$B + L$ NON-CONSERVATION AS A SEMI-CLASSICAL PROCESS*

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I discuss the problem of computing $B + L$ violation at high energies as that of solving classical differential equations. These equations involve boundary conditions at initial and a final times which are anti-Feynman, and involve solving non-linear differential equations which are complex, even if the original Hamiltonian was real.

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1. Introduction

Electroweak theory was constructed so that the symmetries of the classical equations of motion require that the classical baryon plus lepton number current is conserved,

$$\partial_\mu J^\mu_{B+L} = 0 \,. \tag{1}$$

The theory is constructed in this way because of the very long lifetime of the proton $\tau_p \geq 10^{32}$ yr. If baryon number was violated at the level of the classical equations of motion, we would naively expect a proton lifetime of the order of a typical weak decay time. The limits on the proton lifetime on the other hand naively require that the scale of $B+L$ violation in elementary processes be larger than about 10^{15} GeV.

On the other hand, it has been known for some time that in first order in quantum corrections, $B + L$ is no longer conserved [1] and

$$\partial_\mu J^\mu_{B+L} = 2N_{\text{fam}} \; \partial_\mu K^\mu_{\text{CS}} \,, \tag{2}$$

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where the Chern–Simons current is given as

$$\partial_\mu K^\mu = \frac{\alpha_{\mathrm{W}}}{8\pi} F F^d , \tag{3}$$

where N_{fam} is the number of quark families, F is the electroweak field strength and $F^d_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\sigma\rho}F^{\sigma\rho}$. We will work in this paper in the limit where the Weinberg angle is zero, that is, we will ignore the effects of the U(1) field of SU(2)×U(1)

We can understand the factor of N_{fam} in the above expression since the electroweak Lagrangian is blind to family, and therefore all families must be produced equally. In the same way, the theory is blind to the difference between quark and lepton so the process must produce three quarks and 1 lepton for each family, that is the change in baryon number must equal that of lepton number. Electroweak theory however know the difference between upper and lower components of a doublet and therefore the minimal process need and in fact does involve only one particle of the two in each doublet.

By the anomaly equation, we see that the only way baryon number can be changed is by having a non-zero value of $\int d^4x \, FF^d$. If we formulate the computation of physical quantities as a Euclidean path integral, in order to generate a baryon number violating amplitude, we must have contributions from the sector of the theory where this integral is non-zero. Moreover, the anomaly equation involves an integer change in Chern–Simons charge. One can prove that the topological charge change is in fact integer, and the process involving the least baryon number change has change in Chern––Simons charge of 1 unit. Because of the factor of α_{W} in front of FF^d in the expression for the topological charge, the fields strength with such integer values of the change in Chern–Simons charge are therefore of order $1/g$. The contribution to the path integral is therefore of order

$$\int [dA] \, e^{-S[A]} \sim e^{-k/\alpha_{\mathrm{W}}} , \tag{4}$$

where k is some constant.

The contributions to the path integral which generate $B + L$ violation can be evaluated in weak coupling and are the instanton solutions for SU(2) Yang–Mills theory [2]. Long ago, 't Hooft computed their contribution to $B + L$ violation and found that the amplitude for this process is of order [3]

$$| A_{B+L} |^2 \sim e^{-4\pi/\alpha_{\mathrm{W}}} \sim 10^{-170} . \tag{5}$$

This formula is valid for elementary processes which involve only a few W and Z bosons. The result is so small that it would never be of any practical importance.

An amusing feature of the instanton amplitude is that it is a finite action solution of the Euclidean equations of motion. The Euclidean equations of motion differ from those of Minkowski space only by the sign of the potential energy. In order to get a finite action solution, one must therefore find a solution which goes from the top of a hill of the inverted potential to the top of another hill. This can only happen if the original potential had degenerate minima. The instanton solution takes the field between these different minima. It is straightforward to show that the action associated with the instanton action is nothing more than the WKB suppression factor associated with tunneling through a barrier. The tunneling is done at zero energy. 't Hooft's estimate of the rate is therefore just a WKB tunneling computation of the rate of $B + L$ violation for processes which have an energy small compared to the height of the barrier. The barrier just separates sectors of the theory with different values of the Chern–Simons charge. We have seen that these different sectors are degenerate in energy. They may be related to one another by gauge transformations which are topologically non-trivial. This follows because a topologically trivial gauge transformation would allow the different sectors of the theory to be joined together by a small field, yet we know that to get an integer value of the Chern–Simons charge requires a field of strength $1/g$.

The existence of energy barriers which separate the different topological sectors of the theory tell us that the picture of baryon number violation at energies of the order of or higher than that of the top of the barrier must be much different than that which describes low energy processes. Naively, if we have energy higher than that of the barrier, we expect that there exist classical non-forbidden processes which allow the change of $B + L$. Before we discuss such processes, let us first discuss some properties of the barrier.

Manton and Klinkhammer proposed a simple way to compute the energy barrier in electroweak theory [4]. If there is an energy barrier in ordinary classical mechanics, a particle placed at the top of such a barrier will not move. If it is perturbed it will begin to roll down the hill. There is therefore a static classical solution of the equations of motion which is unstable under small perturbations which describes the particle sitting at the top of the barrier. Such a solution is called a sphaleron, and its energy is the energy of the barrier.

The sphaleron solution of electroweak theory has been found, and the energy of the barrier computed. Because such a solution is classical, it will have many quanta of order $1/\alpha$ in it. Its size will be given by a typical electroweak size scale M_W. The energy is of the form

$$E = A \frac{2M_W}{\alpha_W}, \qquad (6)$$

where A is a quantity which depends on the ratio of Higgs mass to W boson

mass. It is of order 1 for all values of M_H/M_W, and was computed by Manton and Klinkhammer [4]. It is of the order of 10 TeV, so that one can probe this energy scale in cosmology at a relatively low temperature scale, and in SSC experiments.

There are two processes of interest at energies higher or of the order of the height of the barrier. There are finite temperature transition processes relevant for cosmological processes such as baryon number violation [5, 6]. There are high energy collision processes where the particles in the initial state have energies higher than that of the top of the barrier [7-9].

On the other hand, the argument we constructed above that such processes are necessarily of order e^{-k/α_W} seems to be convincing. We found it was necessary to change Chern–Simons charge. On the other hand the action for any process is bounded from below by the action for a single instanton. To see this recall that the Higgs part of the electroweak action is positive definite, so for purpose of setting a lower bound on the action, we can ignore its contribution. Using the identity that

$$(F \pm F^d)^2 = 2(F^2 \pm FF^d), \tag{7}$$

shows that

$$\int F^2 \geq \left| \int FF^d \right|. \tag{8}$$

For the instanton $F = \pm F^d$ so the lower bound is saturated. The contribution of sectors with topological charge is always $\leq e^{-2\pi/\alpha_W}$.

On the other hand, when one computes the rate for baryon number violation at high temperature, the rate is not exponentially suppressed in contradiction with the above arguments. How can this be? The evasion of the above argument is a consequence of coherent multiparticle emission. To understand this recall that the instanton field is of order $1/g$. The rate for instanton induced processes involving N fields is therefore of order

$$|A|^2 \sim (1/\alpha_W)^N e^{-4\pi/\alpha_W}$$
$$\sim e^{-4\pi/\alpha_W - N \ln(\alpha_W)},$$

so that for large enough $N \sim 1/\alpha_W$ the enhancement due to the factors of $1/\alpha_W$ due to the external vector boson lines begin to dominate.

In general, this suggests that non-perturbative phenomena may become important at high energy. Suppose we consider multiparticle emission in weak coupling. For each particle emitted, there is a factor of α_W. On the other hand, coherence requires that there is a factor of N for each external line since we are assuming the field of each external line adds coherently to that of all the other particles involved in the process. We have

$$|A|^2 \sim e^{N \ln(N\alpha_W)}, \tag{10}$$

so that when $N \sim 1/\alpha_W$ non-perturbative multiparticle production may become big.

2. Formulating the problem

In the previous section, we saw that non-perturbative many particle processes in weakly coupled theories can become large. The example of finite temperature electroweak theory provides an example where this is the case.

This does not necessarily imply that high energy processes are also large. If we consider a finite temperature process, we can have a large number of particles in the initial state going into a large number in the final state. The total energy can be large because there are a large number of particles. Each particle need only carry a small amount of energy. It is this process which is large at high temperature.

For high energy, there are only two particles in the initial state, and they therefore must carry a high energy. We might expect that such processes would be suppressed by form factors. The surprise is that when such processes are computed to lowest order in weak coupling, there are no form factors, and the largeness of the finite temperature amplitudes would seem to imply that the high energy amplitudes are also large.

To understand how this works, we repeat the analysis of Ringwald [7–9]. To compute an instanton induced amplitude to lowest order in weak coupling for scalar fields, we have

$$\langle \phi(p_1) \cdots \phi(p_N) \rangle = \phi_{\text{inst}}(p_1) \cdots \phi_{\text{inst}}(p_N). \quad (11)$$

Upon going to the residue of the pole at $p_i^2 = -m^2$, we get point like amplitudes. The amplitude has no form factor. Analyzing vector amplitudes is more complicated but the same conclusion holds [10].

Therefore if the amplitudes for high energy $B + L$ violation are not large at high energy, then the weak coupling expansion must fail. This is not too surprising since we are dealing with processes involving $N \sim 1/\alpha_W$ particles, and we would naively expect that perturbation theory breaks down. Perturbation expansions are only expected to be asymptotic expansions valid if the order of the expansion N is $N \ll 1/\alpha_W$.

It is amusing how the expansion breaks down. Final state interactions among soft particles are of order α_W, but there are N^2 ways to join lines together. The final state corrections are of order $\alpha_W N^2 \sim 1/\alpha_W$. The initial final interactions involve only N particles, so we would naively expect these corrections to be of order 1. Mueller showed that there was an additional factor proportional to the energy of the hard particle times that of the soft, which again makes hard soft processes of order $E_1 E_2 \sim 1/\alpha_W$ [11]. The

hard-hard scattering process is naively of order α_W, but the Muller factor is $E_1 E_2 \sim 1/\alpha_W^2$ so that the overall factor is $1/\alpha_W$. Therefore all the corrections are large, and of the same order of magnitude.

It can be proven that at least for the soft-soft interactions on this amplitude, and presumably for the hard-soft and hard-hard interactions, the corrections to the lowest order process exponentiate and the cross section for $B + L$ violation is of the form [12, 13]

$$\sigma \sim \exp(-4\pi F(E/E_{sph})/\alpha_W)\,, \tag{12}$$

where E_{sph} is the sphaleron energy of electroweak theory. The weak coupling expansion resummed in this way is equivalent to a low energy expansion valid when $E \ll E_{sph}$. The issue of whether or not the rate for $B + L$ violation is large at high energy is the issue of whether or not F has a zero. This issue is clearly outside the scope of naive weak coupling expansions.

3. A possible solution of the problem

Although it is true that the naive weak coupling expansion has broken down, it is not clear that there is no well defined semi-classical approximation to the computation of scattering amplitudes for large numbers of particles. For any such approximation to be sensible, we must of course require that the corrections to this approximation be small.

Using the example of a 1 dimensional integral, we can see that such a semi-classical approximation may exist. Consider the integral

$$I = \int dx \; x^N \; \exp\left(-\frac{(x^2 - v^2)^2}{g^2}\right)\,. \tag{13}$$

We will consider the case where $g \ll 1$ and $N \gg 1$. The naive weak coupling expansion would correspond to doing a stationary phase approximation on the exponential, that is, expanding around $x = \pm v$. This is a good approximation so long as $N \sim 1$. For $N \gg 1$, the proper way to do the stationary phase approximation is to exponentiate the power of x and extremize

$$-\frac{(x^2 - v^2)^2}{g^2} + N\ln(x)\,. \tag{14}$$

It is easy to check that this extremization yields a good approximation to the integral for large N.

This suggests what may be done for field theory [14], that is we write an expression for a Green's function as

$$\int [d\phi] \; \phi(p_1) \cdots \phi(p_N) \; e^{iS} = \int [d\phi] \; \exp\left\{iS + \ln(\phi(p_1) \cdots \phi(p_N))\right\}\,. \tag{15}$$

We are forced to write this expression in Minkowski space for as we will soon see, the equations of motion we will derive will not be Wick rotatable into Euclidean space. The reason for this is simple to understand: the equations of motion including the exponentiated external lines describe not vacuum to vacuum transitions, but configurations with particles in the initial and final states. In this case, there will be anti-Feynman pieces to the boundary conditions, and such terms may not be Wick rotated.

The equation of motion resulting by varying the above effective action are

$$-i\frac{\delta S}{\delta \phi(x)} = \sum_i \frac{e^{ip_i x}}{\phi(p_i)} \,. \tag{16}$$

As we approach the residue of the pole, $p_i^2 = -m^2$, we see that the right hand side of the equation vanishes. On the other hand, in the asymptotic region for large x, such a term would be coupled to the free wave equation which also vanishes,

$$-i(-\partial^2 + m^2)\phi(x) = (-\partial^2 + m^2)\sum_i \frac{e^{ip_i x}}{\phi_{\text{res}}(p_i)} \,, \tag{17}$$

where ϕ_{res} is the residue of the pole of the Fourier transformed field. We see therefore that these source terms act as a boundary condition for the fields requiring that asymptotically

$$\lim_{t \to \pm\infty} \phi(x) = i\sum_i \frac{e^{ip_i x}}{\phi_{\text{res}}(p_i)} \,. \tag{18}$$

That is, we have anti-Feynman boundary conditions for particles in the initial state with a strength which must be determined self-consistently by knowing the residue of the pole of the Fourier transformed field [15]. The sign of the limit in time must be determined by the sign of the energy E_i, that is the sum on the right hand side of the previous equations runs only over negative energies for positive times and *vice versa*.

A remarkable feature of the above equations is that one can show that at least to three loops expanding around the above solution, the corrections are small. If we take the instanton solution and expand around it, we generate the Mueller correction.

The problem with the above equation is that it is difficult to solve. Although we started with a real field, we see that the solution of the above equation with the anti-Feynman boundary conditions must in general be complex. Although the solution is complex, there is still a conserved energy function. Because this energy functional is no longer positive definite since

the field is complex, one can take a field which has a singularity, but due to cancellations among the various terms in the energy functional the energy is conserved. The general solution of the above problem therefore has singularities. Although it may be possible to find a contour in the complex time plane which avoids the singularity, one does not know the nature, position or number of the singularities until the equations have been solved [16].

These singularities lead to an amusing phenomenon. Near a singularity, a smooth field may be converted into a rapidly oscillating one. This is precisely what must happen in a high energy collision where high energy particles produce many soft quanta.

It would seem to be a straightforward task to numerically integrate the above equations. It is not. Closed boundary value problems with singularities at internal points are difficult to solve when the position nature and number of the singularities are known. Here we do not know the position, number, and in some cases the nature of the singularity. In addition, one must do a calculation to very high accuracy to handle the spectrum of Fourier modes needed.

Although the treatment used above was tailored for computing scattering amplitudes, one could also write down expressions for the cross section and try to evaluate them in a semi-classical approximation [13, 16]. This leads in the end to a set of equations similar to the above, which might or might not be simpler to analyze. The problems with the singularities and the complexity of the fields is however generic to both problems, and their resolution demands solving the complicated numerical problem outlined above, a problem which is difficult and may prove impossible to solve to the needed precision.

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