INSTABILITY OF THE CLASSICAL SPIN FIELD WITH HOPF INDEX*

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Summarization of our investigations of the stability of the three dimensional, classical spin field with Hopf index is presented. Example of a variational approach to functionals with topological terms is given. A nonnumerical method of analysis of a nonlinear evolution equations is presented.

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1. Variational approach to functionals with topological terms

Usual way of finding a conditional minimum of a functional is by means of Lagrange multipliers. Difficulty in applying this method to the case of functional with topological terms lies in the invariance of such terms under local modifications, as such invariant terms does not contribute to the final equations because its variational derivative vanishes.

A possible alternative to the usual variational calculation has been presented in [1] and further developed in [2]. The main idea of the new method is to find new coordinates, based on derivatives of original variables, which leads to nonlocal variations of the considered functional. The variational derivative of the topological term with respect to such variables does not vanish, and consequently, it is possible to derive equations for a conditional minimum of such a functional.

The described method was applied to the case of the two dimensional σ model and three dimensional Heisenberg model. Detailed calculations are showed in [2], here we will show just a sketch of the Heisenberg model calculations.

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2. Heisenberg model of the three dimensional spin field

The Heisenberg model is defined on the compactified space $\mathcal{R}^3$, which is isomorphic to the three dimensional sphere S^3. On this sphere we define a spin field as a map to the two dimensional sphere S^2:

$$S : S^3 \mapsto S^2 . \tag{1}$$

The homotopy group of such a map is $\Pi_3(S^2)$, which is isomorphic with the additive group of integer numbers Z. Thus, the map (1) may be classified using Hopf index as a class number [2–4]. Hopf index can be defined as:

$$q = -\int_{\mathcal{R}^3} d^3x \, A_\mu J^\mu , \tag{2}$$

where

$$J^\mu = \frac{1}{8\pi} \epsilon^{\mu\nu\lambda} \epsilon_{abc} S^a \partial_\nu S^b \partial_\lambda S^c \tag{3}$$

and A satisfies

$$J^\mu = \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda . \tag{4}$$

Hamiltonian of the map (1) has the form of the following integral:

$$H = \int_{\mathcal{R}^3} (\nabla S)^2 d^3x . \tag{5}$$

Using the method described in the first paragraph and new variables:

$$\omega = u + iv = \frac{S^x + iS^y}{1 + S^z} . \tag{6}$$

$$\begin{cases} S^x = \dfrac{2\,\mathrm{Re}(\omega)}{1 + \omega\omega^*} \\ S^y = \dfrac{2\,\mathrm{Im}(\omega)}{1 + \omega\omega^*} \\ S^z = \dfrac{1 - \omega\omega^*}{1 + \omega\omega^*} \end{cases} \tag{7}$$

$$\alpha_\mu = \frac{u_{,\mu}}{(1 + u^2 + v^2)}$$

$$\beta_\mu = \frac{v_{,\mu}}{(1 + u^2 + v^2)} ,$$

it is possible to rewrite formulas for the Hamiltonian and the Hopf index in terms of variables α_μ, β_μ:

$$H = \int_{\mathcal{R}^3} d^3x \, (\alpha_\mu \alpha^\mu + \beta_\mu \beta^\mu) \, , \tag{8}$$

$$q = -\frac{1}{\pi^2} \epsilon_{\mu\rho\sigma} \epsilon^{\mu\nu\lambda} \epsilon^{\rho\eta\xi} \int_{\mathcal{R}^3} d^3x \, x^\sigma \alpha_\nu(\boldsymbol{x}) \beta_\lambda(\boldsymbol{x}) \int_0^1 \alpha_\eta(t\boldsymbol{x}) \beta_\xi(t\boldsymbol{x}) t dt \, . \tag{9}$$

Finally we come [2] to the overdetermined set of equations for a configuration expressed in terms of variables α, β:

$$\begin{cases} \boldsymbol{\alpha} = -\frac{\lambda}{2\pi} (\boldsymbol{I} \times \boldsymbol{x}) \times \boldsymbol{\beta} \\ \boldsymbol{\beta} = \frac{\lambda}{2\pi} (\boldsymbol{I} \times \boldsymbol{x}) \times \boldsymbol{\alpha} \end{cases} , \tag{10}$$

where:

$$\boldsymbol{I} = \int_0^\infty [\boldsymbol{\alpha}(t\boldsymbol{x}) \times \boldsymbol{\beta}(t\boldsymbol{x})] \, t \, \mathrm{sgn}(1-t) dt \, . \tag{11}$$

This set of equations has only one solution:

$$\boldsymbol{\alpha} = \boldsymbol{\beta} = 0 \, , \tag{12}$$

with Hopf index equal to zero.

Above result means that there are no stationary configurations of spin field with nonvanishing Hopf index.

3. The Landau–Lifshitz equation

The result presented in the preceding section means that, if there exist configurations with nonvanishing Hopf index they must be time-dependent. Evolution of the spin field in the Heisenberg model is governed by the Landau–Lifshitz equation [5]. In stereographic coordinates (6) equation takes the following form:

$$\frac{\partial \omega}{\partial t} = i \left(\Delta \omega - \frac{2\omega^* (\nabla \omega)^2}{1 + \omega \omega^*} \right) . \tag{13}$$

This is a second order, nonlinear, partial differential equation for complex function of 3+1 variables and as such it is hardly easy to handle. There are several approaches to this type of equation. First is, of course, analytical solving, which is, being the most general, very difficult. Analytical methods have not succeeded in finding solutions of this equation so far. An alternative approach is use of numerical methods which have two main disadvantages for this type of problem. First, it is difficult to preserve topological properties (*i.e.* Hopf index) of the calculated solution. The other limitation is rather technical, as the problem is inherently three dimensional, it needs big amounts of computing power and memory to be solved numerically[1]. Yet another approach is use of perturbative methods. Perturbative methods work fine if we have some anchoring point, this is the case for $q = 0$, but this is not our case. We are interested in the case of $q \neq 0$ where we have no anchoring point.

In this situation we chose to use automated analytical calculations to find some clues to the shape of the solution.

4. Nonnumerical approach to the evolution equation [6]

The equation (13) may be rewritten, using definition of the derivative, in the approximate form:

$$\frac{\omega_{t+\Delta t} - \omega_t}{\Delta t} = i\left(\Delta\omega - \frac{2\omega^*(\nabla\omega)^2}{1+\omega\omega^*}\right) \tag{14}$$

and then converted to the recursive formula for the time evolution of the configuration:

$$\omega_{t+\Delta t} = \omega_t + i\Delta t\left(\Delta\omega_t - \frac{2\omega_t^*(\nabla\omega_t)^2}{1+\omega_t\omega_t^*}\right) . \tag{15}$$

This formula may be then iterated analytically in order to get the shape of the configuration in the analytical form. One can prove that this procedure preserves the Hopf index of the initial configuration.

As the starting configuration we have used simple map with Hopf index equal to one [3]:

$$\omega = \frac{\alpha + i\beta}{\gamma + i\delta}\,, \tag{16}$$

[1] One needs 128 megabytes of memory for a 256^3 lattice

where

$$\left\{\begin{aligned} \alpha &= \frac{2x}{1+x^2+y^2+z^2} \\ \beta &= \frac{2y}{1+x^2+y^2+z^2} \\ \gamma &= \frac{2z}{1+x^2+y^2+z^2} \\ \delta &= \frac{1-x^2-y^2-z^2}{1+x^2+y^2+z^2} \,. \end{aligned}\right. \tag{17}$$

Configurations with higher values of the Hopf index may be produced from (16) by taking appropriate power of the (16).

We were able to calculate two first iterations of the (15). The resulting formulas are very long and thus not suitable to quote here, instead we have produced plots of the energy density of the configuration. To further visualize considered field configuration we have produced ray traced "wire frame" model of some field lines of the starting configuration (16). This model is shown on the Fig. 1. Next five pictures show plots of energy density in the XZ plane[2] for configuration after one and two steps of iteration of the formula (15) with various values of time parameter.

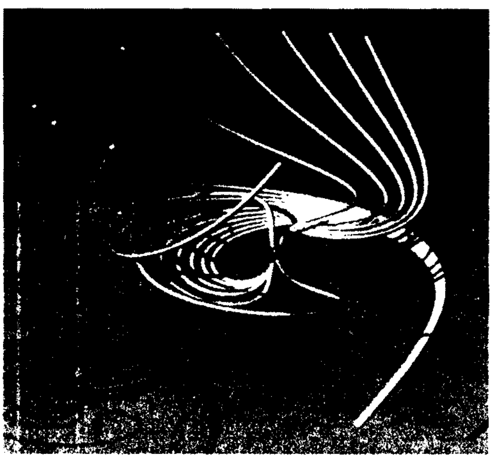

Fig. 1. Field lines of the Hopf configuration of spin field.

[2] Energy density has cylindrical symmetry.

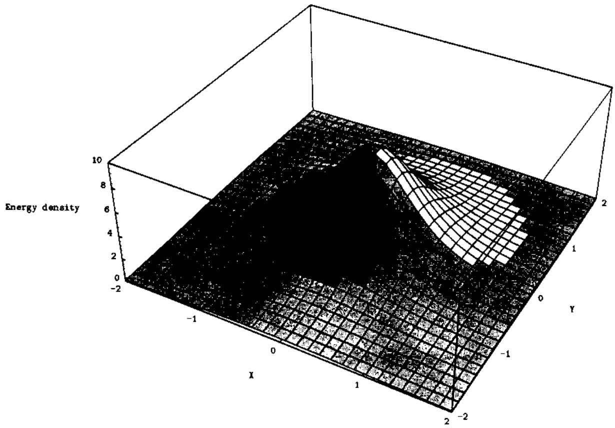

Fig. 2. Energy density of the Hopf configuration for $t = 0$.

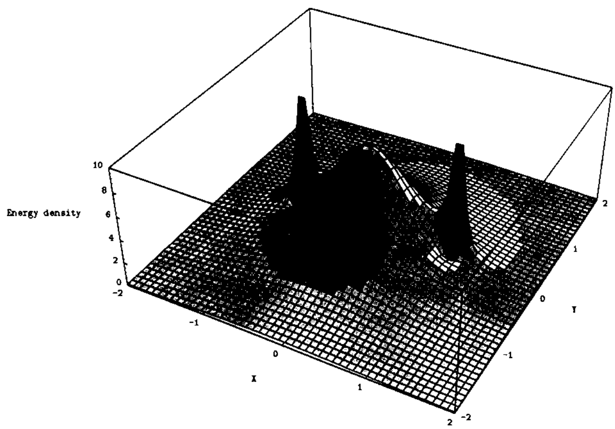

Fig. 3. Energy density of the Hopf configuration for $t = 0.1$ in the first step.

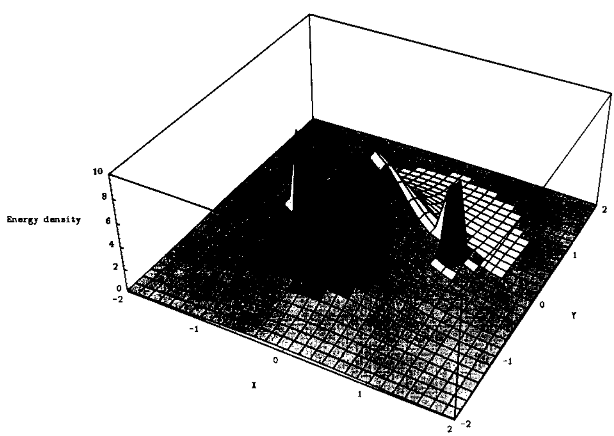

Fig. 4. Energy density of the Hopf configuration for $t = 0.1$ in the second step.

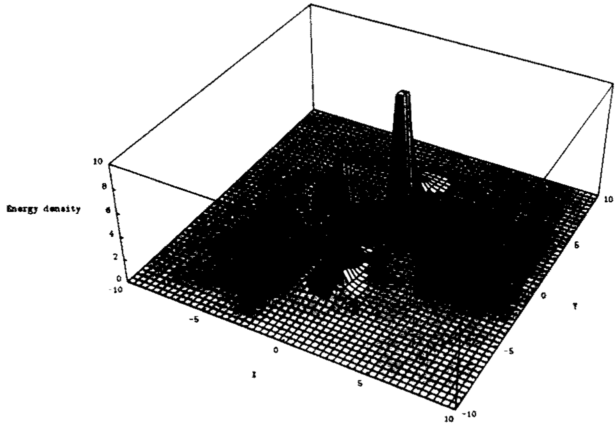

Fig. 5. Energy density of the Hopf configuration for $t = 200$ in the first step.

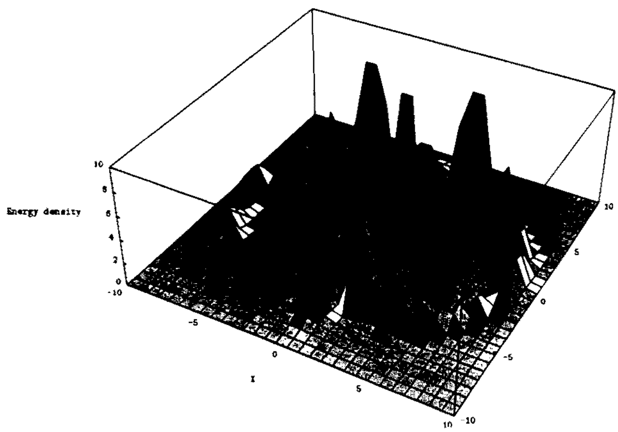

Fig. 6. Energy density of the Hopf configuration for $t = 200$ in the second step.

5. Conclusion

We have shown that the only possible configuration of the spin field in the Heisenberg model must be time dependent. We have also shown the method of nonnumerical analysis of nonlinear evolution equations. Qualitative results shown in this paper are similar to the solution of the similar problem for the evolution of the electromagnetic field [7]. Configuration from Figs 2–6 shows similar behavior to the one exhibited by "knotted solution" from [7]. Both solutions have regions of high energy density which spread in space with constant velocity. This similarity may indicate that this behavior is generated rather by geometry of the configuration then by the equation of motion.

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