SEMICLASSICAL PION RADIATION*

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The lecture is an introduction to understanding pion production in semiclassical terms.

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1. Introduction

Pions are the lightest hadrons and therefore they are the first to be produced both in the decay of resonances and in high energy hadron collisions. Moreover, in many circumstances a semiclassical approach to pion production is justified parametrically. A semiclassical treatment of a problem means usually a considerable simplification which, if applicable, is not what one should step over. In this lecture I discuss certain conceptual problems in connection with the pion radiation. To be specific, I deal with these problems considering two important physical cases: *(i)* decay of the Δ resonance into πN and *(ii)* multi-pion production in high energy collisions. The first part of the paper is based on a work with Victor Petrov [1] while the second is based on a work with Jean-Paul Blaizot [2].

2. Pion radiation by a rotating Skyrmion

In this Section we shall show that pion radiation can be treated much as the electro-magnetic one. In particular, we shall demonstrate that, similar to the case of electro-magnetism;

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1) Accelerated sources of isospin cannot exist; they loose energy, angular momentum and isospin through pion radiation;
2) The Goldstone nature of pions is an analogue of gauge invariance; in particular, the pion radiation is very similar to the "dipole" radiation in classical electrodynamics, proportional to the square of the acceleration;
3) Classical pion radiation is in a correspondence (*à la* N. Bohr) with the quantum theory, where comparable.

To demonstrate that V. Petrov and I [1] have considered the classical radiation by a rotating chiral soliton — a Skyrmion, to be short. The concrete dynamical realization of the soliton will be irrelevant here – a nice model to keep in mind is the Chiral Quark Soliton Model [3–5] which proved to be successful in explaining quantitatively most of the nucleon characteristics (see, *e.g.*, [6, 7] and references therein). The most successful predictions of the chiral soliton models are however those which are independent of the concrete dynamics but which rather rely on the symmetry of the hedgehog pion field [8], and that is what we shall use here. From that angle the naive Skyrme model [9, 10] suits us not worse than a much more powerful Chiral Quark Soliton Model.

The effective pion lagrangian can be written as

$$\mathcal{L}_{\text{eff}}[\pi] = \frac{F_\pi^2}{4}\,\mathrm{Tr}\,\partial_\mu U^\dagger \partial_\mu U \;+\; \text{higher derivatives}, \tag{2.1}$$

where U is a unitary 2×2 matrix composed of the pion fields:

$$U = \exp\left(\frac{i\pi^a \tau^a}{F_\pi}\right). \tag{2.2}$$

We assume that the effective pion action admits a static hedgehog solution:

$$U_0(\boldsymbol{r}) = \exp\left[i(\boldsymbol{n}\cdot\boldsymbol{\tau})P(r)\right], \tag{2.3}$$

where $P(r)$ is the so-called profile function of the soliton which starts from $\pi = 3.14\ldots$ at the origin, has the characteristic scale of a nucleon, r_0, and decreases as A/r^2 at large distances.

In order to provide nucleons with quantum numbers one has to consider an SU(2)-rotated soliton:

$$U(\boldsymbol{r},t) = R(t)U_0(\boldsymbol{r})R^\dagger(t). \tag{2.4}$$

The quantization of the rotation is similar to that of the spherical top [10]; the hedgehog form of the static pion field (2.3) guarantees that all rotational states have isospin T equal to spin J, and their wave functions are given by the Wigner D-functions [5]

$$\Psi^{J=T}_{J_3,T_3}(R) = \sqrt{2J+1}(-1)^{J+T_3} D^{J}_{-T_3,J_3}(R)\,, \tag{2.5}$$

with the familiar rotational spectrum

$$\mathcal{M}_J = \mathcal{M} + \frac{J(J+1)}{2I}\,, \tag{2.6}$$

where $\mathcal{M}$ is the rest mass of the soliton and I is its moment of inertia. The $T = J$ quantization for hedgehogs is a welcome feature since in nature this is what we observe for nucleons ($T = J = 1/2$) and Δ ($T = J = 3/2$). However we do not have decisive evidence of the existence of exotic resonances with $T = J = 5/2, \ldots$ though claims of their discovery appear from time to time. The reason why the higher rotational states do not exist is to some extent "accidental": their decay width is too large to allow identification with a resonance. Let me start by reminding how to calculate the decay rate of a soliton rotational state.

The pion-soliton coupling in terms of the soliton spin–isospin orientation matrix R is given by [10–12]

$$-\frac{3g_{\pi NN}}{2\mathcal{M}}\,\frac{1}{2}\,\mathrm{Tr}(R^{\dagger}\tau^{a}R\sigma_i)\,ik_i\,, \tag{2.7}$$

where k_i is the pion 3-momentum and a is its isotopic component. Sandwiching (2.7) between initial and final state wave functions (2.5) we get for the $J \to J-1$ transition amplitude squared (averaged over the initial and summed over final spin and isospin states):

$$\left(\frac{3g_{\pi NN}}{2}\right)^2 \frac{2J-1}{2J+1}\,\frac{k^2}{3}\,. \tag{2.8}$$

To get the decay width one has to multiply (2.8) by the phase space factor. We obtain

$$\Gamma_{J\to J-1} = \frac{3g^2_{\pi NN}}{8\pi\mathcal{M}^2}\,\frac{2J-1}{2J+1}\,\frac{\mathcal{M}_{J-1}}{\mathcal{M}_J}\left[\frac{(\mathcal{M}_J^2 - \mathcal{M}_{J-1}^2 + m_\pi^2)^2 - 4\mathcal{M}_J^2 m_\pi^2}{4\mathcal{M}_J^2}\right]^{3/2}. \tag{2.9}$$

Neglecting the pion mass m_π we have at $J \gg 1$:

$$\Gamma = \frac{3g^2_{\pi NN}}{8\pi\mathcal{M}^2}\left(\frac{J}{I}\right)^3. \tag{2.10}$$

Numerically, if we take the experimental values $g_{\pi NN} \approx 14$, $\mathcal{M}_{1/2} \approx 940\,\mathrm{MeV}$, $\mathcal{M}_{3/2} \approx 1240\,\mathrm{MeV}$ (hence $1/I \approx 200\,\mathrm{MeV}$), we get for the width of the Δ resonance $\Gamma_\Delta \approx 120\,\mathrm{MeV}$ in good agreement with the experiment. The asymptotic formula (2.10) gives a much larger width $\Gamma_\Delta^{\mathrm{asymp}} \approx 600\,\mathrm{MeV}$, which is not surprising since $J = {}^3\!/_2$ can be hardly considered to be much greater than unity; also the non-zero pion mass is essential for the real-world numerics. For the exotic $J = T = {}^5\!/_2$ state with $\mathcal{M}_{5/2} \approx 1700\,\mathrm{MeV}$ Eq. (2.9) predicts a width $\Gamma_{5/2} \approx 760\,\mathrm{MeV}$ which explains why it is hard to observe such a state. We see that this huge width is in a sense accidental, being due to the large pion-nucleon coupling. In order to elucidate an important theoretical point, I shall proceed as if higher rotational states were possible. My purpose is to derive Eq. (2.10) from a *classical pion radiation theory*.

First of all we have to describe the rotation classically, which should be possible for $J \gg 1$. We shall see, however, that strictly stationary rotating solitons cannot exist: they have to loose energy through radiation of the classical pion field. A similar theorem is well known in classical electrodynamics.

Let us consider the far-distance tail of the rotating soliton field assuming that the pion field is already small, so that one can linearize the equation of motion. Then it is just the d'Alembert equation. Assuming that the soliton rotates with the angular velocity ω around the z axis meaning that the field depends on r, θ and $\phi' = \phi - \omega t$, the d'Alembert equation can be written as

$$\Delta\pi^a - \omega^2 \frac{\partial^2}{\partial\phi'^2}\pi^a = 0\,. \tag{2.11}$$

We expand solutions of this equation in the complete set of functions,

$$\pi^a(r,\theta,\phi') = \sum_{l,m} R^a_{lm}(r) Y_{lm}(\theta,\phi')\,, \tag{2.12}$$

where the radial functions satisfy the equation

$$\left[\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \frac{l(l+1)}{r^2} + m^2\omega^2\right] R^a_{lm}(r) = 0\,. \tag{2.13}$$

Its solutions are spherical Bessel functions $j_l(m\omega r)$ and $y_l(m\omega r)$ [13]. Their asymptotics are $j_l(z) \sim \sin(z - l\pi/2)/z$ and $y_l(z) \sim -\cos(z - l\pi/2)/z$. Hence the rotating pion field (2.12) has to fall off as $1/r$ at large distances. Such behaviour is typical for the radiation field. The energy and angular momentum of such field diverge at $r \to \infty$:

$$E = \frac{F_\pi^2}{2}\int d^3r\left[(\partial_0\pi^a)^2 + (\partial_i\pi^a)^2\right] = \infty\,,$$

$$J_3 = -F_\pi^2 \int d^3r \partial_0 \pi^a \partial_\phi \pi^a = \infty . \tag{2.14}$$

This is because we calculate here, in fact, not the energy and the angular momentum of a soliton but rather of a soliton together with its radiation field. The "proper" energy and angular momentum of the soliton itself can be found by subtracting the contribution of the radiation to these quantities. Let us show how it is done in the simplest case of a spherically-symmetric soliton — a hedgehog given by Eq. (2.3). We shall consider slow rotations, $\omega r_0 \ll 1$, where r_0 is the typical size of the hedgehog. At $\omega r_0 \sim 1$ the problem looses sense: in this case the radiation is so strong that one cannot speak of a stationary rotation.

At $r \gg r_0$ the pion field of the soliton is small, and one can use the asymptotic form of the hedgehog (2.3) replacing the profile function $P(r)$ by its asymptotics, $P(r) \to A/r^2$, $A = \text{const } r_0^2$. In the range $r_0 \ll r \ll 1/\omega$ we have for the pion field of a rotating soliton

$$\begin{aligned}
\pi^1 &= \frac{A}{r^2} \sin\theta \cos(\phi - \omega t) , \\
\pi^2 &= \frac{A}{r^2} \sin\theta \sin(\phi - \omega t) , \\
\pi^3 &= \frac{A}{r^2} \cos\theta .
\end{aligned} \tag{2.15}$$

On the other hand, at $r \gg r_0$ the pion field must satisfy the d'Alembert equation, $\partial^2 \pi^a = 0$, to which the general non-linear equation of motion for the soliton is reduced when the fields are small. In addition, at $r \to \infty$ the pion field must satisfy the so-called radiation condition,

$$\frac{\partial \pi^a}{\partial r} + \frac{\partial \pi^a}{\partial t} = 0 . \tag{2.16}$$

In principle, one could as well look for a field having the form of incoming waves at infinity. That would correspond to a situation when one keeps the rotation by pumping energy from infinity. But we are now interested in a free soliton loosing its energy and not *vice versa*. Solving the d'Alembert equation (2.11) with the boundary conditions (2.15) we get

$$\begin{aligned}
\pi^1 &= \frac{A}{r^2} \sin\theta (\cos\alpha + \omega r \sin\alpha) , \\
\pi^2 &= \frac{A}{r^2} \sin\theta (\sin\alpha - \omega r \cos\alpha) , \\
\pi^3 &= \frac{A}{r^2} \cos\theta , \quad \alpha = \omega r + \phi - \omega t .
\end{aligned} \tag{2.17}$$

At $r \ll 1/\omega$ these equations reduce to (2.15). However, at $r \gg 1/\omega$ Eq. (2.17) exhibits a $1/r$ falloff in accordance with a general result above.

Let us find the intensity of the outgoing waves. Using the general expression for the stress–energy tensor,

$$T_{\mu\nu} = \frac{F_\pi^2}{2}\left(\mathrm{Tr}\,\partial_\mu U \partial_\nu U^\dagger - \tfrac{1}{2} g_{\mu\nu}\,\mathrm{Tr}\,\partial_\alpha U \partial_\alpha U^\dagger\right), \tag{2.18}$$

we get for the momentum flow at large distances:

$$T_{0i} \to F_\pi^2 \partial_0 \pi^a \partial_i \pi^a \to \frac{F_\pi^2 A^2 \omega^4 \sin^2\theta n_i}{r^2}. \tag{2.19}$$

In a slightly different context this formula was derived some time ago by Lorentz [14], coinciding in its angular and frequency dependence with the intensity of the electromagnetic *dipole* radiation. The coincidence is not accidental: owing to its Goldstone nature the pion field couples to the isospin source through a gradient; the same is true for the electromagnetic field and is due to gauge invariance. In the non-relativistic limit we are now considering, only the lowest, dipole component of the radiation survives in both cases.

According to the energy–momentum conservation law, $\partial^\mu T_{\mu\nu} = 0$, the energy loss owing to radiation is

$$-\frac{dE}{dt} \equiv W = \lim_{r\to\infty} \int d\Sigma_i T_{0i} r^2$$
$$= F_\pi^2 A^2 \omega^4 2\pi \int_0^\pi d\theta \sin^3\theta = \frac{8\pi F_\pi^2 A^2 \omega^4}{3}. \tag{2.20}$$

We notice that the coefficient A is directly related [9, 10] to the nucleon axial constant,

$$g_A = \frac{8\pi A F_\pi^2}{3}, \tag{2.21}$$

which, in its turn, is related to the pion–nucleon coupling constant through the Goldberger–Treiman relation,

$$g_{\pi NN} = \frac{g_A \mathcal{M}}{F_\pi}, \tag{2.22}$$

where $\mathcal{M}$ is the nucleon mass. Using these relations, the radiation intensity can be rewritten as

$$W = \frac{3g_{\pi NN}^2 \omega^4}{8\pi \mathcal{M}^2}. \tag{2.23}$$

The radiation carries away not only energy but also the angular momentum of the soliton:

$$\frac{dJ_3}{dt} = \lim_{r\to\infty} \int d\Sigma_i \epsilon_{3jk} x_j T_{ik} = F_\pi^2 \int d\Sigma \frac{\partial \pi^a}{\partial x_i} \frac{\partial \pi^a}{\partial \phi} = -\frac{W}{\omega}. \tag{2.24}$$

The proper energy and angular momentum of a rotating soliton can be found by subtracting volume integrals of the energy density and of the angular momentum density, respectively [14]:

$$\begin{aligned} E^{\text{prop}} &= \int d^3r T_{00}^{\text{prop}} = \int d^3r (T_{00} - n_i T_{i0}), \\ J_3^{\text{prop}} &= \int d^3r \epsilon_{3jk} T_{oj}^{\text{prop}} x_k = \int d^3r \epsilon 3jk (T_{0j} - n_i T_{ij}) x_k. \end{aligned} \tag{2.25}$$

It can be seen from Eq. (2.17) that at $r \to \infty$ the integrands in Eq. (2.25) behave as follows:

$$\begin{aligned} T_{00} - n_i T_{i0} &= \frac{F_\pi^2 A^2}{r^4} \left(\frac{6}{r^2} + \omega^2 \sin^2\theta \right), \\ \epsilon 3jk (T_{0j} - n_i T_{ij}) x_k &= \frac{F_\pi^2 A^2}{r^4} 2\omega \sin^2\theta. \end{aligned} \tag{2.26}$$

As a result both quantities, E^{prop} and J_3^{prop} are convergent as anticipated. Moreover, it can be shown on general grounds that these quantities satisfy familiar relations for slowly rotating bodies,

$$\begin{aligned} E^{\text{prop}} &= \mathcal{M} + \frac{I\omega^2}{2} + O(\omega^4), \\ J_3^{\text{prop}} &= I\omega + O(\omega^3), \end{aligned} \tag{2.27}$$

where $\mathcal{M}$ is the soliton mass at rest and I it its moment of inertia. Equations (2.26) are evidently in accordance with (2.27) giving in fact the large-distance contributions to E^{rest} and I. It is important that, owing to the "dipole" nature of the pion radiation, the energy loss is $\sim \omega^4$ (see Eq. (2.23)). To this accuracy the rotation can be regarded as approximately stationary and hence the $O(\omega^2)$ correction to the rest mass makes sense.

We are now in a position to calculate the lifetime of a rotating soliton which looses energy owing to classical pion radiation. According to the Bohr correspondence principle, the lifetime is defined as the time t during which a highly excited state looses a portion of its energy corresponding to a transition to the nearest lower state. In our case it is the transition from a rotational state J to the one with $J-1$. Using the expression for the radiation intensity (2.23) and the relation $\omega = J/I$ we get

$$\Gamma = \frac{1}{t} = \frac{W}{\mathcal{M}_J - \mathcal{M}_{J-1}} = \frac{3g_{\pi NN}^2}{8\pi \mathcal{M}^2}\left(\frac{J}{I}\right)^3 \tag{2.28}$$

which coincides exactly with the quantum-mechanical result (2.10)! The same result follows also from Eq. (2.24): one can determine the lifetime as the time during which the soliton looses one unit of its angular momentum,

$$\Gamma = \left|\frac{dJ}{dt}\right| \frac{1}{J-(J-1)} = \frac{3g_{\pi NN}^2}{8\pi \mathcal{M}^2}\left(\frac{J}{I}\right)^3 . \tag{2.29}$$

We leave it to the reader to check that the time during which the rotating soliton looses one unit of its isospin owing to pion radiation, is also given by the same formula.

To summarize: we have demonstrated that the pion radiation is similar to the electromagnetic radiation by the accelerated charge. A weekly accelerated source of isospin radiates a "dipole" pion field (which is due to the Goldstone nature of pions), and the classically computed lifetime of that source is in correspondence with the exact quantum-mechanical calculation of the decay rate. Of course, if the pion source is relativistic one would expect that all "multipoles" in the pion radiation enter the game.

Before I move to the applications of these ideas to high energy collisions, let me briefly discuss the fate of the rotating soliton, as one increases the angular momentum J. As seen from Eq. (2.10) the condition that the radiation width Γ_J remains much less than the mass of the rotating state $\mathcal{M}_J$ is

$$J \ll N_c \quad \text{or} \quad \omega r_0 \ll 1 , \tag{2.30}$$

where N_c is the number of colours and r_0 is the characteristic size of the soliton. Simultaneously, it is the condition that the form of the soliton is not affected by the rotation. It is amusing that in the theoretical limit $N_c \to \infty$ there is a window for the angular momenta, $1 \ll J \ll N_c$, where not only the chiral soliton is a classical object but the decay of the rotational excitations of this object can be described by classical pion radiation theory.

At $J \sim N_c$ or, equivalently, at $\omega r_0 \sim 1$ the classical radiation of pions by a spherically-symmetric soliton blows up to such extent that the widths become comparable to the masses. Simultaneously, the centrifugal forces become comparable to the binding forces inside the soliton, and the soliton expands in the direction perpendicular to the rotation axis. The angular velocity of an expandable cigar-like soliton *decreases* with the growth of J [1] — this is a well-known result for a string whose endpoints are moving with the speed of light. As a result the lifetime of a rotating soliton becomes stable or even slightly decreasing with J. That is the way how the chiral soliton survives at high angular momenta. Naturally, the quantization rule $T = J$ is lost once the soliton looses its hedgehog form, that is at $J \sim N_c$.

Unfortunately, the crossover region $J \sim N_c$ is too complicated to be studied analytically; an insight of what happens there can be found in a paper by Blaizot and Ripka [15]. A significant simplification is achieved at $J \gg N_c$. In this case a string-like analytical solution of the equation of motion has been found [1]. Victor Petrov and I have shown that the chiral solitons with large angular momenta lie on linear Regge trajectories. Knowing the transverse pion field distribution inside the string we have found the string tension or, equivalently, the Regge slope

$$\alpha' = \frac{1}{8\pi^2 F_\pi^2} \simeq 1.45\,\mathrm{GeV}^{-2} \tag{2.31}$$

being but a factor of 1.5 larger than the phenomenological value of $.9\,\mathrm{GeV}^{-2}$. The discrepancy is possibly eliminated by quantum corrections to the string tension.

Finally, let me mention that the decay widths of highly excited states lying on linear Regge trajectories can be also estimated from the classical pion radiation theory [1].

3. Pion radiation in high energy collisions

In the previous section I have described classical pion radiation by a rotating chiral soliton. There is another physical example where we could expect that classical pion radiation is applicable — I mean multi-pion production in high energy collisions. Indeed, high energy collisions, especially central collisions of heavy ions, provide a strong source of the isospin current and are thus a strong source of pion fields. Various *dynamical* ideas about the classical pion field formation in proton-proton and/or heavy ion collisions have been suggested recently [16–19]; however at this stage we prefer to keep as far as possible from the concrete dynamical questions.

If the number of pion quanta per unit phase space volume is greater than one the pion radiation should be described semiclassically. In simple terms it means that there should be a dominant "antenna pattern" for pions. However, pions carry small isospin (= 1) which is hence always a quantum-mechanical characteristics, even in the academic limit of infinite pion multiplicity. Therefore, there cannot be any pattern for the observable π^+, π^- or π^0 mesons. The semiclassical nature of pion production would result only in certain *correlations* in the production of pions with definite isospin components. The main point of this section is that rather peculiar correlations are indicated by the symmetry considerations only [2] — the role of dynamics is to guarantee that the pion radiation is so strong that it can be treated semiclassically; testing the correlations is a way to check that hypothesis. These correlations, quite distinct from the much studied Bose–Einstein ones, originate from correlations between spatial and isospin coordinates in the solution of the nonlinear field equations. As mentioned in the Introduction, in this section I follow our recent work with Jean-Paul Blaizot.

Mathematically, one can write the amplitude of N pions production with the help of the Lehmann–Symanzik–Zimmermann formula, where the N-point pion Green function is presented through the functional integral over the pion fields:

$$\mathcal{A}^{a_1 \ldots a_N}(k_1 \ldots k_N) = \lim_{k_n^2 \to m_\pi^2} \int D\pi^a \int DJ^a W[J] \exp\left(iS[\pi] + i\int d^4x \pi^a J^a\right) \times \prod_n^N \int d^4 x_n e^{ik_n x_n}(-\partial_{x_n}^2 - m_\pi^2)\pi^{a_n}(x_n). \tag{3.1}$$

Here $S[\pi]$ is the effective pion action and J is the source formed by the colliding nuclei; one has to integrate over the sources with some weight functional $W[J]$ which summarizes the dynamics of the collision. If the source J is in a sense "large", the functional integral can be evaluated in the saddle-point approximation, where the saddle-point pion field is the solution of the equations of motion,

$$\frac{\delta S[\pi]}{\delta \pi^a(x)} + J^a(x) = 0\,, \tag{3.2}$$

supplemented with the radiation condition at large distances, which at $m_\pi \to 0$ reads

$$\frac{\partial \pi^a}{\partial t} + \frac{\partial \pi^a}{\partial r} = 0\,. \tag{3.3}$$

Let us denote the solution of these equations as $\pi^a(\mathbf{r}, t)$. In the leading WKB approximation one replaces the pion fields everywhere in Eq. (3.1) by this saddle-point field. In the next approximation quantum fluctuations about the classical field $\pi^a(\mathbf{r}, t)$ should be also taken into account.

We use two assumptions:

1) the collision is axially-symmetrical;
2) the collision is isotopically-symmetrical.

The first assumption is well justified at least for central collisions with zero impact parameter. Without having a detailed dynamical vision of the collision processes it is hard to foresee how far in the impact parameter can one proceed with this assumption. The second would be an exact statement in case of zero isospin of colliding nuclei like $S+S$. The derivation below can be generalized to non-zero isospins of the nuclei using the projection method, however we do not think that there should be significant departures from our predictions for the correlations in the case of, say, Au + Au or Si + Ag collisions.

The first assumption means that the saddle-point pion radiation field can be sought in the axially-symmetrical "hedgehog" form:

$$\pi^a(\boldsymbol{r},t) = O^{ai}\left(\boldsymbol{n}_{\perp i}P_\perp(\rho,z,t) + \boldsymbol{n}_{\parallel i}P_\parallel(\rho,z,t)\right), \tag{3.4}$$

where $P_{\perp,\parallel}$ are functions of the distance ρ from the beam axis, of the distance z from the collision point and of time t; $\boldsymbol{n}_{\parallel(\perp)}$ are unit vectors parallel (perpendicular) to the beam axis and O^{ai} is an arbitrary 3×3 orthogonal matrix. (In principle, axial symmetry does not contradict higher harmonics in the transverse plane with $\boldsymbol{n}_\perp$ replaced by a unit vector with the components $(\cos m\phi,\ \sin m\phi)$ where ϕ is the azimuthal angle and m is an integer. We shall work with $m=1$, *i.e.* assume that $\boldsymbol{n}_\perp$ points in the radial direction, and introduce the arbitrariness in the choice of m only in our final results). The type of correlations between space and isospin degrees of freedom is a familiar feature of solutions of nonlinear field equations as illustrated for example by the Skyrme model. The second assumption means that the saddle point is degenerate in the global isospin rotation O^{ai}, so that the functional integral in Eq. (3.1) reduces to the integration over all possible orientations of the pion field in the isospin space. The action and the source-weight functionals are taken at the saddle-point values of J^a and π^a and provide an overall normalization factor $\sqrt{\mathcal{N}}$ which may be a function of the total 4-momentum of the pions P_μ, but now we are not interested in this factor.

At large distances/time the isospin source J^a dies out and one can also neglect the non-linearity of the pion effective action. Therefore, Eq. (3.2) reduces to the free Klein–Gordon equation at large distances, and we are guaranteed that the Fourier transform of Eq. (3.4) has a pole at $k_0^2 - k_z^2 - \boldsymbol{k}_\perp^2 = m_\pi^2$, which cancels out in the LSZ leg amputation procedure (see Eq. (3.1)). We have therefore:

$$\lim_{k^2\to m_\pi^2}\int d^4x e^{ikx}(-\partial_x^2 - m_\pi^2)\pi^a(x) =$$
$$O^{ai}\left(k_{\perp i}F_\perp(k_\perp, k_\|) + k_{\| i}F_\|(k_\perp, k_\|)\right) \equiv O^{ai}F_i(k)\,, \quad (3.5)$$

where $F_{\perp,\|}(k_\perp, k_\|)$ are related through Fourier transformation to $P_{\perp,\|}$ of Eq. (3.4). Since $P_{\perp,\|}$ are real the functions $F_i(k)$ are purely imaginary, with $F_i^\star(k) = F_i(-k) = -F_i(k)$.

Squaring the amplitudes, summing over the isospin $a = 1,2,3$ of the pions and multiplying by the phase space factor, one gets for the N pion production cross-section:

$$\sigma^{(N)} = \frac{\mathcal{N}}{N!}\int dO_1 dO_2 \prod_{n=1}^{N}\int \frac{d^4k_n}{(2\pi)^4}2\pi\delta_+(k_n^2 - m_\pi^2)(2\pi)^4\delta^{(4)}(P_\mu - \Sigma k_{n\mu})$$
$$\times\, O_1^{ai}F_i(k)O_2^{aj}F_j^\star(k), \quad \delta_+(k_n^2 - m_\pi^2) = \delta(k_n^2 - m_\pi^2)\theta(k_0)\,, \quad (3.6)$$

where $k_{n\mu}$ are individual 4-momenta of the produced pions, $O_{1,2}$ are isospin orientation of the pions in the amplitude and the conjugate amplitude, respectively, P is the total 4-momentum of the produced pions; the factorial accounts for the identical particles.

Writing the 4-momentum conservation restriction as

$$(2\pi)^4\delta^{(4)}(P - \sum k_n) = \int d^4R \exp\left(i(P\cdot R) - i\sum(k_n\cdot R)\right)\,, \quad (3.7)$$

we get factorized integrals over the momenta of produced pions:

$$\int \frac{d^4k}{(2\pi)^4}2\pi\delta_+(k^2 - m_\pi^2)e^{-i(k\cdot R)}F_i(k)F_j^\star(k) = \mathcal{F}_{ij}(R_0, \boldsymbol{R})\,. \quad (3.8)$$

This tensor is further on contracted with the *relative* orientation matrix $O_{12}^{ij} \equiv O_1^{ai}O_2^{aj} = (O_1^T O_2)^{ij}$. We note that in integrating over the SO(3) rotations one can use the Haar measure property, $dO = d(CO) = d(OC)$, where C is an arbitrary orthogonal matrix, so that

$$\iint dO_1 dO_2 = \iint dO_1 dO_{12}\,, \qquad \int dO = 1\,. \quad (3.9)$$

The total cross-section being a sum of $\sigma^{(N)}$ over N becomes thus a series for an exponent, and we get

$$\sigma^{\rm tot} = \mathcal{N}\int dO_{12}\int d^4R \exp\left[i(P\cdot R) + \mathcal{F}_{ij}(R_0, \boldsymbol{R})O_{12}^{ij}\right]\,. \quad (3.10)$$

Let us use the frame in which the total momentum of the produced pions is zero, $\boldsymbol{P} = 0$, $P_0 = E$ where E is the total energy of the pions. Since both E and the function $\mathcal{F}$, proportional to the probability of the pion production, are presumably large, one can integrate over R_0, $\boldsymbol{R}$ and O_{12} by the saddle-point method. Let us parametrize the relative orientation matrix O_{12} in terms of a unit 4-vector u_μ $\left(u_\mu^2 = 1, \tau_\mu = (1, -i\tau)\right)$,

$$O_{12}^{ij} = \tfrac{1}{2}\,\mathrm{Tr}(u_\mu \tau_\mu \tau^i u_\nu \tau_\nu^+ \tau^j) = (1 - 2\mathbf{u}^2)\delta ij + 2u_i u_j + 2u_0 u_k \epsilon_{ijk}\,. \quad (3.11)$$

We expect the saddle point to be at

$$\boldsymbol{u} \approx 0\,, \quad \boldsymbol{R} \approx 0\,, \quad R_0 \sim \frac{1}{E}\,. \quad (3.12)$$

Expanding the exponent in Eq. (3.10) around this point, and using the explicit form of $\mathcal{F}$ given by Eq. (3.8), it can be shown that at small R_0 the saddle-point condition is indeed satisfied. Without knowing the explicit form of the functions $F_i(\boldsymbol{k})$ we cannot prove that there are no other saddle-points but we shall disregard such possibilities. Note that the maximum at $\boldsymbol{u} = 0$ corresponds to $O_{12}^{ij} = \delta^{ij}$, *i.e.* to the case when the pion isospin orientation in the conjugate amplitude is the same as in the direct amplitude — not an unnatural result.

We thus write the total cross-section as

$$\sigma^{\mathrm{tot}} \approx \mathcal{N} \int dR_0 \exp\left[i(ER_0) + \mathcal{F}_{ii}(R_0, \mathbf{0})\right]. \quad (3.13)$$

In what follows we shall use this relation to remove the unknown normalization factor $\mathcal{N}$.

We next turn to the 1,2,... particle *inclusive* cross-sections. All of them are directly derived from Eq. (3.6) where one skips integration over 1,2,... momenta and summation over the 1,2,... isotopic subscripts. Thus, the 1-particle inclusive cross-section is given by (we use the abbreviation $(d\boldsymbol{k}) = d^3\boldsymbol{k}/2k_0(2\pi)^3$):

$$\frac{d\sigma^{(A)}}{(d\boldsymbol{k})} = u_b^{(A)} u_{b'}^{\star(A)} \int dO_1 dO_2 O_1^{bi} O_2^{b'j} F_i(\boldsymbol{k}) F_j^\star(\boldsymbol{k})$$

$$\times \mathcal{N} \sum_{N=1}^{\infty} \frac{1}{(N-1)!} \prod_n^{N-1} \int (d\boldsymbol{k}_n) \left(F(\boldsymbol{k}_n) O_{12} F^\star(\boldsymbol{k}_n)\right) (2\pi)^4 \delta^{(4)}(P - k - \sum k_n). \quad (3.14)$$

Here $u_b^{(A)}$ are the isospin "polarization" vectors:

$$u_b^{(\pi^0)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_b , \quad u_b^{(\pi^+)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}_b , \quad u_b^{(\pi^-)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}_b , \qquad (3.15)$$

to be contracted with the isospin orientation matrices $O_{1,2}$. The superscript A refers to the isospin component of the observed pion; there is no summation in A.

The 2-particle inclusive cross-section for production of the pion of sort A_1 with momentum $\boldsymbol{k}_1$ and of the pion of sort A_2 with momentum $\boldsymbol{k}_2$ is

$$\frac{d\sigma^{(A_1A_2)}}{(d\boldsymbol{k}_1)(d\boldsymbol{k}_2)} = u_b^{(A_1)} u_{b'}^{*(A_1)} u_c^{(A_2)} u_{c'}^{*(A_2)}$$
$$\times \int dO_1 dO_2 O_1^{bi} O_2^{b'j} F_i(\boldsymbol{k}_1) F_j^*(\boldsymbol{k}_1) O_1^{ck} O_2^{c'l} F_k(\boldsymbol{k}_2) F_l^*(\boldsymbol{k}_2) \mathcal{N} \sum_{N=2}^{\infty} \frac{1}{(N-2)!}$$
$$\times \prod_n^{N-2} \int (d\boldsymbol{k}_n) \left(F(\boldsymbol{k}_n) O_{12} F^*(\boldsymbol{k}_n)\right) (2\pi)^4 \delta^{(4)}(P - k_1 - k_2 - \sum k_n) , \qquad (3.16)$$

and so on. Writing the 4-momentum conservation δ function with the help of an auxiliary integral as in Eq. (3.7), we again obtain the exponential series, so that

$$\frac{d\sigma^{(A)}}{(d\boldsymbol{k})} = u_b^{(A)} u_{b'}^{*(A)} \int dO_1 dO_{12} O_1^{bi} O_1^{b'k} O_{12}^{kj} F_i(\boldsymbol{k}) F_j^*(\boldsymbol{k})$$
$$\times \mathcal{N} \int d^4 R \exp\left[i(P \cdot R) - i(k \cdot R) + \mathcal{F}_{ij}(R) O_{12}^{ij}\right] , \qquad (3.17)$$

where we used $O_2^{b'j} = O_1^{b'k} O_{12}^{kj}$. If the momentum of the observed pion is negligible as compared to the total momentum P of the pions and if the integration over O_{12} and R is performed about the presumably steep saddle point given by Eq. (3.12), we obtain:

$$\frac{d\sigma^{(A)}}{(d\boldsymbol{k})} \simeq u_b^{(A)} u_{b'}^{*(A)} \int dO_1 O_1^{bi} O_1^{b'j} F_i(\boldsymbol{k}) F_j^*(\boldsymbol{k}) \sigma^{\text{tot}}$$
$$= u_b^{(A)} u_{b'}^{*(A)} \tfrac{1}{3} \delta^{bb'} \delta^{ij} F_i(\boldsymbol{k}) F_j^*(\boldsymbol{k}) \sigma^{\text{tot}} = \tfrac{1}{3} F_i(\boldsymbol{k}) F_i^*(\boldsymbol{k}) \sigma^{\text{tot}} . \qquad (3.18)$$

We see that the inclusive cross-section is directly related to the square of the Fourier transform of the classical pion radiation field, — a most natural result. Also naturally, we find identical cross-sections for π^+, π^- and π^0 production. The average multiplicity is obtained by integrating the inclusive cross-section over $\boldsymbol{k}$ and summing over $A = \pi^+, \pi^-, \pi^0$:

$$\langle N \rangle = \int (d\boldsymbol{k}) F_i(\boldsymbol{k}) F_i^\star(\boldsymbol{k}) = \mathcal{F}_{ii}(R_\mu = 0)\,. \tag{3.19}$$

It should be kept in mind that if this integral is not convergent by itself at large $\boldsymbol{k}$ it should be cut at least at P in accordance with the more precise Eq. (3.17).

In Eq. (3.17) we used the following formula for averaging over isospin rotations:

$$\int dO O^{bi} O^{b'k} = \tfrac{1}{3} \delta^{bi} \delta^{b'k}\,. \tag{3.20}$$

To calculate the 2-particle inclusive cross-section we need a formula for averaging over four matrices:

$$\begin{aligned} \int dO O^{bi} O^{b'j} O^{ck} O^{c'l} = {} & \tfrac{1}{30} \delta^{ij} \delta^{kl} (4\delta^{bb'} \delta^{cc'} - \delta^{bc} \delta^{b'c'} - \delta^{bc'} \delta^{b'c}) \\ & + \tfrac{1}{30} \delta^{ik} \delta^{jl} (-\delta^{bb'} \delta^{cc'} + 4\delta^{bc} \delta^{b'c'} - \delta^{bc'} \delta^{b'c}) \\ & + \tfrac{1}{30} \delta^{ik} \delta^{jl} (-\delta^{bb'} \delta^{cc'} - \delta^{bc} \delta^{b'c'} + 4\delta^{bc'} \delta^{b'c})\,. \end{aligned} \tag{3.21}$$

One can check this formula by applying various contractions and reducing it to Eq. (3.20); an alternative method is to note that O^{bi} is a Wigner D function for isospin 1, and using the Clebsch–Gordan machinery.

Starting from Eq. (3.16) and repeating the same steps as above we get for the 2-particle inclusive cross-section:

$$\begin{aligned} & \frac{d\sigma^{A_1 A_2}}{(d\boldsymbol{k}_1)(d\boldsymbol{k}_2)} = u_b^{(A_1)} u_{b'}^{\star(A_1)} u_c^{(A_2)} u_{c'}^{\star(A_2)} \\ & \times \tfrac{1}{30} \left[2\delta^{bb'} \delta^{cc'} (2V - W) + (\delta^{bc} \delta^{b'c'} + \delta^{bc'} \delta^{b'c})(-V + 3W) \right] \sigma^{\text{tot}} \\ & \qquad = \frac{\sigma^{\text{tot}}}{30} \begin{cases} 3V + W \\ 4V - 2W \\ 2V + 4W \end{cases} = \begin{cases} \sigma^{(1)} \\ \sigma^{(2)} \\ \sigma^{(3)} \end{cases}, \end{aligned} \tag{3.22}$$

where we have introduced the abbreviation:

$$V = |F_i(\mathbf{k}_1)|^2 |F_j(\mathbf{k}_2)|^2, \qquad W = |F_i(\mathbf{k}_1) F_i(\mathbf{k}_2)|^2 . \tag{3.23}$$

The upper line in Eq. (3.22) (case 1) refers to the $\pi^+\pi^-, \pi^-\pi^+, \pi^+\pi^+$ or $\pi^-\pi^-$ production, the second line (case 2) refers to four other combinations, $\pi^+\pi^0, \pi^-\pi^0$, $\pi^0\pi^+, \pi^0\pi^-$, while the last line (case 3) corresponds to the $\pi^0\pi^0$ production. It should be noted that, if the two observed pions are not identical, they are distinguished by the momenta $\mathbf{k}_1, \mathbf{k}_2$. The three possibilities in Eq. (3.22) reflect three possible isospin states ($T = 0, 1, 2$) which can be formed by a pair of pions [1]. Since, however, the three cross-sections are expressed through only two functions, we get a relation:

$$\sigma^{(2)} + \sigma^{(3)} = 2\sigma^{(1)} . \tag{3.24}$$

Let us now investigate Eq. (3.22). If one sums up all 9 possible combinations of pion pairs, one gets

$$\frac{d\sigma^{\rm all}}{(d\mathbf{k}_1)(d\mathbf{k}_2)} = |F_i(\mathbf{k}_1)|^2 |F_j(\mathbf{k}_2)|^2 \sigma^{\rm tot} \tag{3.25}$$

which is independent of the angle between the two pions. Further on, integrating Eq. (3.25) over the momenta $\mathbf{k}_{1,2}$ and recalling Eq. (3.19) for the average multiplicity, one finds

$$\langle N(N-1)\rangle = \frac{1}{\sigma^{\rm tot}} \int (d\mathbf{k}_1)(d\mathbf{k}_2) \frac{d\sigma^{\rm all}}{(d\mathbf{k}_1)(d\mathbf{k}_2)} = \langle N \rangle^2 , \tag{3.26}$$

which is the dispersion law of the Poisson distribution. It should be stressed though that for charged or neutral pions separately there is no Poisson distribution! Also, we would expect a deviation from the Poisson distribution at the end point of the spectrum, where the 4-momentum conservation law from a more accurate Eq. (3.16) imposes additional correlations.

The structure denoted as W depends on the azimuthal angle between the two pions,

$$W = \left[\cos(\phi_1 - \phi_2) k_1^\perp k_2^\perp F_\perp(k_1) F_\perp(k_2) + k_1^\parallel k_2^\parallel F_\parallel(k_1) F_\parallel(k_2)\right]^2 , \tag{3.27}$$

while

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[1] I would like to thank M.Polyakov and M.Praszałowicz who helped me to check Eq. (3.22)

$$V = \left[(k_1^\perp)^2 F_\perp^2(k_1)) + (k_1^\parallel)^2 F_\parallel^2(k_1)\right] \left[(k_2^\perp)^2 F_\perp^2(k_2)) + (k_2^\parallel)^2 F_\parallel^2(k_2)\right] . \tag{3.28}$$

Out of the three cross-sections mentioned in Eq. (3.22) one can construct angle-dependent and -independent combinations:

$$2\sigma^{(3)} - \sigma^{(2)} = 4\sigma^{(1)} - 3\sigma^{(2)} = \frac{\sigma^{\rm tot}}{3} W(k_1, k_2) ,$$
$$2\sigma^{(1)} + \sigma^{(2)} = 2\sigma^{(2)} + \sigma^{(3)} = \frac{\sigma^{\rm tot}}{3} V(k_1, k_2) . \tag{3.29}$$

Eqs (3.22)–(3.29) summarize our result for double-inclusive pion production. However, it might be useful to make a prediction which is independent of the dynamics hidden in the Fourier-transformed pion fields $F_{\perp,\parallel}$. To this end we restrict ourselves to pions with zero rapidity, *i.e.* to the case $k_1^\parallel = k_2^\parallel = 0$, so that $W/V = \cos^2(\phi_1 - \phi_2)$. This quantity is obtained by taking the ratio of differential double-inclusive cross-sections, say,

$$\frac{4\sigma^{(1)} - 3\sigma^{(2)}}{2\sigma^{(2)} + \sigma^{(3)}} = \cos^2(\phi_1 - \phi_2) . \tag{3.30}$$

Another way to isolate the azimuthal angle dependence is to normalize to the single-inclusive cross-sections. For example, we predict at $k_{1,2}^\parallel = 0$:

$$\left(\sigma^{\rm tot} \frac{d\sigma^{\pi^+\pi^-}}{(dk_1)(dk_2)} - \frac{9}{10} \frac{d\sigma^{\pi^+}}{(dk_1)} \frac{d\sigma^{\pi^-}}{(dk_2)}\right) \left(\frac{d\sigma^{\pi^+}}{(dk_1)} \frac{d\sigma^{\pi^-}}{(dk_2)}\right)^{-1} = \frac{3}{10} \cos^2(\phi_1 - \phi_2) . \tag{3.31}$$

At this point an experimentalist may derive correlations for his own favorite (charged or neutral) pairs of pions. Let us recall finally that the axial symmetry does not contradict higher harmonics in the transverse plane, with the replacement $\cos(\phi_1 - \phi_2) \to \cos m(\phi_1 - \phi_2)$ where m is an integer.

To conclude: we have investigated consequences of the hypothesis that pions are produced semiclassically in high-energy heavy-ion collisions. Literally, the above results for the isospin-azimuth correlations imply central collisions and zero isospin of the colliding nuclei. (The second requirement is technical, and can be avoided). We have also tacitly assumed that the source for pion production is coherent. From this angle it might be that proton-proton collisions are more "neat", rather than the heavy ion ones. Without knowing the detailed dynamics of the collisions it is hard to foresee how restrictive are these requirements.

To check the correlations one should analyze the double differential cross-sections of charged and/or neutral pion production at highest possible energies, together with a trigger for central collisions — like high multiplicity of the events, *etc.* Finding experimentally (or rejecting) the above correlations would contribute equally significantly to our understanding of multi-pion production at high energy collisions.

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