CHIRAL RANDOM MATRIX THEORY AND THE SPECTRUM OF THE DIRAC OPERATOR NEAR ZERO VIRTUALITY*

J. VERBAARSCHOT

Department of Physics
SUNY, Stony Brook, New York 11794, USA

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We study the spectrum of the QCD Dirac operator near zero virtuality. We argue that it can be described by a random matrix theory with the chiral structure of QCD. In the large N limit, this model reduces to the low energy limit of the QCD partition function put forward by Leutwyler and Smilga. We conjecture that the microscopic limit of its spectral density is universal and reproduces that of QCD. Using random matrix methods we obtain its exact analytical expression. This result is compared to numerically calculated spectra for a liquid of instantons, and we find a very satisfactory agreement.

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1. Introduction

As has become clear from a wide body of both numerical and theoretical evidence, the $SU(N_f)\times SU(N_f)$ chiral symmetry of the massless QCD Lagrangian is broken spontaneously to $SU(N_f)$, whereas the $U_A(1)$ symmetry is broken explicitly by the anomaly (see [1]). The order parameter that characterizes the presence of chiral symmetry is the chiral condensate. According to the Banks–Casher formula [2], it is proportional to the spectral density of the Euclidean Dirac operator near zero virtuality. Therefore, the behaviour of the spectrum in this region is of fundamental importance to the phenomenon of chiral symmetry breaking.

In this paper we study the spectrum of the QCD Dirac operator near zero virtuality in the thermodynamic limit while keeping the average spacing

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between the eigenvalues fixed, *i.e.*the spectrum is blown up by a factor V_4 as $V_4 \to \infty$. This limit of the spectral density, which has a nontrivial limit as a consequence of a finite value of the chiral condensate, will be called the microscopic spectral density. It measures correlations on the order of one level spacing as opposed to the average spectral density that is obtained by averaging over many levels. We conjecture that the microscopic spectral density is *universal*. This is reminiscent to the universality observed in the theory of critical phenomena where large scale fluctuations wash out the details of the interaction. Because the spectrum near zero virtuality is dominated by the soft modes of the system, we expect that it is entirely determined by the symmetries of the microscopic theory. Indeed, Leutwyler and Smilga [3] were able to express negative moments of the microscopic spectral density in terms of the effective low-energy limit of QCD that only embodies its underlying chiral structure.

A similar splitting of scales between spectral fluctuations on the order of one level spacing and the average spectral density is well-known in the statistical theory of spectra [4] and quantum chaos [5, 6]. In that case one finds that correlations on the microscopic scale are universal if the classical counterpart of the system is chaotic. The correlations only depend on the symmetries of the system and are given by the invariant random matrix ensembles. This suggests to describe the microscopic spectral density of the QCD Dirac operator in terms of a random matrix theory. The relevant ensemble will be called the chiral random matrix ensemble. We have shown [7] that it is equivalent to the low-energy limit of the chiral effective Lagrangian (see Section 3). Moreover, techniques developed in random matrix theory enabled us [8] to obtain an analytical expression for the microscopic spectral density (see Section 4).

Apart from the above arguments, there is additional evidence in favor of the universality of the spectrum near zero virtuality. First, the Leutwyler–Smilga sum rules are not only found in QCD but in the Schwinger model as well [9] and, numerically, for an instanton liquid model of the QCD-vacuum [7]. Second, the chiral random matrix model reproduces all Leutwyler–Smilga sum rules [7]. Third, the microscopic spectral density of a Dirac operator in a liquid of instantons agrees within numerical accuracy with the random matrix result (see Section 6). Fourth, the same microscopic spectral density is found in the Hofstadter model for universal conductance fluctuations [10].

The Leutwyler–Smilga sum rules and the microscopic spectral density are based on the observation that a nonzero value of the chiral condensate implies that the spacing of the small eigenvalues is $\sim 1/V_4$, as opposed to $\sim 1/V_4^{1/4}$ for a noninteracting system. Such accumulation of small eigenvalues is obtained naturally for a liquid of instantons, where the exact fermionic

zero modes can be chosen as basis states to construct a matrix representation of the Dirac operator [11]. Such model can be simulated numerically and results for its spectral density are presented in Section 7.

This paper is organized as follows. In Section 2 we give an outline of the general framework of this paper and introduce the microscopic spectral density. A derivation of the simplest Leutwyler–Smilga sum rule is given in Section 3. In section 4 we formulate the chiral random matrix model for the trivial topological sector of the QCD vacuum. Its equivalence to the low-energy effective partition function is shown in Section 5. The microscopic spectral density is calculated in section 6, and, in Section 7, numerical results for a liquid of instantons are presented. Concluding remarks are made in Section 8.

2. Formulation of the problem

In this paper we study the spectrum of the massless Euclidean Dirac operator defined by the eigenvalue equation

$$(i\gamma\partial + \gamma A)\phi_\lambda = \lambda\phi_\lambda\,, \tag{2.1}$$

where A is an SU(N_c) valued gauge field. The statistical distribution of the eigenvalues is induced through the gauge field configuration by the statistical weight of the Euclidean QCD partition function

$$Z = \sum_\nu e^{i\nu\theta}\langle \prod_{f=1}^{N_f} \prod_{\lambda_n>0} (\lambda_n^2 + m_f^2) m_f^\nu \rangle_{S_\nu(A)}\,, \tag{2.2}$$

where the masses play the role of sources. The product is over the positive eigenvalues of the Dirac operator, and the average $\langle\cdots\rangle_{S_\nu(A)}$ is over gauge field configurations with topological quantum number ν weighted by the gauge field action $S_\nu(A)$. The topological part of the action, $\exp i\theta\nu$, has been displayed explicitly. The eigenvalues occur in pairs, λ and $-\lambda$, with corresponding eigenfunctions ϕ_λ and $\gamma_5\phi_\lambda$. The only exception is $\lambda = 0$, when we have the logical possibility that $\gamma_5\phi_\lambda \sim \phi_\lambda$. Indeed, according to the Atiyah-Singer theorem, gauge field configurations with topological quantum number ν have at least ν zero modes with a definite chirality.

The propagator is given by

$$S(x,y) = \sum_\lambda -\frac{\phi_\lambda(x)\phi_\lambda^*(y)}{\lambda + im}\,, \tag{2.3}$$

where we have suppressed the color and Dirac indices. The chiral condensate $\langle\bar{q}q\rangle$ is defined as the space-time average of $\langle \mathrm{Tr}\, S(x,x)\rangle$ (the brackets

$\langle \cdots \rangle$ without indices denote averaging with respect to the partition function under consideration). According to the Banks–Casher [2] formula we have

$$\langle \bar{q}q \rangle = \frac{1}{V_4} \int d^4x \langle \mathrm{Tr}\, S(x,x) \rangle = i\pi \frac{\langle \rho(0) \rangle}{V_4}, \tag{2.4}$$

where the spectral density $\rho(\lambda)$ is defined as

$$\rho(\lambda) = \sum_{\lambda_n} \delta(\lambda - \lambda_n). \tag{2.5}$$

The condensate can be expressed as a derivative of the partition function

$$\langle \bar{q}q \rangle = \lim_{m_f \to 0} \lim_{V \to \infty} \frac{i}{V_4} \frac{d}{dm_f} \log Z(m_f). \tag{2.6}$$

The spectral density at zero virtuality has to be defined with care: the thermodynamic limit has to be taken before the chiral limit, *i.e.*,

$$\rho(0) = \lim_{m \to 0} \lim_{V_4 \to \infty} \frac{1}{\pi} \sum_{\lambda} \frac{m}{\lambda^2 + m^2}. \tag{2.7}$$

In this way it is possible to obtain a nonzero spectral density $\langle \rho(0) \rangle$ even in the case that zero eigenvalues are absent at all *finite* volumes. If this leads to a nonzero value of the chiral condensate, this phenomenon is referred to as the spontaneous breaking of chiral symmetry. A necessary requirement for this to happen is that

$$\langle \rho(0) \rangle \sim V_4, \tag{2.8}$$

or, put differently, the spacing between the eigenvalues near zero virtuality is $\sim 1/V_4$.

This number should be contrasted with the spacing of the small eigenvalues for a non-interacting system which is $\sim 1/V^{1/4}$ (obtained from the solution of the free Dirac equation in a hypercubic box of volume V_4). A natural explanation for the accumulation of eigenvalues near zero is provided by the properties of gauge field configurations of a liquid of instantons. Each isolated instanton, a localized field configuration by itself, gives rise to one localized zero mode. When the space-time volume is saturated with an equal number of instantons and anti-instantons the low-lying modes can be expressed as linear combinations of the exact zero modes and form the zero mode zone [12]. For a rather dilute ensemble we therefore expect that the number of small eigenvalues is $\sim V_4$.

As was observed by Leutwyler and Smilga, Eq. (2.4) implies the existence of a family of new sum rules. The simplest one involves the sum

$$\frac{1}{V_4^2}\sum_{\lambda_n\neq 0}\langle\frac{1}{\lambda_n^2}\rangle_\nu \tag{2.9}$$

which should converge to a finite limit for $V_4 \to \infty$.

The most important consequence of chiral symmetry breaking is the appearance of a massless pion in the chiral limit ($m \to 0$). In other words, the pion correlator at zero momentum (note that $S(y,x) = -\gamma_5 S^\dagger(y,x)\gamma_5$)

$$\Pi(p=0) = \int d^4y \langle \mathrm{Tr}\, S(x,y)S^\dagger(x,y)\rangle\,, \tag{2.10}$$

diverges as $\sim 1/m$. This follows from the chiral Ward identity

$$\Pi(p=0) = \frac{1}{2im}(\langle \mathrm{Tr}\, S(x,x)\rangle - \langle \mathrm{Tr}\, S^\dagger(x,x)\rangle)\,, \tag{2.11}$$

which allows us to express this correlator in the spectral density:

$$\Pi(p=0) = \frac{1}{mV_4}\pi\langle\rho(0)\rangle\,, \tag{2.12}$$

where we have used the translational invariance of $\langle \mathrm{Tr}\, S^\dagger(x,x)\rangle$. From Eq. (2.4) it then immediately follows that a finite value of the condensate leads to a massless pion in the chiral limit. As always, it is understood that the thermodynamic limit is to be taken before the limit $m \to 0$.

Because of the importance of the small eigenvalues in the spontaneous breaking of chiral symmetry, we propose to study the spectrum near zero virtuality in the thermodynamic by enlarging it proportional to V_4. In this way the spacing of the eigenvalues remains $\mathcal{O}(1)$ for $V_4 \to \infty$. We call this limit the microscopic limit. Formally, it is defined as

$$\rho_S(x) = \lim_{V_4\to\infty}\frac{1}{V_4}\langle\rho(\frac{x}{V_4})\rangle_\nu\,, \tag{2.13}$$

in the sector of topological charge ν. All Leutwyler-Smilga sum rules can be expressed in the microscopic spectral density and similarly defined spectral correlation functions. For the sum (2.9) we find

$$\int dx \frac{\rho_S(x)}{x^2}\,. \tag{2.14}$$

By using random matrix theory we will obtain an analytical expression for $\rho_S(x)$ in the zero topological charge sector.

3. Derivation of Leutwyler–Smilga sum rules

According to Leutwyler and Smilga [3], the mass dependence of the low-energy limit of the QCD partition function is given by

$$\frac{Z(m,\theta)}{Z(m=0,\theta)} = \int\limits_{\det U=1} \mathcal{D}U \exp\left(\frac{V_4\Sigma}{2}\operatorname{Tr}(mU^{-1}\exp(-i\theta/N_f) + mU\exp(i\theta/N_f)\right). \tag{3.1}$$

The integration is over $SU(N_f)$ with the Haar measure, m is the quark mass matrix, which is taken diagonal, and Σ is the absolute value of chiral condensate. For one flavor, when the integration is absent, this ratio simplifies to

$$\frac{Z(m,\theta)}{Z(m=0,\theta)} = \exp(mV_4\Sigma\cos\theta). \tag{3.2}$$

Expanding this ratio in powers of m and comparing the coefficients to those of the same expansion of the full QCD partition function provides us with an infinite family of sum rules. The simplest sum rule is derived from the $\mathcal{O}(m^2)$ term. In the zero topological charge sector (projected out by integrating over θ) the result is

$$\sum_{\lambda_n>0}\left\langle\frac{1}{\lambda_n^2V_4^2}\right\rangle_{\nu=0} = \frac{\Sigma^2}{4N_f}, \tag{3.3}$$

where the case $N_f = 1$ follows immediately from Eq. (3.2).

4. The chiral random matrix model

In this section we construct a random matrix model for the spectrum of the Dirac operator near zero virtuality in the sector of zero total topological charge. The underlying assumption is that the microscopic spectral density is universal, and only follows from the symmetries of the system. For N_f flavors, the partition function that reflects the chiral structure of QCD is given by [13, 14, 7]

$$Z = \int \mathcal{D}T P(T) \prod_f^{N_f} \det\begin{pmatrix} m_f & iT \\ iT^\dagger & m_f \end{pmatrix}, \tag{4.1}$$

where the integral is over the real and imaginary parts of the matrix elements of the arbitrary complex $N/2 \times N/2$ matrix T, *i.e.*, $\mathcal{D}T$ is the Haar measure. In agreement with the maximum entropy principle [16] the distribution function of the overlap matrix elements $P(T)$ is chosen Gaussian

$$P(T) = \exp(-\frac{N\Sigma^2}{2}\mathrm{Tr}TT^\dagger)\,. \tag{4.2}$$

The symplectic structure is a manifest consequence of chiral symmetry, and implies that the quark eigenvalues occur in pairs. The density of the modes that define the matrix representation of the Dirac operator, N/V_4, is taken equal to 1, which allows us to identify the space-time volume and the total number of modes N.

The partition function (4.1) is inspired by the instanton liquid approximation to the QCD-vacuum [17, 12], where the fermion determinant is approximated by its value in the space of the fermionic zero modes. In that case, the overlap matrix elements T_{ij} are a function of the collective coordinates of the instantons. Therefore, they are not statistically independent, and, as we will see in Section 7, the average level density differs from a semi-circle, the result that can be obtained for the ensemble (4.1).

Although in this section the random matrix model has been defined for zero topological charge only, it can be easily extended [7] to the general case including an arbitrary value of the QCD $\theta-$angle. The essential observation is that if T is an $N \times M$ rectangular matrix, the matrix in Eq. (4.1) has exactly $|N - M|$ zero eigenvalues for $m_f = 0$. For simplicity we restrict ourselves in this paper to square overlap matrices.

5. Reduction of the partition function

In order to evaluate the partition function (4.1) the determinant is written as an integral over Grassmann variables

$$\prod_f \det \begin{pmatrix} m_f & iT \\ iT^\dagger & m_f \end{pmatrix} = \int \prod_f \mathcal{D}\psi^f \mathcal{D}\phi^f \exp \sum_f \begin{pmatrix} \psi^{f*} \\ \phi^{f*} \end{pmatrix} \begin{pmatrix} m_f & iT \\ iT^\dagger & m_f \end{pmatrix} \begin{pmatrix} \psi^f \\ \phi^f \end{pmatrix}, \tag{5.1}$$

where the measure of the Grassmann integration is as usual

$$\mathcal{D}\psi^f = \prod_i d\psi_i^f d\psi_i^{f*}\,, \tag{5.2}$$

and the conjugation * is the conjugation of the second kind (*i.e.*, $\psi^{**} = -\psi$) [18, 19]. The integral over T is gaussian and can be performed easily. In the partition function this results in the factor

$$\exp \frac{2}{N\Sigma^2} \psi_i^{f*} \psi_i^g \phi_j^{g*} \phi_j^f\,, \tag{5.3}$$

which represents a 4-fermion interaction.

The quartic term can be written as a sum of two squares

$$\psi_i^{f*}\psi_i^g\phi_j^{g*}\phi_j^f = \tfrac{1}{4}(\psi_i^{f*}\psi_i^g + \phi_i^{g*}\phi_i^f)(\psi_j^{f*}\psi_j^g + \phi_j^{g*}\phi_j^f) - \tfrac{1}{4}(\psi_i^{f*}\psi_i^g - \phi_i^{g*}\phi_i^f)(\psi_j^{f*}\psi_j^g - \phi_j^{g*}\phi_j^f). \tag{5.4}$$

Each of the two squares can be linearized with the help of a Hubbard–Stratonovich transformation [19]. This allows us to perform the Grassmann integrations at the expense of the introduction of the new real valued integration variables σ^{fg} and $\bar{\sigma}^{fg}$, respectively. Apart from an irrelevant overall constant, the partition function reduces to

$$Z = \int \mathcal{D}\sigma\mathcal{D}\bar{\sigma}\det{}^{N/2}(\sigma + i\bar{\sigma} + m)\det{}^{N/2}(\sigma - i\bar{\sigma} + m) \times \exp\left(-\frac{N\Sigma^2}{2}\operatorname{Tr}(\sigma + i\bar{\sigma})(\tilde{\sigma} - i\tilde{\bar{\sigma}})\right). \tag{5.5}$$

As always, the measure of the integral over the matrices σ and $\bar{\sigma}$ is the Haar measure. The diagonal mass matrix is denoted by m. This integral greatly simplifies for $N_f = 1$. This case is discussed in Appendix A where we obtain Leutwyler–Smilga sum rules for finite values of N.

The complex matrix $\sigma + i\bar{\sigma}$ can be decomposed in 'polar coordinates' as [20]

$$\sigma + i\bar{\sigma} = U\Lambda V^{-1}, \tag{5.6}$$

where U and V are unitary matrices and Λ is a diagonal real positive definite matrix. Since the r.h.s has N_f more degrees of freedom than the l.h.s., one has to impose constraints on the new integration variables. This can be achieved [20] by restricting U to the coset $U(N_f)/U(1)^{N_f}$, where $U(1)^{N_f}$ is the diagonal subgroup of $U(N_f)$. In terms of the new variables the partition function reads

$$Z = \int J(\Lambda)\mathcal{D}\Lambda\mathcal{D}U\mathcal{D}V \times\det{}^{N/2}(U\Lambda V^{-1} + m)\det{}^{N/2}(V\Lambda U^{-1} + m)\exp(-\frac{N\Sigma^2}{2}\operatorname{Tr}\Lambda^2), \tag{5.7}$$

where the integral over U is over $U(N_f)/U(1)^{N_f}$ and the integral over V is over $U(N_f)$.

For N_f flavors we have N_f condensates which break down the symmetry of the action to $U(N_f)/U(1)^{N_f}$ leaving us with N_f^2 Goldstone modes. When we allow for fluctuations of the topological charge the phase of the

determinant becomes massive and the usual number of $N_f^2 - 1$ Goldstone modes is recovered.

The term proportional to m plays the role of a small symmetry breaking term. The integrals over the nonzero modes will be performed by a saddle point integration at $m = 0$, whereas the integrals over the soft modes will be accounted for exactly for a fixed value of mN. The nonzero mode part of the partition function is given by

$$Z(m=0) = \int J(\Lambda) d\Lambda \det{}^N \Lambda \exp(-\frac{N\Sigma^2}{2} \operatorname{Tr} \Lambda^2) . \tag{5.8}$$

The leading order contribution in $1/N$ of the integral over Λ can be obtained by a saddle point approximation. The saddle point equations for the Λ integrals read

$$\Lambda_i = \pm\lambda . \tag{5.9}$$

The negative solution is not inside the integration manifold and can be omitted.

At the saddle point in Λ, the $U-$dependence can be absorbed into V. The $U-$integration yields a finite irrelevant constant. The remaining integral over V which is over $U(N_f)$ can be split into an integral over the phase of the determinant and an integral over $SU(N_f)$. We treat m as a small parameter and expand the determinants up to first order in m. The result for the m dependent part of the partition function is

$$\frac{Z(m)}{Z(m=0)} = \int_0^{2\pi} d\alpha \int_{\det V=1} \mathcal{D}V \exp\left(\frac{N\Sigma}{2} \operatorname{Tr}(mV^{-1}\exp(-i\alpha/N_f) + mV\exp(i\alpha/N_f))\right) , \tag{5.10}$$

which coincides with the result for zero topological charge derived by Leutwyler and Smilga [3] for the QCD partition function using chiral perturbation theory. As was shown in [7] this result can be easily generalized to an arbitrary topological charge sector.

The value of the chiral condensate can be obtained from Eq. (2.6). In the case of equal positive masses we have

$$\langle \bar{q}q \rangle_{\nu=0} = \lim_{m\to 0} \lim_{N\to\infty} \frac{\Sigma}{2N_f} \langle \operatorname{Tr}(V + V^{-1}) \rangle_{\nu=0} . \tag{5.11}$$

For $N_f = 1$ the integral over V is absent, and the sign of the quark condensate is independent of the sign of m. For more than one flavor the order

of the limits allows us to perform the V integral by a saddle point approximation. In the case of equal positive masses the saddle point is at $V = 1$ which allows us to identify the parameter Σ and the chiral condensate

$$\langle \bar{q}q \rangle = \Sigma . \tag{5.12}$$

This completes the reduction of the partition function.

As observed in [3], for more than one flavor the value of the condensate depends on the sign of the quark mass. For example, in the case of two flavors with equal negative masses and $\theta = 0$, the saddle point is at $V = -1$.

6. Spectral density of the random matrix model

In this section we obtain the spectral density of the random matrix partition function (4.1). We use the orthogonal polynomial method developed by Wigner, Dyson and Mehta [21, 22] in the context of the invariant random matrix ensembles. The first step is to rewrite the matrix integration in the partition function (4.1) in polar coordinates. For an arbitrary complex matrix this amounts to the transformation [20]

$$T = U\Lambda V^{-1} , \tag{6.1}$$

where U and V are unitary matrices and Λ is a positive definite diagonal matrix. Since the r.h.s has $n \equiv N/2$ more degrees of freedom than the l.h.s., one has to impose constraints on the new integration variables. This can be achieved [20] by restricting U to the coset $U(n)/U(1)^n$, where $U(1)^n$ is the diagonal subgroup of $U(n)$. The Jacobian of this transformation, that depends only on the eigenvalues λ_k of Λ, is given by

$$J(\Lambda) = \prod_{k<l} (\lambda_k^2 - \lambda_l^2)^2 \prod_k \lambda_k . \tag{6.2}$$

The integrations over the eigenvalues and the unitary matrices decouple. The latter only result in an overall irrelevant constant factor and can be ignored. The eigenvalue distribution is thus given by

$$\rho_n(\lambda_1, \cdots, \lambda_n) = J(\Lambda) \prod_f \prod_k (\lambda_k^2 + m_f^2) \exp(-n\Sigma^2 \sum_{k=1}^n \lambda_k^2) . \tag{6.3}$$

The spectral density $\rho_1(\lambda)$ is obtained by integration over the remaining $n-1$ eigenvalues

$$\rho_1(\lambda_1) = \int \prod_{k=2}^n d\lambda_k \rho_n(\lambda_1, \cdots, \lambda_n) . \tag{6.4}$$

These integrals can be evaluated with the help of the orthogonal polynomial method (see [21] for references). The main ingredient is to write the product over the differences of the eigenvalues as a Vandermonde determinant, *i.e.*

$$\prod_{k<l}(\lambda_k^2-\lambda_l^2)^2=\begin{vmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ \lambda_1^{2(n-1)} & \cdots & \lambda_n^{2(n-1)} \end{vmatrix}^2 \tag{6.5}$$

which up to a constant can be rewritten in terms of orthogonal polynomials P_k as

$$\begin{vmatrix} P_0(\lambda_1^2) & \cdots & P_0(\lambda_n^2) \\ \vdots & & \vdots \\ P_{n-1}(\lambda_1^2) & \cdots & P_{n-1}(\lambda_n^2) \end{vmatrix}. \tag{6.6}$$

The P_k will be chosen orthogonal according to the weight function

$$\int_0^\infty d(\lambda^2)(\lambda^2+m^2)^{N_f}\exp(-n\Sigma^2\lambda^2)P_k(\lambda^2)P_l(\lambda^2)=\delta_{kl}\,. \tag{6.7}$$

For $m=0$ these polynomials are well known,

$$P_k(s)=\left(n\Sigma^2\frac{k!}{\Gamma(N_f+k+1)}\right)^{1/2}L_k^{N_f}(sn\Sigma^2)\,, \tag{6.8}$$

where the $L_k^{N_f}$ are the generalized Laguerre polynomials.

The determinant can be expanded according its definition. All integrals can be performed immediately by orthogonality and, up to an overall constant, we are left with

$$\rho_1(\lambda)=2\Sigma\sqrt{n}\sum_{k=0}^{n-1}\frac{k!}{\Gamma(N_f+k+1)}L_k^{N_f}(z)L_k^{N_f}(z)z^{N_f+1/2}\exp(-z)\,, \tag{6.9}$$

where z is defined by

$$z=n\lambda^2\Sigma^2\,. \tag{6.10}$$

The normalization constant has been chosen such that $\int d\lambda\rho(\lambda)=N$ (remind that $N=2n$). The sum can be evaluated exactly with the Christoffel–

–Darboux formula (which can be found in any treatise on orthogonal polynomials), resulting in

$$\rho_1(\lambda) = \frac{2\Sigma\sqrt{n}\,n!}{\Gamma(N_f+n)} \times \left(L_{n-1}^{N_f}(z)L_{n-1}^{N_f+1}(z) - L_n^{N_f}(z)L_{n-2}^{N_f+1}(z)\right) z^{N_f+1/2}\exp(-z)\,, \quad (6.11)$$

which, constitutes the exact spectral density of the model (4.1). The microscopic limit is obtained by taking $N \to \infty$ while keeping $N\lambda = x$ fixed (remember that $n = N/2$). This can be achieved from the asymptotic relation

$$\lim_{n\to\infty} \frac{1}{n^\alpha} L_n^\alpha(\frac{x}{n}) = x^{-\alpha/2} J_\alpha(2\sqrt{x})\,, \quad (6.12)$$

where J_α is the ordinary Bessel function of degree α. One notices that to leading order in n the difference in Eq. (6.11) cancels. However, using recursion relations for the generalized Laguerre polynomials, this cancellation can be achieved explicitly, and the asymptotic relation can be applied to the next to leading order terms. The result for the microscopic spectral density is

$$\rho_S(x) = \frac{\Sigma^2 x}{2}(J_{N_f}^2(\Sigma x) - J_{N_f+1}(\Sigma x)J_{N_f-1}(\Sigma x))\,. \quad (6.13)$$

From the asymptotic relation for the Bessel function

$$J_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} \cos(z - \frac{\pi}{2}\nu - \frac{\pi}{4})\,, \quad \text{for} \quad z \to \infty\,, \quad (6.14)$$

we find that

$$\lim_{x\to\infty} \rho_S(x) = \frac{\Sigma}{\pi} \quad (6.15)$$

which agrees with the Banks–Casher relation. As follows from the small x behaviour of

$$\rho_S(x) \sim \frac{\Sigma}{N_f!(N_f+1)!}\left(\frac{\Sigma x}{2}\right)^{2N_f+1} \quad \text{for} \quad x \to 0\,, \quad (6.16)$$

small eigenvalues are strongly suppressed for an increasing number of flavors.

The formula (6.13) reproduces all diagonal sum rules of Leutwyler and Smilga, *e.g.*, the sum

$$\sum_n \frac{1}{N^{2p}\lambda_n^{2p}}\,, \quad (6.17)$$

can be converted into an integral over the microscopic variable $x = \lambda N$ resulting in

$$\int_0^\infty \frac{\rho_S(x)dx}{x^{2p}} = \left(\frac{\Sigma}{2}\right)^{2p} \frac{\Gamma(2p-1)\Gamma(N_f-p+1)}{\Gamma(p)\Gamma(p+1)\Gamma(N_f+p)}. \tag{6.18}$$

The above spectral density thus summarizes all sum rules and yields new sum rules, *e.g.* sum rules for noninteger values of p.

It is possible to derive all higher order spectral correlation functions which generate an infinite family of new spectral sum rules. An explicit expression for the level correlation function is given in [8], and it generates sum rules obtained previously by Leutwyler and Smilga [3].

7. Spectral density for the instanton liquid

The sum over field configurations in the partition function (2.2) can be approximated semiclassically by that of a liquid of instantons. Instead of averaging over all gauge field configurations, we average over the collective coordinates of the instantons only, whereas 1-loop quantum fluctuations about the instantons are included in the measure. The action in (2.2) is the instanton action, which also includes the interaction between instantons. We use the so called streamline [23, 24] interaction supplemented by a core in order to stabilize the instanton liquid. The fermion determinant is calculated in the space spanned by the fermionic zero modes with overlap matrix elements that can be derived from the streamline configuration [15]. More details on the above instanton liquid model can be found in [26].

The numerical simulations were carried out for a liquid of 64 instantons in a Euclidean space time volume of $(2.378)^3 \times 4.756$ in units of $\Lambda_{\rm QCD}^{-4}$. Averages were obtained from 1,000 statistically independent configurations for $N_f = 1$, 2, 3 and 100,000 for $N_f = 0$. Our main results are presented in Figs 1 and 2. The number of flavors is shown in the label of the figures. In Fig. 1 we show the average spectral density $n(\lambda)$ of the Dirac operator in the space of the zero modes (the normalization is $\int_0^\infty n(\lambda)d\lambda = 1$, and λ is in units of $\Lambda_{\rm QCD}$, so $n(\lambda)$ differs from $\langle\rho(\lambda)\rangle$ by a normalization factor). Note that for $N_f = 2$ the thermodynamic limit of the slope $n'(0)$ is zero, whereas for $N_f = 3$ it is negative. This agrees with a recent result of Smilga and Stern [25] according to which the flavor dependence of $n'(0)$ is given by $\sim (N_f^2 - 4)/N_f$.

For $N_f = 0$, 1, 2 we observe a clear separation of the microscopic scale near zero virtuality and the overall average spectral density. The latter

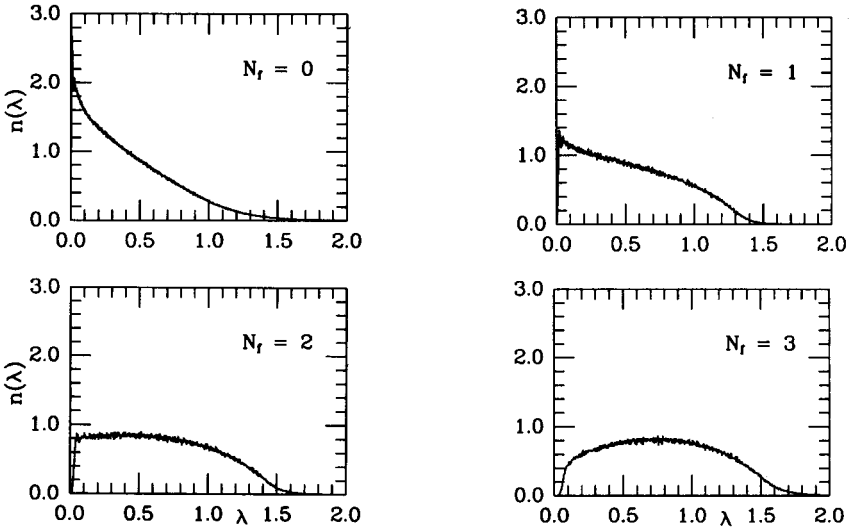

Fig. 1. The eigenvalue density $n(\lambda)$ for $N_f = 0$, 1, 2, and 3. The area below each curve is normalized to 1, and the bin size is 0.00125 for $N_f = 0$ and 0.0025 in the other cases.

is obtained by a fit with a smooth function ignoring the region near zero virtuality. This allows us of obtain a finite value for $\langle\rho(0)\rangle$. The microscopic limit of the spectral density is shown in Fig. 2. The full line is the spectral density of the unfolded spectrum $\langle\rho(\mu)\rangle$ (see Ref. [5] for an exact definition), for which the average spacing between the eigenvalues is equal to one. The dashed curve shows the asymptotic result for the microscopic spectral density in the same units which agrees well with the random matrix result. This is even more remarkable if one realizes that the average spectral density is very different from the random matrix result (a semi-circle). It should be stressed that the microscopic spectral density contains no free parameters. For $N_f = 3$ it is no longer possible to extract the thermodynamic limit of $\langle\rho(0)\rangle$ and an unambiguous comparison with the microscopic spectral density cannot be made. The strong suppression of small eigenvalues is consistent with Eq. (6.16).

We observe that the spectrum is very stiff. The peak occurs because the smallest eigenvalue is always close to its average position. As is also the case for the other invariant random matrix ensembles, the fluctuations are much less than for a random sequence of eigenvalues, although it looks like that the oscillations in the asymptotic spectral density are not reproduced by the liquid of instantons. This may be a finite size effect.

The Leutwyler–Smilga sum rules can be checked for a liquid of instantons. The results completely agree with the exact results in [3] (see [7] for a detailed comparison).

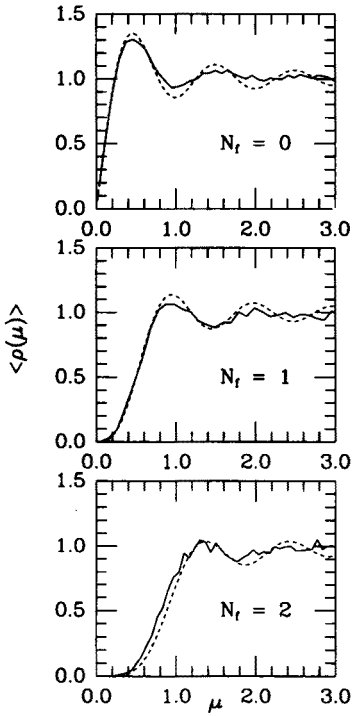

Fig. 2. The microscopic limit of the spectral density in Fig 1. We show the density for the unfolded spectrum where the *average* position of the n'th eigenvalue is at $\mu = n$. The dashed line represents the exact microscopic spectral density $\rho_S(\mu)$ for the same normalization, *i.e.* $\Sigma = \pi$.

8. Conclusions

We have studied the spectrum of the QCD Dirac operator near zero virtuality with the help of the microscopic spectral density, which is obtained by enlarging the spectrum by V_4 as $V_4 \to \infty$. The existence of this limit is an immediate consequence of the spontaneous breaking of chiral symmetry.

We have conjectured that the microscopic spectral density is universal. This should be contrasted with the average spectral density which depends on the details of the system. The latter varies on a scale of $\mathcal{O}(V_4^0)$, whereas the first shows fluctuations on the microscopic scale of $\mathcal{O}(1/V_4)$. The implicit assumption is that both scales can be separated. On the basis of the universality we have formulated a random matrix theory that apart form the chiral structure of the QCD Dirac operator has no other information content.

It was shown that this random matrix theory is equivalent to the extreme low-energy limit of the QCD partition function. Therefore all Leutwyler–Smilga sum rules are reproduced. Using the orthogonal polynomial tech-

nique developed by Wigner, Dyson and Mehta, we were able to obtain the exact analytical result for the microscopic spectral density.

In general it is very hard to obtain exact numerical spectra for a field theory. However, in the instanton liquid model for the QCD vacuum we were able to obtain sufficient statistics for the microscopic spectral density, and we found a very satisfactory agreement with the random matrix model. In view of this, it would be very interesting to compare our results to spectra obtained from lattice QCD.

Appendix A

In this appendix we derive the sum rules for one flavor and finite values of N. For $N_f = 1$ the partition function (5.5) simplifies to

$$Z = \int d\sigma d\bar{\sigma}(\sigma + i\bar{\sigma} - m)^{N/2}(\sigma - i\bar{\sigma} - m)^{N/2} \exp(-\frac{N\Sigma^2}{2}(\sigma + i\bar{\sigma})(\sigma - i\bar{\sigma})), \tag{A.1}$$

where an irrelevant overall constant has been suppressed. At finite N the pre-exponential factors can be expanded as a binomial series which provides us with an expansion in powers of m. The coefficients are elementary integrals, and for the $m-$dependent part of the partition function we find

$$\frac{Z(m)}{Z(0)} = 1 + \frac{m^2N^2\Sigma^2}{4} + \frac{m^4N^4\Sigma^4}{64}(1 - \frac{2}{N}) + \cdots. \tag{A.2}$$

It should be noted that no approximations have been made. On the other hand, the fermion determinant can be written as a product over the eigenvalues which leads to the expansion

$$\frac{Z(m)}{Z(0)} = 1 + m^2 \left\langle \sum_n \frac{1}{\lambda_n^2} \right\rangle_{\nu=0} + m^4 \frac{1}{2} \left\langle \sum_{n \neq n'} \frac{1}{\lambda_n^2 \lambda_{n'}^2} \right\rangle_{\nu=0} + \cdots, \tag{A.3}$$

where the average is with respect to the massless partition function. By equating the coefficients of the powers of m^2 we obtain sum rules for the inverse powers of the eigenvalues that are valid for any value of N. The sum rule (3.3) for $N_f = 1$ is reproduced. It is valid for any value of N. The second sum rule is modified by the factor $(1 - 2/N)$. For $N = 2$ we find zero which is correct because in this case there are no terms that contribute to the sum $n' \neq n$ in (A.3).

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