EXTRACTING PHYSICS FROM LATTICE ARTEFACTS*

A. Di Giacomo

Dipartimento di Fisica dell'Universita' e INFN
Pisa, Italy

(Received November 4, 1993)

Critical slowing down of local algorithms in lattice gauge theories can be used to extract physics at different scales of length. We show that renor- malization constants of the topological susceptibility of QCD vacuum can be determined numerically, without any use of perturbation theory, by a heating and cooling technique based on the above mentioned principle. The comparison with perturbative computations is discussed.

PACS numbers: 12.38. Gc

1. Introduction

Gauge theories on a lattice were first formulated by Wilson in 1974 [1], and were part of a research strategy, trying to separate physics at different scales of length [2]. In 1979 Creutz [3] demonstrated by numerical simulations that continuum physics can be extracted from the lattice. Since then a number of results have been obtained on lattice by Monte Carlo simulations, in particular

(i) It has been established that lattice is a regulator of continuum QCD. At the fixed point $g^2 = 0$ lattice and continuum QCD both belong to the same class of universality.

(ii) Color is confined. A string tension exists between static $Q\bar{Q}$ pairs [3]. The string consists of chromoelectric flux tubes [4, 5], joining the two particles.

(iii) A deconfining phase transition takes place at $T_c \sim 160\,\mathrm{MeV}$ which is first order in a theory without light quarks (the so called quenched approximation). T_c is somewhat smaller and the order of the transition

* Presented at the XXXIII Cracow School of Theoretical Physics, Zakopane, Poland, June 1–11, 1993.

not yet established for the more realistic theory including the propagation of light quarks [6].

(iv) Chiral SU(3)⊗SU(3) is spontaneously broken to vector SU(3), and $\langle\bar{\psi}\psi\rangle$ is the order parameter. The symmetry is restored above the deconfinement temperature T_c [7].

(v) The breaking of axial $U_A(1)$ due to the chiral anomaly [8] can explain the high mass of the η' (Witten, Veneziano) [9, 10], thus solving the so called [11] $U_A(1)$ problem [12].

The result *(i)* means that numerical simulations on the lattice do represent the theory in its full dynamical content: every physical quantity can, in principle, be computed on lattice from first principles. The results *(ii)*-*(v)* and many others provide evidence that QCD does indeed describe the real world.

An immediate question is then: are present limitations in the results obtained from the lattice only due to computer power, or do they also reflect limitations in our understanding of theory?

I will show in the following that a crucial point is our poor understanding of non perturbative field theory. On the one hand what we know on continuum field theory is mainly based on perturbation theory, and as a consequence phenomena like confinement, vacuum condensation, dimensional transmutation, Gribov copies ... are out of reach. On the other hand extracting non perturbative effects from numerical simulations on lattice requires concepts like Wilson Operator Product Expansion (OPE) or renormalization theory which, strictly speaking, are only understood in perturbation theory. A number of non trivial questions in field theory are thus encountered in the process of extracting physics from the lattice.

There are two main attitudes in lattice community:

A. A "phenomenological" approach which consists in computing quantities of direct experimental interest [hadron spectrum, weak decay amplitudes, α_s from the spectra of heavy quarkonia ...]

B. A "theoretical" approach, which tries to get information on the mechanisms and on field theoretical aspects [mechanisms of colour confinement, structure of the ground state, Gribov copies, string configurations ...].

In this talk I will focus on topological aspects of QCD, a subject which is somehow in between A and B: it is related to the non perturbative structure of vacuum, to the mechanism of breaking of $U_A(1)$ and the role of instantons, but at the same time to problems of direct phenomenological relevance like the spin content of the nucleon. The main points of this lecture will be

a) A discussion of renormalization of the lattice regularized QCD. Lattice QCD is a cut-off version of the theory. What is provided by the numerical simulations are unrenormalized correlation functions, similar to what would be Pauli–Villars cut-off Green functions in QED. To get physics the cut-off must be removed by a suitable identification of multiplicative and additive renormalizations.

b) A systematic use of the so called "critical slowing down" in lattice simulations, to separate physics at different length scales in the spirit of Ref. [2]. Any local updating algorithm in Monte Carlo simulations rapidly brings to equilibrium short wavelength modes, with wavelength $\lambda \sim a$, a being the lattice spacing. Modes corresponding to longer wavelengths $\lambda \sim \xi a$ reach equilibrium at a much slower rate. It is easy to realize that the thermalization time t at a distance ξa is governed by a typical mechanism of diffusion [13]

$$t \sim \xi^{\alpha}\,, \qquad \alpha \simeq 2\,. \tag{1.1}$$

Since continuum is reached at the critical point, where the correlation length goes large with respect to lattice spacing ($\xi \gg 1$), the phenomenon is called critical slowing down, and represents an inconvenience of numerical simulations. We will instead make systematic use of it and of the renormalizations as described above to extract physics. In a sense, by the use of Eq. (1.1) we will separate distances in terms of computer time.

Before doing that I will present the very basic elements of lattice gauge theories for the benefit of non experts [Sec. 2], as well as an outline of the physics involved [Sec. 3].

2. Lattice lexicon

Lattice is a discretization of space-time to a cubic array of points Fig. 1. The building block of the theory is the link $U_\mu(n)$ [1]

$$U_\mu(n) = \exp\left[igaA_\mu(n)\right]\,, \tag{2.1}$$

$$A_\mu(n) = \sum_a T^a A^a_\mu(n)\,. \tag{2.2}$$

The link is the parallel transport from the site n to the neighbouring site in direction μ. g is the bare coupling constant, T^a are the generators of the gauge group in the fundamental representation. A gauge transformation is defined as a set of group elements $\Omega(n)$, one for each site, and the gauge transformed of $U_\mu(n)$ is

$$U_\mu(n) \to \Omega(n) U_\mu(n) \Omega^\dagger(n+\mu)\,. \tag{2.3}$$

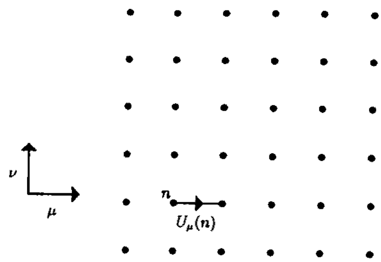

Fig. 1.

Continuous paths can be formed with links, *e.g.* the plaquette $\Pi_{\mu\nu}(m)$, as shown in Fig. 1: a path is defined as the ordered product of the links.

$$\Pi_{\mu\nu}(m) = U_\mu(m)U_\nu(m+\mu)U_\mu^\dagger(m+\nu)U_\nu^\dagger(m) . \tag{2.4}$$

The trace of a closed path is gauge invariant.

As $a \to 0$

$$\Pi_{\mu\nu}(m) \simeq 1 + iga^2 G_{\mu\nu}(m) - \tfrac{1}{2} g^2 a^4 G_{\mu\nu}G_{\mu\nu} + \dots , \tag{2.5}$$

$G_{\mu\nu}$ being the field strength.

Wilson (euclidean) action is

$$S_W[U] = -\beta \sum_{n,\mu<\nu} \mathrm{Tr}\,[1 - \Pi^{\mu\nu}(n)] \underset{a\to 0}{\simeq} a^4 \sum_{n,\mu,\nu,a} \tfrac{1}{4} G^a_{\mu\nu}(n) G^a_{\mu\nu}(n) + \mathcal{O}(a^6) , \tag{2.6}$$

$\beta = 2N/g^2$ for a gauge group SU(N) plays the role of Boltzman factor $1/kT$. The theory has a fixed point at $\beta \to \infty$, corresponding to a higher order phase transition in the language of statistical mechanics. As $\beta \to \infty$ the correlation length tends to infinity in units of lattice spacing a. In formulae

$$a = \frac{1}{\Lambda_L} f(\beta) ,$$

$$f(\beta) \underset{\beta\to\infty}{\simeq} \left(\frac{\beta}{2Nb_0}\right)^{b_1^2/2b_0} \exp\left(-\frac{\beta}{4Nb_0}\right) \left\{1 + \mathcal{O}\left(\frac{1}{\beta}\right)\right\} , \tag{2.7}$$

here b_0 and b_1 are the first two non trivial coefficients of the perturbative expansion of the β function

$$b_0 = \tfrac{1}{3} \frac{11 - N_f}{(4\pi)^2} , \qquad b_1 = \tfrac{1}{3} \frac{102 - 38N_f}{(4\pi)^4} . \tag{2.8}$$

The sign of b_0 reflects asymptotic freedom and is what makes the lattice spacing exponentially small in physical units at the fixed point $\beta \to \infty$, or makes one Fermi much larger than a. Λ_L is the physical scale of the regularization scheme. The generating functional (partition function in the language of statistical mechanics) is

$$W = \int \Pi_{\mu,n} dU_\mu(n) \exp\left[-S_W(U) - S_Q(U)\right] , \tag{2.9}$$

$$e^{-S_Q(U)} = \int d\bar{\psi}(n) d\psi(n) \exp\left[-\sum_{n,m} \bar{\psi}(n)\mathcal{D}_{nm}\psi(m)\right] , \tag{2.10}$$

$$-S_Q = \mathrm{Tr}\log\mathcal{D} ,$$

is the action due to quarks. dU_μ is the Haar measure of the group. W is gauge invariant and finite: lattice discretization makes W a ordinary integral from a functional integral. This integral is computed by Monte Carlo method. Notice that

(i) In the process preparing a significant sample of configurations by a local Monte Carlo updating procedure, short wavelengths are thermalized rapidly, while the computer time for bringing to equilibrium modes at wavelength ξa is longer and roughly proportional to ξ^2[Eq. (1.1)] (critical slowing down). Local minima of the action, which correspond to classical solutions of the equations of motion, as are the instantons, can produce metastabilities which make the thermalization time even longer.

(ii) W is the generating functional of the regularized Green functions. They have then to be renormalized.

3. Topology in QCD

3.1. The $U_A(1)$ problem

The celebrated current algebra of Gell-Mann [14] was abstracted from the massless free quark model: the idea of Gell-Mann was that, whatever the strong interaction lagrangian, it should preserve that basic symmetry.

QCD, which was proposed ten years later as the theory of strong interactions, does in fact preserve SU(3)⊗SU(3) chiral symmetry.

$$\partial^\mu V^a_\mu(x) = 0 , \quad V^a_\mu = \bar{\psi}(x)\gamma_\mu \frac{\lambda^a}{2}\psi(x) ,$$

$$\partial^\mu A^a_\mu(x) = 0 , \quad A^a_\mu = \bar{\psi}(x)\gamma_\mu\gamma^5 \frac{\lambda^a}{2}\psi(x) , \tag{3.1}$$

λ^a are the SU(3) flavour generators. As required by phenomenology vector symmetry is realized a la Wigner–Weyl, whilst axial symmetry is realized a la Nambu–Goldstone, *i.e.* it is spontaneously broken. Lattice simulations demonstrate that, from first principles [7]. Vector U(1) is also a symmetry in QCD, like in free quark model and corresponds in nature to baryon number conservation.

$$\partial_\mu V^\mu(x) = 0\,, \qquad V^\mu = \bar{\psi}\gamma^\mu\psi\,. \tag{3.2}$$

The axial $U_A(1)$ current is conserved in Gell-Mann's free massless quark model

$$\partial_\mu A^\mu(x) = 0\,, \qquad A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi\,, \tag{3.3}$$

but $U_A(1)$ is broken by anomaly in QCD, where

$$\partial_\mu A^\mu(x) = N_f Q(x)\,. \tag{3.4}$$

N_f is the number of light quark species ($N_f = 3$) and

$$Q(x) = \frac{g^2}{64\pi^2} G^{\mu\nu} G^*_{\mu\nu}\,, \tag{3.5}$$

is the so called topological charge density. This fact opens a possibility of solving the so called $U_A(1)$ problem [11]. $U_A(1)$ must be broken in nature: if it were not broken its generator Q_A would commute with the Hamiltonian of strong interactions. As a consequence on a proton at rest $|p\rangle$

$$HQ_A|p\rangle = Q_A H|p\rangle = mQ_A|p\rangle\,, \tag{3.6}$$

implying that the state $Q_A|p\rangle$, which has opposite parity to the proton, would have the same mass: in nature there is no evidence for parity doublets. If $U_A(1)$ were spontaneously broken, the mass of the corresponding Goldstone boson, η', should obey the inequality [11] $m_{\eta'} \leq 3m_\pi$, which is again not the case in nature. The appearance of the anomaly (3.4) in QCD, as contrasted with the symmetry of the free quark model, eliminates the contradiction. In fact, it was shown by use of $1/N_c$ expansion [9, 10] that the η' mass can be explained if

$$\chi \frac{2N_f}{f_\pi^2} = m_{\eta'}^2 + m_\eta^2 - 2m_K^2\,, \tag{3.7}$$

χ is the topological susceptibility of the ground state of a pure gluon gauge theory; N_f is the number of light flavours, f_π the π decay constant. Inserting numbers in Eq. (3.7) gives

$$\chi = (180\,\text{MeV})^4\,. \tag{3.7$'$}$$

The assumption

$$\frac{1}{\chi} m_{\eta'}^2 \left|\frac{d\chi}{dq^2}\right|_{q^2=0} \ll 1\,, \tag{3.8}$$

is implied in the derivation [10]. By χ we mean $\chi(q^2 = 0)$ and

$$\chi(q^2) = \int d^4x e^{iqx} \langle 0|\mathrm{T}\left(Q(x)Q(0)\right)|0\rangle\,. \tag{3.9}$$

The idea of Refs [9, 10] is that in the limit $N_c \to \infty$ $\mathrm{U_A}(1)$ is a symmetry, and is spontaneously broken: $\langle\bar{\psi}\psi\rangle$ is the corresponding order parameter. The mass of the η' is produced by non leading corrections in $1/N_c$ which displace the pole of the Goldstone boson. In a pure gauge theory one should observe a restoration of the $\mathrm{U_A}(1)$ symmetry above a certain temperature. The obvious question is if that temperature is the same at which chiral SU(3)⊗SU(3) is restored, which, in turn, seems to coincide with the deconfining phase transition [see Sec. 1]

3.2. The spin content of the proton

The general parametrization of the matrix elements between proton states $|\vec{p}\rangle$ and $|\vec{p}'\rangle$ of the $\mathrm{U_A}(1)$ axial current (3.3) is

$$\langle\vec{p}'|A^\mu(0)|\vec{p}\rangle = G_A(q^2)\bar{u}(\vec{p}')\gamma^\mu\gamma^5 u(\vec{p}) + G_P(q^2)q^\mu\bar{u}(\vec{p}')\gamma^5 u(\vec{p})\,, \tag{3.10}$$

$q^2 = (p' - p)^2$ is the momentum transfer. Recent experiments [15] on deep inelastic scattering of polarized electrons on polarized protons, have measured $G_A(0)$, finding a small number, compatible with 0

$$G_A(0) = .120 \pm .094 \pm .138\,. \tag{3.11}$$

This result creates problems in the naive interpretation of the spin of the proton in terms of spins of the quarks (Spin crisis) (See *e.g.* Ref. [16]). Using Eq. (3.4) for the anomaly

$$\langle\vec{p}'|\partial_\mu A^\mu(0)|\vec{p}\rangle = \langle\vec{p}'|N_f Q(0)|\vec{p}\rangle \underset{q^2\to 0}{\simeq} 2M_p G_A(0)\bar{u}(\vec{p}')\gamma^5 u(\vec{p})\,. \tag{3.12}$$

A measurement of $G_A(0)$ from first principles, amounts then to measure the matrix element between proton states of the density of topological charge.

4. Topology on the lattice

How can topology be defined on the lattice, where continuity does not exist?

4.1. Geometrical approach [17, 18]

The idea is to interpolate discrete lattice configurations by continuous configurations on which the Chern number can be read: the hope is that at large β's, when the scale of strong interactions is large compared to lattice spacing, large instantons will dominate, and continuum will be reached. That hope turns out to be wrong [19, 20]: instantons of size a dominate at large β: they are called dislocations. No clear solution to this difficulty has been found.

4.2. Field theoretic approach [21]

A lattice version of $Q(x)$ is defined, *e.g.* as

$$Q_L(x) = -\frac{g^2}{32\pi^2} \sum_{\pm\mu\nu\rho\sigma} \varepsilon^{\mu\nu\rho\sigma} \mathrm{Tr}\left\{\Pi_{\mu\nu}(x)\Pi_{\rho\sigma}(x)\right\} \underset{a\to 0}{\simeq} Q(x)a^4 + \mathcal{O}(a^6) . \tag{4.1}$$

The limit $a \to 0$ above is the formal limit of the operator. Different Q_L's can be defined, which only differ by irrelevant $\mathcal{O}(a^6)$ or higher terms. Q_L is a cut-off version of $Q(x)$. The general theory of renormalization gives for the operator $Q_L(x)$ in the limit $\beta \to \infty$ when $a \to 0$ in physical units, or in the limit in which the cutoff is removed [22]

$$Q_L(x) \underset{\beta\to\infty}{\simeq} Z(\beta)Q(x)a^4(\beta) + \mathcal{O}(a^6) , \tag{4.2}$$

the rule being that any regularized operator mixes with all local operators with the same quantum numbers and equal or lower dimension (multiplicative and additive renormalizations), apart from irrelevant terms of higher dimension which vanish with higher powers of the cut-off. $Q(x)$ is a pseudoscalar of dimension 4, and no other gauge invariant operator of equal or lower dimension can be constructed out of the field operators with the same quantum numbers. The lattice topological susceptibility can be defined as

$$\chi_L = \sum_n \langle Q_L(n) Q_L(0)\rangle = \frac{\langle Q_L^2\rangle}{L^4} , \tag{4.3}$$

$$Q_L = \sum_n Q_L(n) . \tag{4.4}$$

Here again, by use of the general rules of renormalization theory [21]

$$\chi_L \underset{\beta\to\infty}{\simeq} Z^2(\beta)\chi a^4(\beta) + M_G(\beta)G_2 a^4(\beta) + M_\psi(\beta)G_\psi a^4(\beta) + P(\beta)\langle 0|I|0\rangle + \mathcal{O}(a^6)\,, \quad (4.5)$$

$Z(\beta)$ is the same multiplicative renormalization which appears in Eq. (4.2); the other terms are generated by the singularity occurring when the two points of the correlation function coincide.
G_2 is known as gluon condensate, and is given by

$$G_2 = \frac{\beta(g)}{g b_0}\frac{1}{4\pi^2}\langle G^a_{\mu\nu}(0)G^a_{\mu\nu}(0)\rangle\,, \quad (4.6)$$

G_ψ is the chiral condensate, which is present if fermions are included in the theory

$$G_\psi = \sum_f \langle m_f \bar{\psi}_f \psi_f\rangle\,. \quad (4.7)$$

The sum index f runs on light quarks.
I is the identity operator.

Eq. (4.5) must be correct if lattice QCD belongs to the same class of universality as continuum QCD. In principle $Z(\beta)$, $M_G(\beta)$, $M_\psi(\beta)$, $P(\beta)$ are well defined functions: they have to be determined to extract the physical quantities χ, G_2, G_ψ from the numerical determinations of $\chi_L(\beta)$.

Since renormalizations are dominated by short range fluctuations [they correspond to ultraviolet divergences in the limit of zero lattice spacing], one expects that they can be computed in perturbation theory, due to asymptotic freedom. Non perturbative effects are, however, expected to enter at some order [23].

A related problem is the validity of the perturbative expansion as an asymptotic series [24]. All these problems can be investigated on lattice if these renormalization functions can be determined directly from numerical simulations and are then compared with the perturbative expansion.

The status of the perturbative calculation of different renormalization functions both in QCD and in $2d$ $O(3)\sigma$ model is presented in Table I.

TABLE I

	SU(2)	SU(3)	O(3) σ model
$Z(\beta) = 1 + \frac{z_1}{\beta} + \frac{z_2}{\beta^2} + \ldots$	$z_1 = -2.1448$ [a]	$z_1 = -5.4$ [a]	$z_1 = -.6839$ $z_2 = -.0598$ [b]
$M_\psi(\beta) = \frac{d_4}{\beta^4} + \ldots$		$d_4 = 1.5875\,10^{-4}$ [c]	
$M_G(\beta) = \frac{e_2}{\beta^2} + \ldots$	$e_2 = 1.874\,10^{-3}$ [d]	$e_2 = 6.3226\,10^{-3}$ [d]	
$P(\beta) = \frac{c_3}{\beta^3} + \frac{c_4}{\beta^4} + \ldots$	$c_3 = 2.648\,10^{-4}$ [e] $c_4 = .70\,10^{-4}$ [f]	$c_3 = 3.5759\,10^{-3}$ [e] $c_4 = 8.4193\,10^{-4}$ [e] $c_4^Q = 6.1923\,10^{-4}$ [c]	 $c_4 = 6.832\,10^{-5}$ $c_5 = 5.722\,10^{-5}$ [b]

[a] [22], [b] [40], [c] [27], [d] [21, 25], [e] [26], [f] [41]

The mixing to the chiral condensate is negligible [27], both because it is OZI forbidden, and because G_ψ is much less than G_2 due to the smallness of current quark masses. In what follows we will neglect it. According to the general discussion of Secs 1 and 2 (Eq. (1.1)) any local algorithm which freezes the system (like cooling [28] or smearing [29]) will rapidly thermalize to large β's local fluctuations (modes of wavelength $\sim a$) while longer wavelength will take a longer time, and configurations which correspond to local minima of the action like the instantons, which are protected by topology, will be metastable and survive a much longer time. In formulae a β_{eff} will be produced for short range fluctuations along the cooling process, which becomes larger and larger as cooling goes on, whilst long range modes will for a while preserve the original β: in particular the topological charge will be preserved. Hence [see Eq. (4.5)]

$$\chi_L \simeq Z^2(\beta_{\text{eff}})\chi a^4 + M_G(\beta_{\text{eff}})G_2 a^4 + P(\beta_{\text{eff}})\,. \tag{4.8}$$

Now [21]

$$M_G(\beta_{\text{eff}}) \underset{\beta\to\infty}{\simeq} \mathcal{O}\left(\frac{1}{\beta^2_{\text{eff}}}\right), \qquad P(\beta_{\text{eff}}) \underset{\beta\to\infty}{\simeq} \mathcal{O}\left(\frac{1}{\beta^3_{\text{eff}}}\right),$$

so after a few cooling steps the corresponding terms rapidly go to zero, and $Z^2(\beta)$ can be measured and compared to the perturbative computations. After a few steps $Z^2(\beta) = 1$ and

$$\chi_L \underset{\beta\to\infty}{\simeq} \chi a^4\,,$$

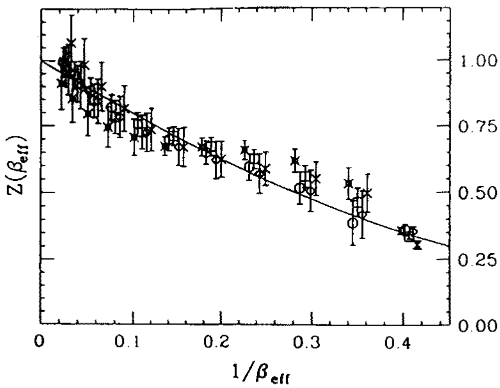

Fig. 2. $Z(\beta)$ **determined by cooling.**

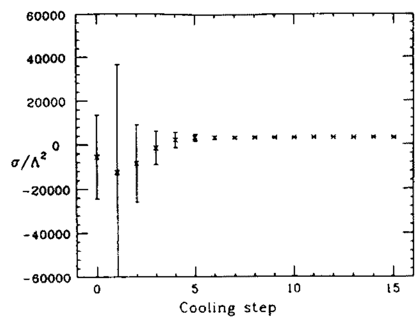

Fig. 3. The SU(2) string tension σ as extracted from $\chi(7,7)$ along cooling, ($\beta = 2.5$). Statistical errors are strongly reduced as short range fluctuations are eliminated. Data correspond to a sample of 50 configurations [4].

χ can then be extracted [21, 32]. Fig. 2 shows how $Z \to 1$ along cooling. β_{eff} is determined by using the density of action as a thermometer for local fluctuations. This method of freezing local fluctuations, while preserving long range phenomena can be used for more general purposes than the original cooling procedure, [28], which was limited to deep freezing of topologically protected quantities. The string tension σ can be measured by looking at Wilson loops of a size R, T along cooling [4] at a stage of cooling where short range fluctuations have already been frozen but distance R has not yet been reached in the sense of Eq. (1.1). The result will be a drastic reduction of fluctuations, allowing a good measurement of σ on a small sample of configurations [see Fig. 3–4]. At the same time field configurations produced by the propagating pair of quarks can be studied, polished of the local quantum fluctuation [4, 30]. By the same procedure gauge invariant correlations of field strengths at large distance can be measured [33] which are relevant for models of colour confinement based on stochastic

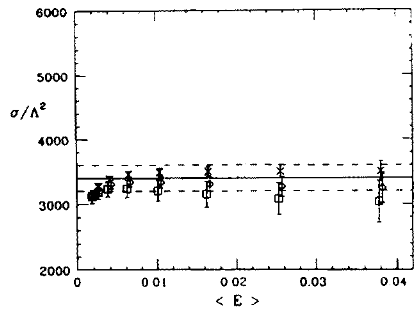

Fig. 4. σ obtained from $\chi(6,6)$ (crosses), $\chi(6,7)$ (diamonds), $\chi(7,7)$ (squares, same data as in Fig. 3) at $\beta = 2.5$, along cooling. They are plotted versus $\langle E \rangle$ (the internal energy density) [4]. $\langle E \rangle$ decreases with cooling. The right most point corresponds to cooling step 5, the second going to left to cooling step 6 and so on. The solid line (and the two dotted lines) show σ as measured without cooling and the corresponding error [31].

background fields [34, 35, 36].

5. Making use of Eq. (1.1) to determine numerically the renormalizations

5.1. Determination of $Z(\beta)$ [37]

The method, inspired by Ref. [39], has first been tested on $O(3)$ σ model [38]. An instanton is put on lattice by hand: the topological charge measured by any method is 1. The configuration is then thermalized to any given value of β, by a local algorithm, and Q_L is measured: the effect of thermalization will be to bring to equilibrium short wavelength modes rapidly, whilst long range correlations and the topology need a longer time to thermalize, according to the general discussion of Secs 1, 2. Since renormalization is produced by short range correlations, one expects that, after a number of steps which is independent of the physical correlation length, (*i.e.* of β), the operator Q_L will be renormalized and will reach a plateau.

$$Q_L \simeq Z(\beta) Q_L(0) . \tag{5.1}$$

By $Q_L(n)$ we denote the measured value of Q_L at the n-th step of cooling. This is in fact what happens [See Fig. 5]. The procedure can be repeated at different values of β and the result can be compared with the perturbative expansion:

$$Z = 1 + \frac{z_1}{\beta} + \frac{z_2}{\beta^2} , \tag{5.2}$$

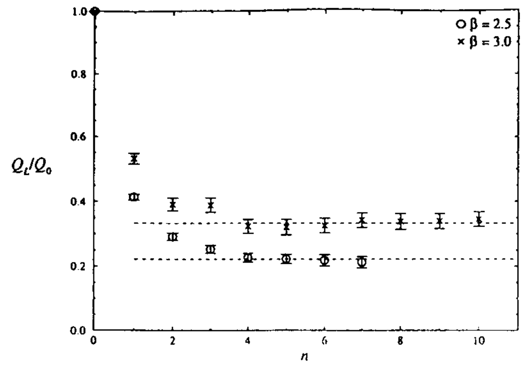

Fig. 5. Heating an instanton and measuring Q_L. Gauge group SU(2) [37].

z_1 is known by analytic computation [22]. Within our errors, which are typically of few percent, Z is consistent with an expansion up to terms β^{-2} like Eq. (5.2) and

$$\begin{cases} z_2 = .48 \pm .04 & \text{for SU(2)} \\ z_2 = 3.3 \pm 1.2 & \text{for SU(3)} \end{cases}$$

5.2. Determination of the mixing to G_2 and to the identity

The method was introduced in Refs [27] and [37]. We start from a flat (zero field) configuration, with all links $U(n) = 1$, and we thermalize at a given β with a local algorithm. We expect thermalization to set in through different stages.

(i) Local fluctuations, with $\lambda \sim a$ are rapidly thermalized, after a number of steps independent of β. A plateau will appear where the topological susceptibility will only get contribution from the mixing to the identity, $P(\beta)$ in Eq. (4.5), since long range modes giving rise to gluon condensation are not yet thermalized, and $\chi = 0$ because the number of instantons is still zero.

(ii) When, according to Eq. (1.1) wavelengths of the order of the physical correlation length are thermalized, a new plateau will appear, higher with respect to the first plateau by an amount equal to the mixing to G_2 in Eq. (4.5).

(iii) After a much longer time the equilibrium value for χ_L will eventually be reached.

This is shown in Fig. 6 for SU(2). From the position of the plateaux, $P(\beta)$, $M_G(\beta)a^4G_2$ and $Z^2\chi a^4$ can be determined. $P(\beta)$ is the exact mixing coefficient to the identity and can be compared with the first few terms of

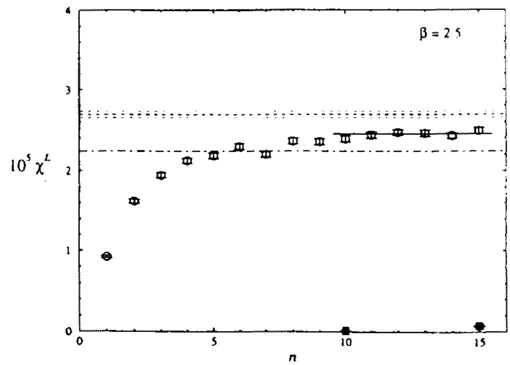

Fig. 6. Heating zero field configuration. The first plateau is $P(\beta)$; the second is $P(\beta) + M(\beta)\,G_2$. The upper line is the equilibrium value of χ_L, which is reached as $n \to \infty$. At $n = 10, 15$ a fast cooling is made to insure that no instantons have yet been produced by the heating process. Gauge group SU(2) [37].

the perturbative expansion. [See Table I.] A clear indication emerges that the perturbative expansion is an asymptotic expansion, and that, in the ranges of β of interest a few terms provide a good approximation, within the statistical errors.

From the mixing to G_2, assuming for $M(\beta)$ the first significant term of its perturbative expansion [21, 25] G_2/Λ_L^4 can be determined. It agrees with previous determinations [42, 43, 47], and, once Λ_L is eliminated, *e.g.* by comparison with σ/Λ^2, the result is about $\sim$ 3 larger than SVZ [44]. Finally χ/Λ^4 can be determined: it agrees with previous determinations [21, 28, 32], and with the value of Eqs (3.7), (3.7') thus confirming again the analysis of Refs [9,10]. A test of the inequalities (3.8) by similar procedures [45, 46] is also positive, giving

$$\frac{1}{\chi}\left|\frac{d\chi}{dq^2}\right|_{q^2=0} m_{\eta'}^2 \simeq 10\% . \tag{5.3}$$

Altogether the solution of the $U_A(1)$ problem proposed in Ref. [9, 10] is thus confirmed from first principles.

5.3. The spin of the proton

The determination of $Z(\beta)$ explained above, allows to extract any matrix element of Q from the corresponding matrix element of Q_L, being

$$Q_L \simeq Z(\beta) Q . \tag{5.4}$$

In particular, to measure the matrix element (3.12) it will be sufficient to measure $\langle \vec{p}'|Q_L(0)|\vec{p}\rangle$. Combining Eq. (3.12) and (5.4)

$$G_A(0)\bar{u}(\vec{p}')\gamma^5 u(\vec{p}) \sim \frac{1}{Z(\beta)2M_p}\langle \vec{p}'|Q_L(0)|\vec{p}\rangle \,. \tag{5.5}$$

Work is in progress.

6. Conclusions

(i) Huge renormalizations exist from lattice to continuum.

(ii) Non perturbative exact numerical determinations of renormalizations are well approximated by few terms of the perturbative expansion. This indicates that the latter is an asymptotic expansion.

(iii) We have found, for a series of problems, a way of disentangling physics at different scales of length by using systematically inconveniencies of lattice algorithms (critical slowing down).

(iv) Lattice is far from being a conceptually cheap and mechanical way for computing physics numerically. Research with lattice requires imagination, to overcome difficulties which are related to understanding of physics.

REFERENCES

[1] K.G. Wilson, *Phys. Rev.* **D10**, 2445 (1974).
[2] K.G. Wilson, *Rev. Mod. Phys.* **55**, 583 (1983).
[3] M. Creutz, *Phys. Rev. Lett.* **43**, 553 (1979).
[4] M. Campostrini, A. Di Giacomo, M. Maggiore, H. Panagopoulos, E. Vicari, *Phys. Lett.* **B 225**, 403 (1989); A. Di Giacomo, M. Maggiore, S. Olejnik, *Phys. Lett.* **B 236**, 199 (1990); *Nucl. Phys.* **B 347**, 441 (1990).
[5] R.W. Haymaker, J. Wosiek, *Phys. Rev. Rapid Comm.* **D36** 3297 (1987).
[6] For a recent review see B. Peterson in *Lattice 92*, *Nucl. Phys.* **B 30**, (Proc. Suppl.) 66 (1993).
[7] J. Kogut, M. Stone, H.W. Wyld, W.R. Gibbs, J. Shigemitsu, S.H. Shenker, D.K. Sinclair, *Phys. Rev. Lett.* **50**, 393 (1983).
[8] G. 't Hooft, *Phys. Rev. Lett.* **37**, 8 (1976); *Phys. Rev.* **D 14**, 3432 (1976).
[9] E. Witten, *Nucl. Phys.* **B 156**, 269 (1979).
[10] G. Veneziano, *Nucl. Phys.* **B 159**, 213 (1979).
[11] S. Weinberg, *Phys. Rev.* **D 11**, 3583 (1975).
[12] For a review of lattice results see eg. A. Di Giacomo in, *Nucl. Phys.* **B 23**, (Proc. Suppl.) 191 (1991).
[13] For a review see eg. D. Sokal *Nucl. Phys.* **B 20**, (Proc. Suppl.) 55 (1991).
[14] M. Gell-Mann, *Phys. Rev.* **125**, 1065 (1962); *Physics* **1**, 64 (1963).
[15] J. Ashman *et al.*, *Phys. Lett.* **B206**, 364 (1988); *Nucl. Phys.* **B 328**, 1 (1989).

[16] R.D. Carlitz in *Proceedings of the XXVI Int. Conf. on High Energy Phys.*, Dallas 1992.
[17] M. Lüscher, *Comm. Math. Phys.* **85**, 29 (1982).
[18] A.S. Kronfeld, A.L. Laursen, G. Schierholz, U. Wiese, *Nucl. Phys.* **B 292**, 349, 412 (1987).
[19] B. Berg, M. Lüscher, *Nucl. Phys.* **B 190**, 412 (1981).
[20] D.J.R. Pugh, M. Teper, *Phys. Lett.* **B 218**, 326 (1989).
[21] M.Campostrini, A. Di Giacomo, H. Panagopoulos, E. Vicari, *Nucl. Phys.* **B 329**, 683 (1990).
[22] M.Campostrini, A. Di Giacomo, H. Panagopoulos, *Phys. Lett.* **B 212**, 206 (1988).
[23] V.A. Novikov, M.A. Shifman, M.A. Vainshtein, A.I. Voloshin, *Nucl. Phys.* **B 525**, 299 (1983).
[24] See eg. *Large order behavior of perturbation theory* ed. J.C. Le Guillou, J. Zinn Justin, North Holland, 1990.
[25] B. Alles, M. Giannetti, *Phys. Rev.* **D44**, 513 (1992).
[26] P. Di Vecchia, K. Fabricius, G.C. Rossi, G. Veneziano, *Nucl. Phys.* **B 192**, 382 (1981).
[27] B. Allés, A. Di Giacomo, *Phys. Lett.* **B 294**, 269 (1992).
[28] B. Berg, *Phys. Lett.* **B 104**, 475 (1981); H. Ilgenfritz, M.L. Laursen, M. Müller-Preussker, G. Schierholz, H. Schiller *Nucl. Phys.* **B 268**, 633 (1986).
[29] The Ape Collaboration, Albanese *et al. Phys. Lett.* **B 192**, 163 (1987).
[30] A. Di Giacomo, these Proceedings, p. 215.
[31] C. Michael, M. Teper, *Nucl. Phys.* **B 305**, 453 (1988).
[32] M. Teper, *Phys. Lett.* **B 202**, 553 (1988).
[33] A. Di Giacomo, H. Panagopoulos *Phys. Lett.* **B**, 133 (1992).
[34] H.G. Dosh, *Phys. Lett.* **B 190**, 177 (1987).
[35] H.G. Dosh, Yu. A. Simonov, *Phys. Lett.* **B 205**, 339 (1988).
[36] Yu. A. Simonov, *Nucl. Phys.* **B 324**, 67 (1989).
[37] B. Allès, M. Campostrini, A. Di Giacomo, Y. Günduc, E. Vicari, *Phys. Rev.* **D 48**, 2284 (1993).
[38] M. Teper, *Phys. Lett.* **B 232**, 227 (1989).
[39] A. Di Giacomo, E. Vicari, *Phys. Lett.* **B 275**, 429 (1992).
[40] A. Di Giacomo, F. Farchioni, A. Papa, E. Vicari, *Phys. Lett.* **B 276**, 148 (1992); *Phys. Rev.* **D 46**, 4630 (1992).
[41] B. Alles, M. Campostrini, A. Feo, H. Panagopoulos, PISA Preprint IFUP-TH 31/92 and *Nucl. Phys.* **B** in press.
[42] A. Di Giacomo, G.C. Rossi, *Phys. Lett.* **B 100**, 481 (1981).
[43] A. Di Giacomo in *Non Perturbative methods*, ed. S. Narison, World Scientific, 1985.
[44] M.A. Shifman, M.A. Vainshtein, V.I. Zakharov, *Nucl. Phys.* **B 147**, 385, 448, 519 (1979).
[45] G. Briganti, A. Di Giacomo, H. Panagopoulos, *Phys. Lett.* **B 253**, 427 (1991).
[46] A. Di Giacomo, E. Meggiolaro, H. Panagopoulos, *Phys. Lett.* **B 291**, 147 (1992).

[47] M. Campostrini, A. Di Giacomo, Y. Günduc, *Phys. Lett.* **225 B**, 393 (1989).