COMPLEXIFIED TWO-SPINOR THEORY OF POSITIVE-FREQUENCY DIRAC FIELDS

J.G. CARDOSO

Department of Mathematics, Quaid-i-Azam University
Islamabad, Pakistan

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The equations of motion yielding the Van-der-Waarden form of the Dirac free-field theory in complex Minkowski space are actually derived by working out a variational principle that involves a manifestly two-spinor Lagrangian density. We make use of this Dirac Lagrangian density to build up a suitable defining relation for the energy-momentum tensor of the theory which gives rise at once to a set of essentially equivalent expressions for the tensor. In particular, it is shown that one of the structures arising from our definition coincides with the well-known Penrose expression up to a conventional overall sign. A set of new covariant contour integrals for the corresponding energy momentum four-vector and angular-momentum bivector is exhibited. We then show that the entire set of spinor formulae affords us another method of establishing the relevant charge-conservation statement.

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1. Introduction

One of the most striking features of the conventional Dirac theory is the fact that the wave functions entering into the field equations can be split into pairs of opposite helicity two-component spinors carrying the same charge and energy [1–4]. Accordingly, the two-spinor fields appear to describe completely the left-handed and right-handed degrees of freedom of the theory at each point of their domain of definition, regardless of whether their propagation takes place in real or complex Minkowski space. For positive-frequency wave functions, the left-handed degrees of freedom are actually carried by the unprimed components while the right-handed modes are locally represented by the primed fields. For negative-frequency fields, this association turns out to be the other way around. In the former case, the fields propagate in the normal future direction, and are effectively required

[4–7] to have holomorphic extensions into the forward tube $\mathbb{CM}^+$ of complex Minkowski space $\mathbb{CM}$. In the latter case, the relevant requirement is similar to the previous one but the fields now propagate in the past direction, the role played by $\mathbb{CM}^+$ being thus taken over by that involving the backward tube $\mathbb{CM}^-$ of $\mathbb{CM}$. In the absence of electromagnetism, the splitting of the wave equations is accomplished in such a way that the fields occurring in each pair behave themselves as sources for one another. Such a remarkable feature of the theory accordingly tells us that any Dirac pair can still be looked upon as a set of interacting spinor fields. In connection with this situation, the rest-mass of the fields plays, in effect, the role of a coupling constant.

In this paper, we formulate a variational principle to derive the explicit equations of motion that yield particularly the Van-der-Waarden form of the Dirac free-field equation for positive-frequency wave functions on a suitable subset of $\mathbb{CM}^+$ (Section 2). Indeed, the method upon which our basic prescription rests was given earlier [8] in connection with the problem of obtaining the fundamental wave equations for positive-frequency Maxwell-Dirac fields in $\mathbb{CM}$ (see also Ref. [9]). This prescription actually involves making use of an appropriate (two-spinor) version of the usual Dirac Lagrangian density. It shall be seen that, upon carrying through the pertinent procedures, we need not impose any commutation condition on the fields and their variations. Our action is thus defined on a bounded open subset of $\mathbb{CM}^+$ whose closure is compact. All the variations involved in the dynamical statement supposedly vanish on the boundary of the closure. The two-spinor Lagrangian density is particularly used for constructing an appropriate defining expression for the energy-momentum tensor of the theory (Section 3). In addition to being manifestly symmetric, this expression also gives rise directly to an explicit tensor which appears to be divergenceless, but not trace-free. At this stage, the structure coming about coincides with that given by Penrose [4] up to a conventional overall sign. We utilize some elementary spinor identities to re-express it in a form which involves four totally symmetrized pieces. These energy-momentum-tensor structures lead us at once to manifestly covariant contour-integral expressions for the corresponding energy-momentum four-vector and angular-momentum bivector (Section 4). In Section 5, it is particularly observed that our spinor formulae can provide us with a new method of establishing the statement associated with the conservation of charge. The original motivation for elaborating this work is indeed related to the belief that a two-component spinor formulation of the theory might not only enhance in a natural way all the fundamental aspects of the systems, but also bring about their most important features automatically. This belief goes hand-in-hand with the need of rectifying the situation concerning the absence of a simple prescription for deriving explicitly the complexified version of the field equations which control the propagation of Dirac fields.

The natural system of units in which $c = \hbar = 1$ is adopted throughout this work. All the spinor conventions and rules as given in Ref. [4] will also be used herein. We shall briefly review the method referred to above when elaborating Section 2.

2. Dirac free-field theory

The set $\mathbb{CM}^+$ is defined as an open connected subset of $\mathbb{CM}$, which is explicitly given by (see, for instance, Ref. [5])

$$\mathbb{CM}^+ = \left\{x^{AA'} \in \mathbb{CM} | x^{AA'} = \xi^{AA'} - i\eta^{AA'},\ \xi^{AA'} \in \mathbb{RM} \text{ and } \eta^{AA'} \in \mathbb{V}^+\right\}, \tag{2.1}$$

whence $\mathbb{CM}^+ \cong \mathbb{RM} \times \mathbb{V}^+$ and $\mathbb{CM}^+ \cap \mathbb{RM} = \emptyset$. Here, $\mathbb{RM}$ stands for the real slice of $\mathbb{CM}$ and $\mathbb{V}^+$ is the (convex) interior of the future cone of some origin of $\mathbb{RM}$. A positive-frequency Dirac system in $\mathbb{CM}$ is a pair $\left\{\psi_A(x),\ \chi^{A'}(x)\right\}$ of independent massive charged spin-$\pm 1/2$ fields which are taken to be holomorphic mappings throughout $\mathbb{CM}^+$. The charge-helicity conjugation operator $\hat{\tau}$ is an antilinear mapping which locally carries the above pair into $\left\{\tilde{\psi}_{A'}(x),\ \tilde{\chi}^A(x)\right\}$. These conjugate fields bear reversed helicities and carry a charge which is opposite to that carried by the fields entering into the former pair. They also carry negative energy, but possess the same rest-mass as the former fields. When acting on a product of fields, $\hat{\tau}$ also reverses the order of the factors. We must emphatically observe that this conjugation mapping is effectively defined so as to act only upon the field mappings themselves without affecting their arguments at all. Additionally, it satisfies the involutory property $\hat{\tau}^2$=identity.

Let Ω^+ be a bounded open (proper) subset of $\mathbb{CM}^+$ whose closure is compact. We write down the free-field Dirac Lagrangian density on Ω^+ as

$$\begin{aligned}\mathcal{L}_D =& i\big\{\tfrac{1}{2}[\tilde{\chi}_A(x)\nabla^{AA'}\chi_{A'}(x) + \tilde{\psi}_{A'}(x)\nabla^{AA'}\psi_A(x)] \\ &- \tfrac{1}{2}[(\nabla^{AA'}\tilde{\chi}_A(x))\chi_{A'}(x) + (\nabla^{AA'}\tilde{\psi}_{A'}(x))\psi_A(x)] \\ &- \mu[\tilde{\chi}_A(x)\psi^A(x) + \tilde{\psi}_{A'}(x)\chi^{A'}(x)]\big\},\end{aligned} \tag{2.2}$$

with $\nabla^{AA'}$ standing for the ordinary (holomorphic) partial derivative operator $\partial/\partial x_{AA'}$, and $m = \sqrt{2}\mu$ being the rest-mass of the fields. For simplicity, we omit here as elsewhere all the arguments of $\mathcal{L}_D$. Notice that (2.2) satisfies $\tilde{\mathcal{L}}_D = \mathcal{L}_D$, that is to say $\mathcal{L}_D$ is a "real" SL(2,$\mathbb{C}$)$\otimes$SL(2,$\mathbb{C}$)-scalar function on Ω^+. In order to introduce the relevant variational principle, we invoke the prescription provided in [8] according to which one defines the real set

$$\omega^+ = \left\{(\xi^a, \eta^a) | (\xi^a - i\eta^a) \in \Omega^+\right\} \cong \Omega^+, \tag{2.3}$$

whose compact volume is

$$\mathrm{Vol}(\omega^{+}) = \int_{\omega^{+}} d^4\xi \wedge d^4\eta\,, \tag{2.4}$$

where $d^4\lambda = \frac{1}{4!}\mathrm{e}_{abcd}d\lambda^a \wedge d\lambda^b \wedge d\lambda^c \wedge d\lambda^d$, with e_{abcd} being the alternating Minkowskian tensor for the (covariant) standard basis of $\mathbb{CM}$, and λ standing for either ξ or η. The ("real") action is thus written as follows

$$S[\mathcal{L}_D] = \int_{\Omega^{+}} \mathcal{L}_D d^4\xi \wedge d^4\eta\,, \tag{2.5}$$

whereas the relevant dynamical statement reads simply $\delta S[\mathcal{L}_D] = 0$. In fact, this δ-variation is here thought of as being ordinary in the sense that it does not involve any deformation of Ω^{+}. We also assume that is commutes with the ∇-derivative operator which occurs in $\mathcal{L}_D$. Furthermore, all the variations carried by $\delta S[\mathcal{L}_D]$ are effectively regarded as being independent of each other. Hence, carrying out the explicit variation of $\mathcal{L}_D$ and making suitable index substitutions, after performing trivial integrations by parts, we arrive at the expression

$$\begin{aligned}
&\int_{\Omega^{+}} \Big\{\delta\tilde{\psi}_{A'}(x)\left[\frac{\partial\mathcal{L}_D}{\partial\tilde{\psi}_{A'}(x)} - \nabla^{AA'}\frac{\partial\mathcal{L}_D}{\partial\nabla^{AB'}\tilde{\psi}_{B'}(x)}\right] \\
&\quad + \delta\tilde{\chi}_A(x)\left[\frac{\partial\mathcal{L}_D}{\partial\tilde{\chi}_A(x)} - \nabla^{AA'}\frac{\partial\mathcal{L}_D}{\partial\nabla^{BA'}\tilde{\chi}_B(x)}\right] \\
&\quad + \left[\frac{\partial\mathcal{L}_D}{\partial\psi_A(x)} - \nabla^{AA'}\frac{\partial\mathcal{L}_D}{\partial\nabla^{BA'}\psi_B(x)}\right]\delta\psi_A(x) \\
&\quad + \left[\frac{\partial\mathcal{L}_D}{\partial\chi_{A'}(x)} - \nabla^{AA'}\frac{\partial\mathcal{L}_D}{\partial\nabla^{AB'}\chi_{B'}(x)}\right]\delta\chi_{A'}(x)\Big\} d^4\xi\wedge d^4\eta \\
&\quad + \int_{\partial\Omega^{+}} \left[\delta\tilde{\psi}_{A'}(x)\frac{\partial\mathcal{L}_D}{\partial\nabla^{AB'}\tilde{\psi}_{B'}(x)} + \delta\tilde{\chi}_A(x)\frac{\partial\mathcal{L}_D}{\partial\nabla^{BA'}\tilde{\chi}_B(x)}\right. \\
&\quad \left. + \frac{\partial\mathcal{L}_D}{\partial\nabla^{BA'}\psi_B(x)}\delta\psi_A(x) + \frac{\partial\mathcal{L}_D}{\partial\nabla^{AB'}\chi_{B'}(x)}\delta\chi_{A'}(x)\right] d^7\Xi^{AA'} = 0\,,
\end{aligned} \tag{2.6a}$$

where we have made use also of the derivative-operator splitting $\nabla_{AA'} = \frac{1}{2}\left(\frac{\partial}{\partial\xi^{AA'}} + \frac{i\partial}{\partial\eta^{AA'}}\right)$. In (2.6a), $d^7\Xi^{AA'}$ is an element of seven hypersurface area of the boundary $\partial\Omega^{+}$ of Ω^{+} which is expressed as

$$d^7\Xi^{AA'} = \tfrac{1}{2}\left(d^3\xi^{AA'}\wedge d^4\eta + i d^4\xi\wedge d^3\eta^{AA'}\right)\,, \tag{2.6b}$$

with $d^3\lambda_a = \frac{1}{3!}e_{abcd}d\lambda^b \wedge d\lambda^c \wedge d\lambda^d$ (see (2.15) below). Now taking $\delta\Lambda_\bullet(x) = 0$ and $\delta\tilde{\Lambda}_\bullet(x) = 0$ on $\partial\Omega^+$, with $\Lambda_\bullet(x)$ standing for either $\psi_A(x)$ or $\chi_{A'}(x)$, we obtain the equations of motion for the complete theory

$$\frac{\partial\mathcal{L}_D}{\partial\tilde{\psi}_{A'}(x)} - \nabla^{AA'}\frac{\partial\mathcal{L}_D}{\partial\nabla^{AB'}\tilde{\psi}_{B'}(x)} = 0\,, \quad \frac{\partial\mathcal{L}_D}{\partial\tilde{\chi}_A(x)} - \nabla^{AA'}\frac{\partial\mathcal{L}_D}{\partial\nabla^{BA'}\tilde{\chi}_B(x)} = 0\,, \tag{2.7}$$

and

$$\frac{\partial\mathcal{L}_D}{\partial\psi_A(x)} - \nabla^{AA'}\frac{\partial\mathcal{L}_D}{\partial\nabla^{BA'}\psi_B(x)} = 0\,, \quad \frac{\partial\mathcal{L}_D}{\partial\chi_{A'}(x)} - \nabla^{AA'}\frac{\partial\mathcal{L}_D}{\partial\nabla^{AB'}\chi_{B'}(x)} = 0\,. \tag{2.8}$$

Consequently, the free-field theory consists of the explicit field equations on Ω^+

$$\nabla^{AA'}\psi_A(x) = \mu\chi^{A'}(x)\,, \quad \nabla_{AA'}\chi^{A'}(x) = -\mu\psi_A(x)\,, \tag{2.9}$$

along with their conjugates

$$\nabla^{AA'}\tilde{\psi}_{A'}(x) = \mu\tilde{\chi}^A(x)\,, \quad \nabla_{AA'}\tilde{\chi}^A(x) = -\mu\tilde{\psi}_{A'}(x)\,. \tag{2.10}$$

It becomes clear that the terms involving the variations of the conjugate fields in the volume integral of (2.6a) give rise to the equations of motion for the former fields. Obviously, this relationship appears to be reversed when we take the pieces carrying the variations of the positive-frequency fields.

The (local) commutativity of the ∇'s enables one to recover either Dirac pair in a way that appears to be somewhat different from that trivially suggested by the field equations (2.9) and (2.10). This was indeed observed much earlier by Feynman and Gell-Mann [10], and can be easily seen by letting the ∇-operator act appropriately on both sides of any one of Eqs (2.9) and (2.10). For the former left-handed field, for example, we have the simple computation

$$\begin{aligned} 2\nabla_{A'\,B}\nabla^{A'}_A\psi^A(x) &= 2\left(\nabla_{A'\,(A}\nabla^{A'}_{B)} - \nabla_{A'\,[A}\nabla^{A'}_{B]}\right)\psi^A(x) \\ &= -\epsilon_{AB}\Box\psi^A(x) = 2\mu^2\psi_B(x), \quad \Box = \nabla_a\nabla^a\,, \end{aligned} \tag{2.11}$$

which immediately leads to $(\Box + m^2)\psi_A(x) = 0$, whence we can recover the former pair by allowing $\nabla^{AA'}$ to act on any solution of this wave equation. Evidently, we can adopt a similar procedure also for any other relevant field. We thus have the wave equations on Ω^+

$$(\Box + m^2)\Lambda_\bullet(x) = 0\,, \qquad (\Box + m^2)\tilde{\Lambda}_\bullet(x) = 0\,. \tag{2.12}$$

These equations will play an important role later when we consider the procedure for deriving one of the properties of the energy-momentum tensor of the theory (see (3.5) below). We are now able to carry out an analytic continuation of both field pairs into $\mathbb{CM}$, thereby extending adequately the domain whereupon the entire theory was initially written.

An important consequence of the above-exhibited field theory is that it gives rise to the conservation of the four-vector

$$J^{AA'}(x) = e\big[\tilde{\psi}^{A'}(x)\psi^{A}(x) + \tilde{\chi}^{A}(x)\chi^{A'}(x)\big]\,, \tag{2.13}$$

which is the so-called Dirac current density [4], the quantity e being the charge of the former fields [11]. We have, in effect, $\nabla^{AA'}J_{AA'}(x) = 0$ on $\mathbb{CM}$. Thus, the explicit charge integral for the theory is written as

$$Q[\Sigma] = e \oint_{\Sigma} [\tilde{\psi}^{A'}(x)\psi^{A}(x) + \tilde{\chi}^{A}(x)\chi^{A'}(x)]\, d^3x_{AA'}\,, \tag{2.14}$$

where Σ is a suitable three-real-dimensional compact contour lying in $\mathbb{CM}$, and $d^3x_{AA'}$ is the spinor element associated with the (complex) Minkowskian form d^3x_a, which is given by

$$d^3x_{AA'} = \tfrac{i}{3!}\big(dx_{DC'} \wedge dx^{C'}_{A} \wedge dx^{D}_{A'} - dx_{CD'} \wedge dx^{C}_{A'} \wedge dx^{D'}_{A}\big)\,. \tag{2.15}$$

The suitability of Σ is essentially related to the fact that the singularity set of the integrand of (2.14) may fail to have an empty intersection with $\mathbb{CM}^+ \cup \mathbb{CM}^-$. This contour is thus specified so as to yield a useful result whenever the integral entering into (2.14) is effectively evaluated.

It is of some interest to observe that the wave equations involving the fields yield the following "spin-1 field equation" for $J_{AA'}(x)$

$$\begin{aligned}(\Box + 2m^2)J_{AA'}(x) = {} & 2e[(\nabla^{BB'}\tilde{\psi}_{A'}(x))\nabla_{BB'}\psi_A(x) \\ & + (\nabla^{BB'}\tilde{\chi}_A(x))\nabla_{BB'}\chi_{A'}(x)]\,.\end{aligned} \tag{2.16}$$

It should be noticed, also, that $\hat{\tau}J_{AA'}(x) = J_{AA'}(x)$. Evidently, this statement amounts to the same thing as saying that the Dirac current density is a "real" four-vector.

We conclude this section by remarking that the holomorphicity of the positive-frequency fields can actually be stated as

$$\bar{\nabla}_{AA'}\Lambda_{\bullet}(x) = 0 \text{ on } \mathbb{CM}^+\,, \tag{2.17}$$

where $\bar{\nabla}_{AA'} = 1/2(\partial/\partial\xi^{AA'} - i\partial/\partial\eta^{AA'})$ is the antiholomorphic covariant derivative operator on $\mathbb{CM}$. Hence, using the splitting of the fields into their real and imaginary parts

$$\Lambda_{\bullet}(x) = \mathrm{Re}\,\Lambda_{\bullet}(\xi,\eta) - i\,\mathrm{Im}\,\Lambda_{\bullet}(\xi,\eta)\,, \tag{2.18}$$

we obtain the explicit Cauchy–Riemann equations on $\mathbb{RM} \times \mathbb{V}^+$

$$\overset{\xi}{\nabla}_{AA'} \operatorname{Re} \Lambda_\bullet(\xi, \eta) = \overset{\eta}{\nabla}_{AA'} \operatorname{Im} \Lambda_\bullet(\xi, \eta), \tag{2.19a}$$

$$\overset{\xi}{\nabla}_{AA'} \operatorname{Im} \Lambda_\bullet(\xi, \eta) = - \overset{\eta}{\nabla}_{AA'} \operatorname{Re} \Lambda_\bullet(\xi, \eta), \tag{2.19b}$$

where $\overset{\lambda}{\nabla}_{AA'} = \partial/\partial\lambda^{AA'}$ (see (2.4) above). Whence we can express the corresponding harmonicity relations as

$$(\Box_\xi + \Box_\eta) \operatorname{Re} \Lambda_\bullet(\xi, \eta) = 0 = (\Box_\xi + \Box_\eta) \operatorname{Im} \Lambda_\bullet(\xi, \eta), \tag{2.20}$$

with $\Box_\lambda = \overset{\lambda}{\nabla}_{AA'} \overset{\lambda}{\nabla}{}^{AA'}$. Moreover, making use of the operator relation

$$\Box = \tfrac{1}{4}\blacksquare + \tfrac{i}{2} \overset{\xi}{\nabla}_{AA'} \overset{\eta}{\nabla}{}^{AA'}, \tag{2.21}$$

with $\blacksquare = \Box_\xi - \Box_\eta$, we derive the real version of the first of Eqs (2.12)

$$(\blacksquare + 2m^2) \operatorname{Re} \Lambda_\bullet(\xi, \eta) = 0 = (\blacksquare + 2m^2) \operatorname{Im} \Lambda_\bullet(\xi, \eta). \tag{2.22}$$

3. The energy-momentum tensor

We shall now see how to construct an adequate defining two-spinor expression for the Dirac energy-momentum tensor $T_{ab}(x)$ which fulfills the usual requirements of symmetry and divergencelessness. The symmetry of the tensor is effectively brought into the definition from the beginning while each term carried by its explicit divergence ultimately involves only ordered products between conjugate fields. We will show, also, that its trace appears to be proportional to the mass term of $\mathcal{L}_D$.

Roughly speaking, the prescription for building up the expression in question consists in taking symmetrized products involving the derivative pieces that enter into (2.2) and the derivatives of $\mathcal{L}_D$ which are defined with respect to these pieces, ordering the factors appropriately and then adding the products together. We thus have the symmetric expression

$$\begin{aligned}
T_{AA'BB'}(x) = & \frac{\partial \mathcal{L}_D}{\partial \nabla^{BB'} \psi_C(x)} \nabla_{AA'} \psi_C(x) + \frac{\partial \mathcal{L}_D}{\partial \nabla^{AA'} \psi_C(x)} \nabla_{BB'} \psi_C(x) \\
& + \frac{\partial \mathcal{L}_D}{\partial \nabla^{BB'} \chi_{C'}(x)} \nabla_{AA'} \chi_{C'}(x) + \frac{\partial \mathcal{L}_D}{\partial \nabla^{AA'} \chi_{C'}(x)} \nabla_{BB'} \chi_{C'}(x) \\
& + \nabla_{AA'} \tilde{\psi}_{C'}(x) \frac{\partial \mathcal{L}_D}{\partial \nabla^{BB'} \tilde{\psi}_{C'}(x)} + \nabla_{BB'} \tilde{\psi}_{C'}(x) \frac{\partial \mathcal{L}_D}{\partial \nabla^{AA'} \tilde{\psi}_{C'}(x)} \\
& + \nabla_{AA'} \tilde{\chi}_C(x) \frac{\partial \mathcal{L}_D}{\partial \nabla^{BB'} \tilde{\chi}_C(x)} + \nabla_{BB'} \tilde{\chi}_C(x) \frac{\partial \mathcal{L}_D}{\partial \nabla^{AA'} \tilde{\chi}_C(x)} \\
& - \varepsilon_{AB} \varepsilon_{A'B'} \mathcal{L}_D = T_{BB'AA'}(x),
\end{aligned} \tag{3.1}$$

where the ε-piece is written out explicitly as

$$\begin{aligned}
&\varepsilon_{AB}\varepsilon_{A'B'}\mathcal{L}_D \\
&= \frac{i}{2}\Big\{\Big[\tilde{\chi}_{[A}(x)\nabla_{B][B'}\chi_{A']}(x) + \tilde{\psi}_{[A'}(x)\nabla_{B']\,[B}\psi_{A]}(x)\Big] \\
&- \Big[\Big(\nabla_{[B'\,[B}\tilde{\chi}_{A]}(x)\Big)\chi_{A']}(x) + \Big(\nabla_{[B\,[B'}\tilde{\psi}_{A']}(x)\Big)\psi_{A]}(x)\Big] \\
&- \Big[\tilde{\chi}_{[B}(x)\nabla_{A]\,[A'}\chi_{B']}(x) + \tilde{\psi}_{[B'}(x)\nabla_{A']\,[A}\psi_{B]}(x)\Big] \\
&+ \Big[\Big(\nabla_{[A[A'}\tilde{\psi}_{B']}(x)\Big)\psi_{B]}(x) + (\nabla_{[A'[A}\tilde{\chi}_{B]}(x))\chi_{B']}(x)\Big]\Big\}. \qquad (3.2)
\end{aligned}$$

This ε-piece actually vanishes identically but has been spelt out here because it enhances the whole structure of the definition (3.1). It shall become clear that (3.2) appears to be nothing else but the difference between $T_{ab}(x)$ and itself. We will make one further point concerning this situation later in Section 5. It has been convenient to split the mass term into two parts such that Eqs (2.9) have been taken up by one part and Eqs (2.10) by the other. At this stage, a straightforward computation shows us that the pieces coming from the terms which involve explicitly derivatives in (3.1) are all canceled together with those carrying $\nabla_{AB'}$ and $\nabla_{BA'}$. We are thus left with

$$\begin{aligned}
T_{AA'BB'}(x) = \tfrac{i}{2}\big[&\tilde{\psi}_{A'}(x)\nabla_{BB'}\psi_A(x) - (\nabla_{BB'}\tilde{\psi}_{A'}(x))\psi_A(x) \\
&+ \tilde{\psi}_{B'}(x)\nabla_{AA'}\psi_B(x) - (\nabla_{AA'}\tilde{\psi}_{B'}(x))\psi_B(x) \\
&+ \tilde{\chi}_A(x)\nabla_{BB'}\chi_{A'}(x) - (\nabla_{BB'}\tilde{\chi}_A(x))\chi_{A'}(x) \\
&+ \tilde{\chi}_B(x)\nabla_{AA'}\chi_{B'}(x) - (\nabla_{AA'}\tilde{\chi}_B(x))\chi_{B'}(x)\big]. \qquad (3.3)
\end{aligned}$$

In fact, this explicit expression for $T_{ab}(x)$ is identical with that given by Penrose [4], up to a conventional overall sign. Its trace can at once be calculated by using Eqs (2.9) and (2.10). We have, in effect, the "real" scalar function on $\mathbb{CM}$

$$T^{AA'}{}_{AA'}(x) = 2i\mu[\tilde{\chi}_A(x)\psi^A(x) + \tilde{\psi}_{A'}(x)\chi^{A'}(x)]. \qquad (3.4)$$

Now, using once again the local commutativity of the ∇'s and invoking Eqs (2.12), we obtain the identically vanishing divergence

$$\begin{aligned}
\nabla^{AA'}T_{AA'BB'}(x) = \tfrac{i}{2}\big[&\tilde{\psi}_{B'}(x)\Box\psi_B(x) - (\Box\tilde{\psi}_{B'}(x))\psi_B(x) \\
&+ \tilde{\chi}_B(x)\Box\chi_{B'}(x) - (\Box\tilde{\chi}_B(x))\chi_{B'}(x)\big], \qquad (3.5)
\end{aligned}$$

or, alternatively

$$\nabla^{AA'}T_{AA'BB'}(x) = \tfrac{i}{2e}\{(\Box + 2m^2)J_{BB'}(x) - 2e\big[(\nabla^{AA'}\tilde{\psi}_{B'}(x))\nabla_{AA'}\psi_B(x) + (\nabla^{AA'}\tilde{\chi}_B(x))\nabla_{AA'}\chi_{B'}(x)\big]\}\,. \tag{3.6}$$

Indeed, this latter result can be immediately derived by recalling Eq. (2.16).

The symmetry of (3.3) can be brought about in a more transparent way by splitting each of the derivative pieces (see (2.11)). After some elementary manipulations, we thus obtain the structure

$$T_{AA'BB'}(x) = i\{\tilde{\psi}_{(A'}(x)\nabla_{B')(A}\psi_{B)}(x) - (\nabla_{(B(A'}\tilde{\psi}_{B')}(x))\psi_{A)}(x) + \tilde{\chi}_{(A}(x)\nabla_{B)(A'}\chi_{B')}(x) - (\nabla_{(B'(A}\tilde{\chi}_{B)}(x))\chi_{A')}(x) + \tfrac{\mu}{2}\varepsilon_{AB}\varepsilon_{A'B'}\big[\tilde{\chi}_M(x)\psi^M(x) + \tilde{\psi}_{M'}(x)\chi^{M'}(x)\big]\}\,. \tag{3.7}$$

4. Energy-momentum and angular-momentum integrals

We are now in a position to introduce the relevant kinematical integrals. The basic expressions with which we start are indeed the conventional covariant ones, but with $T_{ab}(x)$ being effectively expressed by either (3.3) or (3.7). For the energy-momentum four vector, we have the formal contour integral

$$p_{AA'}[\Gamma] = \oint_{\Gamma} T_{AA'BB'}(x)d^3x^{BB'}\,, \tag{4.1}$$

where Γ stands for a (closed) contour which is defined in a way similar to that of (2.14). Presumably, (4.1) remains unaffected when the contour is continuously deformed without crossing any relevant singularity. The above expression can be particularly used for defining the Dirac angular-momentum bivector $M_{ab}[\Gamma]$. In effect, we have

$$M_{AA'BB'}[\Gamma] = 2\oint_{\Gamma} x_{[AA'}T_{BB']CC'}(x)d^3x^{CC'} = \oint_{\Gamma}[\varepsilon_{A'B'}\mu_{ABCC'}(x) + \varepsilon_{AB}\tilde{\mu}_{A'B'CC'}(x)]d^3x^{CC'}\,, \tag{4.2}$$

with the μ-angular momentum densities being symmetric in AB and $A'B'$, and the square brackets that occur in the integrand of the middle structure denoting skew symmetrization over the index-pairs. These densities actually appear as independent conjugate quantities, being formally given by

$$\mu_{ABCC'}(x) = x_{A'(A}T^{A'}_{B)CC'}(x) = \mu_{(AB)CC'}(x)\,, \tag{4.3a}$$

$$\tilde{\mu}_{A'B'CC'}(x) = x_{A(A'}T^{A}_{B')CC'}(x) = \tilde{\mu}_{(A'B')CC'}(x)\,. \tag{4.3b}$$

Now, replacing (3.3) into (4.3a), a somewhat lengthy calculation leads us to the explicit expression

$$\begin{aligned}\mu_{ABCC'}(x) =& \frac{-i}{2}\Big\{2\Big[x^{B'}{}_{(A}\tilde{\psi}_{(B'}(x)\nabla_{|C|C')}\psi_{B)}(x) \\ &- x^{B'}_{(A'}\left(\nabla_{B)(B'}\tilde{\psi}_{C')}(x)\right)\psi_C(x) + x^{B'}_{(A}\tilde{\chi}_{|C|}(x)\nabla_{B)(B'}\chi_{C')}(x) \\ &- \varepsilon_{(B'|D'|}\varepsilon_{C')E'}x^{B'}_{(A}\left(\nabla^{(D'}_{|C|}\tilde{\chi}_{B)}(x)\right)\chi^{E')}(x)\Big] \\ &- \varepsilon_{C(A}x^{B'}_{B)}\Big[\left(\nabla_{M(B'}\tilde{\psi}_{C')}(x)\right)\psi^M(x) - \tilde{\chi}^M(x)\nabla_{M(B'}\chi_{C')}(x) \\ &+ \mu\varepsilon_{B'C'}\left(\tilde{\chi}_M(x)\psi^M(x) + \tilde{\psi}_{M'}(x)\chi^{M'}(x)\right)\Big]\Big\}\,. \end{aligned} \tag{4.4}$$

It is evident that the corresponding result for (4.3b) can be readily obtained by letting the operator $\hat{\tau}$ act on (4.4). We thus have

$$\begin{aligned}\tilde{\mu}_{A'B'CC'}(x) =& \tfrac{i}{2}\Big\{2\Big[\varepsilon_{(B|D|}\varepsilon_{C)E}x^B_{(A'}(\nabla^{(D}_{|C'|}\tilde{\psi}_{B')}(x))\psi^{E)}(x) \\ &- x^B_{(A'}\tilde{\psi}_{|C'|}(x)\nabla_{B')(B}\psi_{C)}(x) + x^B_{(A'}(\nabla_{B')(B}\tilde{\chi}_{C)}(x))\chi_{C'}(x) \\ &- x^B_{(A'}\tilde{\chi}_{(B}(x)\nabla_{C)|C'|}\chi_{B')}(x)\Big] \\ &- \varepsilon_{C'(A'}x^B_{B')}\Big[\tilde{\psi}^{M'}(x)\nabla_{M'(B}\psi_{C)}(x) - (\nabla_{M'(B}\tilde{\chi}_{C)}(x))\chi^{M'}(x \\ &+ \mu\varepsilon_{BC}(\tilde{\psi}^{M'}(x)\chi_{M'}(x) + \tilde{\chi}^M(x)\psi_M(x))\Big]\Big\}\,. \end{aligned} \tag{4.5}$$

5. Concluding remarks

The two-spinor description of the Dirac systems presented here brings out naturally all the elementary aspects of the complete theory. As far as the theory for positive-frequency fields is concerned, this feature is particularly exhibited by the method of Section 2 which actually allows one to derive the simplest form of the field equations on $\mathbb{CM}^+$ in a direct way. Thus, when we restrict ourselves to writing the theory in $\mathbb{RM}$, the operator character of the fields is enhanced at every stage of the calculations yielding any result of interest. Indeed, it is from this feature along with the structures exhibited in Section 3 that the relevance of our work stems.

Of course, Eqs (2.9) and (2.10) would have arisen automatically from the basic dynamical statement if we had replaced the explicit Lagrangian density (2.2) into $\delta S[\mathcal{L}_D]$ upon working out the variational principle. If this procedure has been effectively adopted, we would have arrived at the conclusion that

$$\begin{aligned}
&\int_{\Omega^+} \{\delta\tilde{\psi}_{A'}(x)(\nabla^{AA'}\psi_A(x) - \mu\chi^{A'}(x)) \\
&+ \delta\tilde{\chi}_A(x)(\nabla^{AA'}\chi_{A'}(x) - \mu\psi^A(x)) + (\mu\tilde{\chi}^A(x) - \nabla^{AA'}\tilde{\psi}_{A'}(x))\delta\psi_A(x) \\
&+ (\mu\tilde{\psi}^{A'}(x) - \nabla^{AA'}\tilde{\chi}_A(x))\delta\chi_{A'}(x)\} d^4\xi \wedge d^4\eta \\
&+ \tfrac{1}{2} \int_{\partial\Omega^+} [\tilde{\psi}_{A'}(x)\delta\psi_A(x) + \tilde{\chi}_A(x)\delta\chi_{A'}(x) - (\delta\tilde{\psi}_{A'}(x))\psi_A(x) \\
&- (\delta\tilde{\chi}_A(x))\chi_{A'}(x)] d^7\Xi^{AA'} = 0\,,
\end{aligned} \tag{5.1}$$

which clearly yields (2.9) and (2.10), provided that all the variations involved are taken to vanish on $\partial\Omega^+$.

As regards the statement concerning the conservation of charge, we should observe that it is equivalent to either of the (conjugate) relations

$$\nabla_{A'[A}(\tilde{\psi}^{A'}(x)\psi_{B]}(x)) = \nabla_{A'[B}(\tilde{\chi}_{A]}(x)\chi^{A'}(x))\,, \tag{5.2a}$$

$$\nabla_{A[A'}(\tilde{\psi}_{B']}(x)\psi^A(x)) = \nabla_{A[B'}(\tilde{\chi}^A(x)\chi_{A']}(x))\,, \tag{5.2b}$$

such that

$$\nabla_{A'A}J_B^{A'}(x) = \nabla_{A'(A}J_{B)}^{A'}(x)\,, \quad \nabla_{AA'}J_{B'}^{A}(x) = \nabla_{A(A'}J_{B')}^{A}(x)\,. \tag{5.3}$$

It must be pointed out that the expression (3.3) would turn out to be entirely given by the contributions coming from the derivative terms carried explicitly by the defining expression (3.1) in case we had first worked out the ε-piece. The replacement of the field equations (2.9) and (2.10) into $\mathcal{L}_D$ "annihilates" the term $\varepsilon_{AB}\varepsilon_{A'B'}\mathcal{L}_D$ identically. The fact that (3.3) differs from Penrose's expression [4] by an overall sign is due either to an arbitrary choice of the (overall) i-factor entering into $\mathcal{L}_D$ or to a purely conventional re-arrangement of the terms appearing in (3.1). This former possibility would entail a modification of the covariant derivative operator if electromagnetism were brought into the picture, but such modification might be compensated by an interchange involving the sign of the charge.

When the theory is set upon $\mathbb{RM}$, the integrals (2.14) and (4.1) turn out to be nothing else but the ordinary covariant integral expressions which are taken over (suitable) simply connected three-dimensional space-like submanifolds of $\mathbb{RM}$. Under this circumstance, the conservation of charge and the property $\nabla_a T^{ab}(x) = 0$ entail, respectively, the formal functional statements

$$\frac{\delta}{\delta\Sigma} Q[\Sigma] = 0 \text{ and } \frac{\delta}{\delta\Gamma} p_{AA'}[\Gamma] = 0\,. \tag{5.4}$$

It should be pointed up that the invariance of the expressions (2.14) and (4.1) under continuous deformations of contours which do not cross any relevant singularity constitutes another version of these conservation laws.

It is worth remarking explicitly that the angular-momentum structures (4.4) and (4.5) can be built up from each other by applying the simultaneous interchange rules

$$\begin{aligned} \tilde{\psi}_{B'}(x)\nabla_{CC'}\psi_B(x) &\longleftrightarrow \tilde{\chi}_B(x)\nabla_{CC'}\chi_{B'}(x) \\ (\nabla_{BB'}\tilde{\psi}_{C'}(x))\psi_C(x) &\longleftrightarrow (\nabla_{BB'}\tilde{\chi}_C(x))\chi_{C'}(x)\,. \end{aligned} \tag{5.5}$$

In addition, it is clearly seen that the μ-angular momentum pieces bear the trace-free property

$$\mu^{A}_{\ ACC'}(x) = 0 = \tilde{\mu}^{A'}_{\ A'CC'}(x)\,, \tag{5.6}$$

which obviously appears to be compatible with the trivial symmetry identities

$$\varepsilon^{AB}\mu_{ABCC'}(x) = 0 = \varepsilon^{A'B'}\tilde{\mu}_{A'B'CC'}(x)\,. \tag{5.7}$$

Actually, the expression (3.3) suggests writing the following alternative structure for $T_{ab}(x)$

$$\begin{aligned} T_{AA'BB'}(x) =& i\Big\{\tfrac{1}{e}\nabla_{(AA'}J_{BB')}(x) - 2\big[(\nabla_{(A(A'}\tilde{\psi}_{B')}(x))\psi_{B)}(x) \\ &+ (\nabla_{(A'(A}\tilde{\chi}_{B)}(x))\chi_{B')}(x)\big]\Big\}\,. \end{aligned} \tag{5.8}$$

Clearly, the trace obtained from (5.8) reads

$$\begin{aligned} T^{AA'}{}_{AA'}(x) =& \tfrac{i}{e}\Big\{\nabla_{AA'}J^{AA'}(x) - 2e\big[(\nabla_{AA'}\tilde{\psi}^{A'}(x))\psi^{A}(x) \\ &+ (\nabla_{AA'}\tilde{\chi}^{A}(x))\chi^{A'}(x)\big]\Big\}\,, \end{aligned} \tag{5.9}$$

which really agrees with that given by Eq. (3.4). It can, therefore, be said that, in the massless limiting case where $\mu = 0$, $T_{ab}(x)$ turns out to be trace-free. In fact, when combined with (5.8), this result affords us an independent method of establishing the divergencelessness of (2.13).

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