuitions in consideration of more complicated cases.

A standard analysis of deep inelastic lepton scattering [94] off a target of momentum P, $P^2 = m^2$, leads to

$$F_2(x,q^2) = \frac{\nu}{2m} \sum_{\lambda W} \left|\langle P\lambda|j^+(0)|W\rangle\right|^2 (2\pi)^3 \delta^4(W - P - q)\,, \tag{8.49}$$

where $j^+(0) = e\bar{\psi}(0)\gamma^+\psi(0)$, $|m\lambda\rangle$ is the physical fermion state at rest (in the abbreviated notation $P = m$ in the rest frame of the target), $Pq = 2m\nu$ and we consider the final scattering states $|W\rangle$ with masses W such that $W^2 = (P+q)^2 > (m+\mu)^2$. Following Ref. [90] we have

$$|W\rangle = \lim_{\epsilon\to 0}\left[1 + \frac{1}{W^- - H_0 + i\epsilon} H_I + \cdots\right]|p\sigma k\rangle\,, \tag{8.50}$$

where $|p\sigma k\rangle = b^\dagger_{p\sigma} a^\dagger_k |0\rangle$ and $W = p + k$. An algebra of expressing the above formula by the T-matrix from Eq. (8.25) gives

$$|W\rangle = \lim_{\epsilon\to 0}\left[1 + \sum_{\sigma'}\int [dp']\left[W^- - p'^-_{m_1} - \frac{g^2\omega^2}{p'^+} + i\epsilon\right]^{-1} |p'\sigma'\rangle\langle p'\sigma'|H_I\right]$$
$$\times \sum_{\sigma''}\int [dp''][dk'']|p''\sigma''k''\rangle\langle p''\sigma''k''|\left[1 + (W^- - H_0 + i\epsilon)^{-1}T\right]|p\sigma k\rangle\,. \tag{8.51}$$

Under the assumption that the momentum transfer is such that $q^+ = 0$ the only term in the current operator which contributes to the matrix elements in Eq. (8.49) is

$$j^+ = \sum_{\lambda_1\lambda_2}\int [dp_1][dp_2] e\bar{u}_{mp_2\lambda_2}\gamma^+ u_{mp_1\lambda_1} b^\dagger_{p_2\lambda_2} b_{p_1\lambda_1}\,. \tag{8.52}$$

Evaluation of the current matrix element between the bound state of Eq. (8.40) and the scattering states of Eq. (8.50) gives

$$\langle P\lambda|j^+|W\rangle = gNe\bar{u}_{mP\lambda}\frac{\not{p} - \not{q} + m}{x(m^2 - \mathcal{M}^2_x)}\gamma^+ u_{mp\sigma} + \bar{u}_{mP\lambda}\frac{Ne}{g}$$
$$\times\left[\gamma^+ + \frac{g^2}{16\pi^3}\int \frac{dz d^2\rho^\perp}{z^2(1-z)}\frac{\not{p}_z + m}{m^2 - \mathcal{M}^2_z}\gamma^+ \frac{\not{p}_z + \not{q} + m}{W^2 - [(p_z+q)_m + k_z]^2 + i\epsilon}\right]$$
$$\times \frac{\tilde{g}^2(W)}{W - \tilde{m}(W) + i\epsilon} u_{mp\sigma}\,, \tag{8.53}$$

where we have explicitly introduced the solution for the final state scattering amplitude and some of the renormalized quantities. Also, $p_z = zP + \rho$, $k_z = (1-z)P - \rho$ as usual, with the minus components calculated from the mass shell condition using the mass values indicated by the corresponding subscripts. $\mathcal{M}_z$ denotes the corresponding invariant mass, $\mathcal{M}_z^2 = (p_z + k_z)^2$, and the same convention applies to symbols with the subscript x instead of z. Substituting Eq. (8.53) into Eq. (8.49) produces nine terms. One of them is a "handbag diagram" and the eight other terms appear as a consequence of the final state interactions. The studies of models of deep inelastic scattering including final state interactions have been recently carried out in ET dynamics by Celenza, Koepf and Shakin [95] but no renormalization issue has been considered by those authors. They used special regulating factors in the interaction vertices which guaranteed convergence of the integrals and dominance of the "handbag diagram" in the Bjorken limit where exact scaling appeared. In the present model one has the quantum mechanical Hamiltonian dynamics at her disposal.

One can observe that the cutoff dependence drops out from the structure function thanks to the same mechanism as in the form factors. However, there is a difference that one of the denominators has W^2 as the mass shell value in place of the physical fermion mass squared. Moreover, the integrals are logarithmically sensitive to Q^2 and scaling is violated. We have an example for that the bound state wave function has a tail in the relative transverse momentum of partons which extends so far that scaling is never achieved and logarithms of Q^2 must appear. This is a first example of a relativistic Hamiltonian analysis of deep inelastic structure functions known to the present author. Numerical studies of scaling violations in the model will be presented elsewhere. The model is an interesting laboratory for studying Hamiltonian dynamics of relativistic elementary particles. In particular, apart from scaling violations, one can study the small x behavior of the model structure functions including various spin configurations of the target fermion and its constituents.

9. Advanced theory

The divergence problems listed in Section 6 included the problem of divergences proportional to exact eigenvalues when one attempted to eliminate the three-particle states and obtain an effective Hamiltonian acting in a two-particle state. These divergences appeared because the Hamiltonian coupled states of limited energies to all states of energies ranging up to the cutoffs. This coupling caused strong divergences in the few particle dynamics. Subtractions led to the eigenvalue dependence of the counterterms and triviality. The natural attempt to solve the problem was an application of the R operation from Ref. [24]. However, the R operation involves small energy denominators in perturbation theory. For example, in the language of Section 7, small energy denominators appear when the free energies above the arbitrary cutoff λ are subtracted from the energies below this cutoff. This makes it practically impossible to isolate divergences and find the counterterms since arbitrarily large and complicated terms are generated due to the small denominators. The new renormalization theory given in Ref. [96] is designed to solve this practical problem. Our quotation of the new method from Ref. [96] is given for completeness of this paper and is almost a copy of the essential part of Ref. [96]. We reproduce it here since we find it impossible to make the presentation understandable and still shorter.

In the new theory the bare Hamiltonian with an arbitrarily large, but finite cutoff, is not reduced to an effective Hamiltonian with a small cutoff but it is transformed by a specially chosen similarity transformation. The similarity transformation has two desirable features. Firstly, the transformed Hamiltonian is band-diagonal: in particular all matrix elements vanish which would otherwise have caused transitions with big energy jumps, such as from a state of bounded energy to a state with an energy of order of the cutoff. At the same time, neither the similarity transformation nor the transformed Hamiltonian, computed in perturbation theory, contain vanishing or near-vanishing energy denominators. Instead, energy differences in denominators can be replaced by energy sums for purposes of order of magnitude estimates needed to determine cutoff dependencies. These two properties make it possible to determine relatively easily the list of counterterms needed to obtain finite low energy results (such as for eigenvalues).

9.1. Similarity renormalization scheme [96]

One considers a Hamiltonian

$$H = H_0 + H_I \tag{9.1}$$

with a cutoff Λ, where H_I includes both a bare interaction and any necessary counterterms. Strong Λ-dependence of the eigenvalues and eigenvectors of H will be generated in perturbation theory unless one has found the structure of the necessary counterterms, because the bare interaction couples states below a fixed energy λ to all states from between λ and Λ with growing strength. The similarity renormalization scheme is based on the observation that if those couplings with large energy jumps were removed, the divergences could not be generated in finite orders of perturbation theory. So, one considers a similarity transformation of the Hamiltonian matrix H such that the transformed Hamiltonian matrix H' has all troublesome elements far away from its diagonal equal zero. Λ-dependence appears in the transformed Hamiltonian in the nonzero matrix elements which are close to diagonal. The cutoff dependence can be removed from these matrix elements by introducing counterterms. The similarity matrix should approach unity when the troublesome interaction is zero, since one works in the basis of eigenstates of H_0 and H_0 is therefore already "band-diagonal" too. Similarity transformations preserve eigenvalues and therefore the transformed Hamiltonian H' has the same spectrum as H.

The new Hamiltonian H' is given by the formula

$$H' = S^{-1} H S \,, \tag{9.2}$$

where S is the similarity matrix which is unitary, $S^{-1} = S^\dagger$. S can be written as

$$S = 1 + T \,, \tag{9.3}$$

where $T \to 0$ when $H_I \to 0$. Unitarity of S implies that

$$T + T^\dagger + T^\dagger T = 0 \,. \tag{9.4}$$

We introduce two parts of T; hermitean, $h = {}^1\!/_2(T+T^\dagger)$, and antihermitean, $a = {}^1\!/_2(T - T^\dagger)$. Eq. (9.4) can be rewritten as

$$h = \tfrac{1}{2}(a^2 - h^2) \,. \tag{9.5}$$

Thus, h and a are not independent and h is of higher order in the bare interaction than a.

The new interaction Hamiltonian is defined as $H'_I = H' - H_0$, so that the new free Hamiltonian is the same as the old one and one has

$$H'_I = H_I + T^\dagger H + HT + T^\dagger HT \,. \tag{9.6}$$

Then, we demand that H'_I is "band-diagonal", which means that the matrix elements of H'_I which are far away from the diagonal are zero. In order to define what is meant by the matrix elements which are far away from the diagonal one introduces notions of diagonal remotum of a matrix M, denoted $DR[M]$, and diagonal proximum of M, denoted $DP[M]$. Diagonal proximum refers to the "band-diagonal" part of a matrix and diagonal remotum to the "far off-diagonal" part.

Suppose an operator $\hat{M}$ has matrix elements $M_{ij} = \langle i|\hat{M}|j\rangle$ between the eigenstates of H_0, $H_0|i\rangle = E_i|i\rangle$. The indices i and j run from 0 to some big number determined by the cutoff Λ which defines the size of the matrix M. The diagonal remotum of M has the same matrix elements as M when i is far away from j in some prescribed way, and the matrix elements $DR[M]_{ij}$ for the close indices i and j are equal zero. Then, $DP[M] = M - DR[M]$, *i.e.* $DP[M]_{ij}$ are the same as M_{ij} when indices i and j are close to each other in the prescribed way. For example, one may choose

$$DP[M]_{ij} = M_{ij} \tag{9.7}$$

when

$$(\beta + 1)|E_i - E_j| \le (\beta - 1)(E_i + E_j) + 2E_0 \,, \tag{9.8}$$

where $\beta > 1$, E_i and E_j are the eigenvalues of H_0 (assumed to be positive), and E_0 is some fixed constant. Eq. (9.8) implies that (9.7) holds when

$$\beta E_j + E_0 > E_i > \frac{E_j - E_0}{\beta} \,. \tag{9.9}$$

Otherwise $DP[M]_{ij} = 0$. The above choice removes the possibility that small energy differences appear in the denominators of perturbation theory. This becomes clear in further discussion.

The new interaction Hamiltonian is demanded to satisfy the proximity condition,

$$H_I' = DP[H_I'] \,, \tag{9.10}$$

which implies the following condition on the similarity matrix through Eq. (9.6),

$$DR\left[H_I + T^\dagger H + HT + T^\dagger HT\right] = 0 \,. \tag{9.11}$$

This equation may be slightly rewritten

$$DR\left[H_I + \{H_0, h\} + [H_0, a] + T^\dagger H_I + H_I T + T^\dagger HT\right] = 0 \,, \tag{9.12}$$

and then satisfied by imposing the following condition on the matrix a,

$$[a, H_0] = DR\left[H_I + \{H_0, h\} + T^\dagger H_I + H_I T + T^\dagger HT\right] \,. \tag{9.13}$$

From this condition one can find the similarity matrix recursively in perturbation theory to all orders in the interaction H_I since we know the eigenstates and eigenvalues of H_0. The commutator of a with H_0 is the lowest order term in Eq. (9.12) involving the similarity matrix and it may be used to begin the recursion. Then, it follows that

$$H_I' = DP\left[H_I + \{H_0, h\} + T^\dagger H_I + H_I T + T^\dagger HT\right] , \tag{9.14}$$

since $DP[a] = 0$ by definition. We observe that the arguments of DR and DP in Eqs (9.13) and (9.14) are the same. We denote this argument by Q.

For the purpose of carrying out the renormalization program in perturbation theory to all orders one rewrites Q in a different form. Namely, one can use the inverse of the similarity relation from Eq. (9.2) and the unitarity condition from Eq. (9.4) to express H_I by H_I' in all terms of Q except for the first term which is H_I itself;

$$Q = H_I - \{H_0, h\} + H_I'(a-h) - (a+h)H_I' + (a+h)H'(a-h) . \quad (9.15)$$

The similarity renormalization scheme is defined recursively in powers of the bare interaction Hamiltonian in the following way.

Let us assume for simplicity that the bare interaction Hamiltonian is proportional to a coupling constant g, with counterterms of higher order:

$$H_I = gV_1 + g^2V_2 + g^3V_3 + \cdots . \quad (9.16)$$

gV_1 is the bare interaction projected on the space limited by the cutoff Λ and V_k for $k = 2, 3, \ldots$ denote the counterterms. Correspondingly:

$$H_I' = gV_1' + g^2V_2' + g^3V_3' + \cdots , \quad (9.17)$$

and

$$a = ga_1 + g^2a_2 + g^3a_3 + \cdots . \quad (9.18)$$

The matrix h is of order g^2,

$$h = g^2h_2 + g^3h_3 + \cdots , \quad (9.19)$$

where, from Eq. (9.5),

$$h_n = \tfrac{1}{2} \sum_{k,l;k+l=n} (a_k a_l - h_k h_l) . \quad (9.20)$$

In order to define the recursion formulae for the counterterms one rewrites Q from Eq. (9.15) as a series in g,

$$Q = gQ_1 + g^2Q_2 + g^3Q_3 + \cdots , \quad (9.21)$$

so that

$$Q_n = V_n - \{H_0, h_n\} + \sum_{k=1}^{n-1} V_k'(a_{n-k} - h_{n-k}) - \sum_{k=1}^{n-1} (a_{n-k} + h_{n-k})V_k'$$
$$+ \sum_{k=1}^{n-1} (a_k+h_k)H_0(a_{n-k}-h_{n-k}) + \sum_{k=1}^{n-2} \sum_{l=1}^{n-1-k} (a_k+h_k)V_l'(a_{n-k-l}-h_{n-k-l}). \quad (9.22)$$

Eq. (9.22) is the basic result. It is seen that the structure of Q_n is generated by terms involving only V'_k, a_k and h_k with $k < n$, except for V_n and one term on the right hand side of Eq. (9.22) which involves h_n. h_n is expressed by lower order terms through Eq. (9.20) and V_n completes the definition of Q_n by eliminating the divergent cutoff dependence of other terms. Now, since all V'_k and a_k for $k = 1, 2, \ldots, n-1$ are generated from Q_k, $k = 1, 2, \ldots, n-1$, we have the recursion which expresses Q_n by $Q_k, k = 1, 2, \ldots, n-1$. Expression (9.22) for Q_n is designed in such a form, that the counterterm V_n appears in it only once and linearly, added to potentially divergent terms. Thus, the necessary structure of V_n which makes Q_n independent of the big Λ can be determined from the knowledge of finite Q's of lower orders and operators appearing in Eq. (9.22).

Finally, one obtains expressions for V'_n and a_n. Namely,

$$V'_n = DP[Q_n]\,, \tag{9.23}$$

and

$$a_{nij} = \frac{DR[Q_n]_{ij}}{E_j - E_i}\,. \tag{9.24}$$

This is the only equation which involves energy denominators. Since $DR[Q_n]_{ij}$ vanishes unless E_i and E_j lie outside the band condition from Eq. (9.8) one can verify that

$$|E_i - E_j| > \frac{\beta - 1}{\beta + 1}(E_i + E_j) + \frac{2E_0}{\beta + 1}\,, \tag{9.25}$$

which confirms the claim that energy differences are of order their sums.

The induction to all orders in g defines the similarity renormalization scheme in perturbation theory. One is interested in construction of the counterterms V_n for $n = 2, 3, \ldots, \infty$.

Knowing the generally valid structure of counterterms, one can use Eqs (9.5), (9.13) and (9.14) to find the new Hamiltonian H' for some values of β and E_0 in Eq. (9.8) in an iterative procedure which is not confined to perturbation theory. In the first approximation one has

$$Q_1 = gV_1\,. \tag{9.26}$$

Neither divergences nor counterterms appear in this case. The first approximation to a new interaction is

$$H'_{I1} = DP[gV_1]\,, \tag{9.27}$$

and the similarity matrix is

$$[a_1, H_0] = DR[gV_1]\,, \tag{9.28}$$

together with

$$h_1 = 0 . \tag{9.29}$$

Thus, the unitarity condition is not manifest in the first approximation. The nonperturbative iterative procedure is defined by

$$\begin{aligned} Q_{N+1} &= H_{I(N+1)} - \{H_0, h_N\} \\ &+H'_{IN}(a_N - h_N) - (a_N + h_N)H'_{IN} + (a_N + h_N)H'_N(a_N - h_N) , \end{aligned} \tag{9.30}$$

where the $(N+1)$-th approximation to the counterterm in $H_{I(N+1)} = gV_1 + V_{N+1}$ is defined to remove all divergences in Q_{N+1}. The complementary recursive relations are

$$H'_{I(N+1)} = DP[Q_{N+1}] , \tag{9.31}$$

$$[a_{N+1}, H_0] = DR[Q_{N+1}] , \tag{9.32}$$

and

$$h_{N+1} = \tfrac{1}{2}\left[a_N^2 - h_N^2\right] . \tag{9.33}$$

If a limit of this procedure when $N \to \infty$ exists, the resulting limiting matrices for large N provide a solution to the renormalization problem for the initial Hamiltonian H. It is seen that the unitarity constraints on the similarity matrix are satisfied only in the limit $N \to \infty$. Eq. (9.33) implies unitarity constraints if $h_{N+1} \sim h_N$. One has to verify this procedure on a case by case basis, since it is not possible to generally determine if it is convergent. When the unitarity condition is satisfied the Hamiltonian H' is hermitean and has the same spectrum as H.

Although H' does not introduce cutoff dependence to a desired order of perturbation theory for a sufficiently large Λ it may still happen that H' has large off-diagonal matrix elements so that its eigenstates may depend on the cutoff Λ in a genuinely nonperturbative way. An example of such situation is considered in Appendix B of Ref. [96].

The bare Hamiltonian may be cut off in various ways, not necessarily only by restricting its domain. One may consider the initial Hamiltonian to have built in diagonal proximity bounds with $E_0 \sim \Lambda$ and/or β very large in comparison to 1. Then, one can reduce the bounds to finite E_0 and β in the same renormalization scheme. The resulting Hamiltonians H' for different parameters E_0 and β are related by a new renormalization group relations. A whole variety of alternative definitions of the diagonal proximity can be considered [97].

An illustration of the similarity renormalization scheme to all orders in perturbation theory in a model familiar from Section 7 is given in Ref. [96].

9.2. One gluon exchange

This Section demonstrates how the troublesome longitudinal logarithmic divergence discussed in Section 6.b cancels out in the one gluon exchange interaction between two fermions in the second order terms in the coupling constant. The one gluon exchange is a sum of two terms originating from two possible orderings of the emission and absorption of the gluon by the fermions. The intermediate state contains the gluon of a longitudinal momentum fraction denoted by z. We are interested in what is left after subtracting the leading singularity z^{-2} for z close to the infrared cutoff δ', *cf.* Section 6.2.

The second order term in the operator Q from Eq. (9.15) is

$$Q_2 = V_2 - \tfrac{1}{2}[H_0 a_1 a_1 - 2a_1 H_0 a_1 + a_1 a_1 H_0] + V_1' a_1 - a_1 V_1' \tag{9.34}$$

and

$$[a_1, H_0] = DR[H_I] , \tag{9.35}$$

where the interaction H_I induces the emission and absorption of gluons by the fermions. The leading divergence in z is to be removed by the operator counterterm V_2. It is the same as in Eq. (6.22) where $z = y - y'$ and V_2 must contain the term indicated in Eq. (6.23). Now, in Section 6.b we have encountered the problem that the remaining logarithmic divergence was proportional to the unknown full Hamiltonian eigenvalue. This problem does not appear now. First of all the denominator in the one gluon exchange does not contain the full Hamiltonian eigenvalue. Moreover, the logarithmic divergence is totally gone. The reason is that the characteristic combination of H_0 and a_1 in the square bracket in Eq. (9.34) and the energy denominators in a_1 combine to the result which is in fact behaving like a constant divided by z for $z \sim 0$. Therefore, the logarithmic divergence cancels out as in the principal value of a constant. This is a solution to a long standing problem in LF Hamiltonian QCD [76]. No counterterm for the logarithmic divergence is necessary.

10. Conclusion

A series of examples of LF Hamiltonian models discussed in this paper leads to the conclusion that the LF Hamiltonian approach provides a new and unexplored opportunity for making progress towards solution of the relativistic bound state problem for elementary particles. The bound states are defined as eigenstates of the Hamiltonian. However, attempts to build LF Hamiltonians in local quantum field theories demonstrate that they involve divergences. These divergences appear in the bound state eigenvalue equations and raise very complex renormalization issues. Furthermore, traditional Lagrangian approaches to renormalization do not apply to the eigenvalue problems of LF Hamiltonians. Fortunately, a new formalism for renormalization of Hamiltonians including determination of the required cutoff-dependent counterterms has been formulated and one can begin systematic studies of the eigenvalue equations for the bound states.

Although the renormalized LF Hamiltonian approach may contribute to the development of relativistic nuclear physics, our primary interest in the formalism is stimulated by its potential applicability to QCD. The basic reasons for the counterterms in LF QCD Hamiltonian to be so interesting are following.

The LF Fock space description of elementary particles is a natural basis for explaining the Feynman parton model of hadrons. This is not yet accomplished because of severe divergences in LF dynamics. There is an urgent need for constructing Hamiltonian counterterms which remove the divergences and understanding the field theoretic LF basis of the parton model for hadrons.

The LF counterterms have complex structure and contain free functions, not just a few free parameters in their finite parts. Physically required values of the functions have to be found in part by fitting experimental data. But, the functions can be severely constrained by symmetry requirements. Full rotational symmetry, which is not explicitly satisfied in the LF formulation, provides the most powerful constraints. In turn, the counterterms are necessary to obtain rotational invariance and may effectively become the key to obtain the full Poincaré covariance of theoretical predictions.

The infrared longitudinal cutoff properties of LF theory suggest another fundamental role for the counterterms. Namely, the longitudinal infrared cutoff in LF dynamics makes it impossible to create particles from a bare vacuum by a translationally invariant Hamiltonian and in addition the number of constituents in a given eigenstate is limited. LF counterterms to the longitudinal infrared cutoff dependence become a possible alternative source for features normally associated in standard ET dynamics with a nontrivial vacuum structure - including spontaneous symmetry breaking and confinement.

Finally, even if it turns out that the LF Hamiltonian counterterms are not able to solve all problems, we may still learn about why they fail.

The renormalized LF Hamiltonian approach suggests the following strategy for approaching QCD. One can write a canonical QCD Hamiltonian imposing general cutoffs in the field expansions into creation and annihilation operators. One can then choose the least complicated bare states to form a core in the Fock space which has the quantum numbers of the bound states to be considered. One can define the diagonal proximity conditions for states which can be created from the core by the canonical QCD interactions. One may then attempt a construction of the effective "band diagonal" Hamiltonian expanding the core states and find counterterms to arising divergences. The finite parts of the counterterms will be used to incorporate confinement and enable fits to the observed hadronic symmetries including Poincaré symmetry and chiral symmetry breaking. Dressing of the bare core states within the diagonal proximum is expected to build the effective constituent quark picture of hadrons.

There is a possibility that the dressing of quarks and gluons will not saturate until a whole LF Fermi sphere of wee quarks and antiquarks in interaction with wee gluons will be formed [98]. Then, one will have to consider new many body LF theories and explicitly discuss excitations of the ground state medium.

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