CHIRAL QCD PHASE TRANSITION IN THE LINEAR AND NONLINEAR SIGMA MODEL*

D. Metzger, H. Meyer-Ortmanns, B.-J. Schaefer

and

H.J. Pirner†

Institut für Theoret. Physik
Universität Heidelberg, Germany

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We analyse the QCD chiral phase transition in the nonlinear and linear σ-model. The strategy is the same in both cases. We fix the parameters of the effective meson theory at temperature $T = 0$ and extrapolate the models to temperatures in the vicinity of the phase transition. The linear σ-model in SU(3)×SU(3) gives a first order phase transition at $T_c = 164$ MeV. At this temperature chiral SU(2)×SU(2) is restored. The discontinuity in the energy density is small $\Delta\epsilon/\epsilon \approx 10\%$. We also calculate meson masses as a function of temperature.

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1. Introduction

Relativistic nucleus–nucleus collisions with cm-energies

$$E_{\rm cm} \geq 20 \text{ GeV/nucleon}$$

create large systems of sizes $R > 20$ fm at freeze-out with $> 10^4$ pions [1]. It is natural to try statistical methods to describe such hadronic fireballs. A good starting point may be to use equilibrium thermodynamics with pions when one is interested in the later stages of these collisions, where the temperature is below possible phase transitions. The low energy interaction of

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pions is fully determined by chiral symmetry [2, 3]. Below $T \approx 120$ MeV this interaction can be parametrized by the nonlinear sigma model O(4) with one scalar (σ) and three pseudoscalar $\vec{\pi}$-fields, which are constrained by the condition $\sigma^2 + \vec{\pi}^2 = f_\pi^2$, where f_π is the pion decay constant. Above this temperature region also heavier hadrons give a non-negligible contribution to the condensates and thermodynamic quantities. One way of including part of the heavier mesons is provided by the choice of SU(3)×SU(3) as chiral symmetry group rather than SU(2)×SU(2). The linear SU(3)×SU(3) sigma model includes a nonet of pseudoscalar (O^-)-fields and a nonet of scalar (O^+)-fields. The spontaneous breaking of the SU(3)×SU(3) symmetry leads to massless (O^-) Goldstone modes. Obviously a massless pseudoscalar octet does not provide an adequate approximation to the experimentally observed meson spectrum. Therefore we include explicit symmetry breaking terms to account for the physical mass values of the octet-fields. A determinant term in the meson fields guarantees the correct mass splitting of the $\eta - \eta'$ masses which is due to the U(1)-anomaly. It reflects the 't Hooft-determinant on the quark level.

The experimental challenge is to measure the equation of state of pions from the inclusive pion spectra. The theoretical task is to calculate this equation of state. For this purpose we need reliable techniques to treat field theories at finite temperature. One can then extrapolate from the measured physics at $T = 0$ to the yet unknown physics at high temperatures. A very accurate treatment of the soft modes with lowest mass is essential at low temperatures. We calculate the partition function $\mathcal{Z}$ in terms of a selfconsistent field which is chosen to extremize $\ln \mathcal{Z}$. It gives effective masses to the meson fields. This saddle point approximation to the partition function corresponds to the leading order of a $1/N$ expansion.

The paper is organized as follows. In Section 2 we discuss the nonlinear σ-model for SU(2) ×SU(2). In Section 3 we calculate the partition function in the linear σ-model for SU(3)×SU(3). Section 4 is devoted to a short discussion.

2. The nonlinear σ-model: SU(2)×SU(2)

The partition function $\mathcal{Z}$ for the SU(2)×SU(2) nonlinear σ-model is given in terms of the $O(4)$-multiplet $(n_0, \vec{n}) = (\sigma, \vec{\pi})$ with $n^2 = \sigma^2 + \vec{\pi}^2$ as:

$$\mathcal{Z} = \int \mathcal{D}n(x) \prod_x \delta(n^2(x) - f_\pi^2) \exp\left\{ - \int_0^\beta d\tau \int_V d^3x \left[\tfrac{1}{2}(\partial_\mu n)^2 - cn_0\right] \right\} . \tag{1}$$

At zero temperature $T := 1/\beta = 0$ the parameters of the model are well known. The pion decay constant f_π equals to 93 MeV. The classical vacuum expectation value of n_0 is determined as $\langle n_0 \rangle = f_\pi$ by minimizing the vacuum energy. Expanding the dependent field $n_0 = \sqrt{f_\pi^2 - \vec{n}^2}$ to leading order in $\vec{n}^2/f_\pi^2$, one obtains the mass of the pion as $m_\pi^2 = c/f_\pi$.

The basic idea of our method is to eliminate the nonlinear constraint $n^2 = \sigma^2 + \vec{\pi}^2 = f_\pi^2$ by the introduction of an auxiliary field $\lambda(x)$ via an integral along the imaginary axis

$$\mathcal{Z} = \int_{a-i\infty}^{a+i\infty} \mathcal{D}\lambda(x) \int \mathcal{D}n(x) \times \exp\left\{ - \int_0^\beta d\tau \int_V d^3x \left[\tfrac{1}{2}(\partial_\mu n)^2 + \lambda(n^2 - f_\pi^2) - c n_0 \right] \right\} . \quad (2)$$

After shifting the zero-th component n_0 to $\tilde{n}_0$

$$\tilde{n} = (\tilde{n}_0, \tilde{\vec{n}}) = (n_0 - \frac{c}{2\lambda}, \vec{n}) ,$$

we obtain a Gaussian action for the $O(N)$ multiplet field $\tilde{n}$, when we evaluate Eq. (2) in a saddle point approximation. The resulting partition function is given as

$$\mathcal{Z} = \int \mathcal{D}\tilde{n}(x) \exp\left\{ - \int_0^\beta d\tau \int_V d^3x \left[\tfrac{1}{2}(\partial_\mu \tilde{n})^2 + \lambda \tilde{n}^2 - \lambda f_\pi^2 - \frac{c^2}{4\lambda} \right] \right\} . \quad (3)$$

Here we have dropped the λ-integration and chosen $\lambda(x) = \lambda = \text{const}$. The optimal choice for λ will be determined later, *cf.* Eq. (11).

Upon Gaussian integration over the four ($N = 4$) $\tilde{n}$-fields we end up with a partition function of a free relativistic Bose gas with $N = 4$ components and effective masses

$$m_{\text{eff}}^2 = 2\lambda . \quad (4)$$

$$\mathcal{Z} = \exp\left\{ -\beta V \left[U_T(m_{\text{eff}}^2, \Lambda) + U_0(m_{\text{eff}}^2, \Lambda) - \lambda f_\pi^2 - \frac{c^2}{4\lambda} \right] \right\} . \quad (5)$$

Here U_T denotes the contribution from thermal fluctuations

$$U_T(m_{\text{eff}}^2, \Lambda) = 4 \frac{1}{\beta} \int_0^\Lambda \frac{d^3k}{(2\pi)^3} \ln\left(1 - \exp\left(-\beta\sqrt{\vec{k}^2 + m_{\text{eff}}^2}\right)\right) \quad (6)$$

and U_0 the zero point energy

$$U_0(m_{\text{eff}}^2, \Lambda) = 4 \int_0^\Lambda \frac{d^3k}{(2\pi)^3} \frac{1}{2} \sqrt{\vec{k}^{\,2} + m_{\text{eff}}^2}\,. \tag{7}$$

We regularize the k-integrations in Eqs (7) and (8) with a cut-off Λ, since we do not believe our $\vec{\pi}$-effective theory to be correct for momenta beyond Λ. At momenta $k > \Lambda$ the compositeness of the pions manifests itself in resonance excitations and/or higher derivative couplings of the pion states, which are neglected. For the numerical calculations we take cut–off values $\Lambda = 700$ MeV, 800 MeV, 1000 MeV.

After regularization we adopt the following renormalization procedure. We define a renormalized potential at arbitrary T according to

$$\begin{aligned} U^{ren}(m_{\text{eff}}^2, \Lambda) =& U_T(m_{\text{eff}}^2, \Lambda) + U_0(m_{\text{eff}}^2, \Lambda) \\ &- \left[U_0(m_\pi^2, \Lambda) + (m_{\text{eff}}^2 - m_\pi^2) \left. \frac{\partial U_0}{\partial m_{\text{eff}}^2} \right|_{m_\pi^2} \right]. \end{aligned} \tag{8}$$

The two subtractions guarantee the two renormalization conditions at $T = 0$

$$m_{\text{eff}}^2(\lambda_0) = m_\pi^2\,, \tag{9}$$

$$< n_0 > = f_\pi\,. \tag{10}$$

It is well known that the nonlinear sigma model is not renormalizable in four dimensions. Therefore higher order divergences can only be compensated by higher order derivative terms in the original action. The coefficients of these higher order terms have to be determined by experiment. We do not include such terms in contrast to Ref. [2]. In Section 3 we will extend the calculation to the linear σ-model (SU(3)×SU(3)) which contains higher masses and strange mesons.

The thermodynamic observables at finite T are obtained from the partition function $\mathcal{Z}$ approximated as

$$\mathcal{Z}(\Lambda, T) = \exp\left\{ -\beta V \left[U^{\text{ren}}(m_{\text{eff}}^2(\lambda^*), \Lambda) - \lambda^* f_\pi^2 - \frac{c^2}{4\lambda^*} \right] \right\}, \tag{11}$$

where $\lambda^*(T)$ extremizes $\ln \mathcal{Z}$ at a given temperature $T \neq 0$. The saddle point equation for λ^* is solved numerically, since in the interesting temperature range the relevant parameter $m_{\text{eff}}/T = \sqrt{2\lambda^*}/T$ can have values in the range $0 \leq m_{\text{eff}}/T < \infty$.

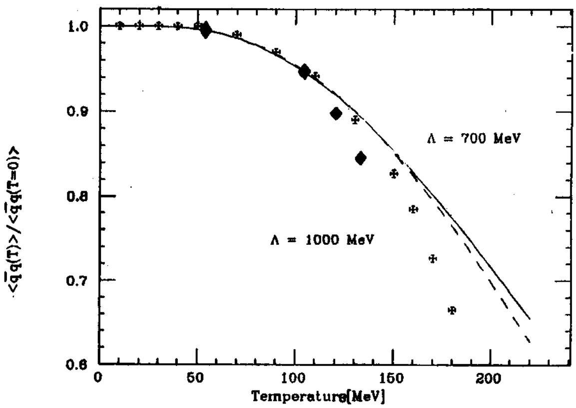

Fig. 1. The order parameter for chiral symmetry breaking $\langle\bar{q}q(T)\rangle/\langle\bar{q}q(T=0)\rangle$ for two different cut-offs $\Lambda = 700$ MeV (solid line) and $\Lambda = 1000$ MeV (dashed line). The diamonds represent the result of the linear σ-model in Section 3.

Let us first study, how the order parameter of chiral symmetry breaking $\langle n_0\rangle$ behaves as a function of temperature. In Fig. 1 we present the result for

$$\frac{\langle n_0(T)\rangle}{\langle n_0(T=0)\rangle} = \frac{\partial \ln \mathcal{Z}(T)}{\partial c}\left(\frac{\partial \ln \mathcal{Z}(T=0)}{\partial c}\right)^{-1} .$$

In quark language this ratio corresponds to the ratio of the quark condensate $\langle\bar{q}q(T)\rangle$ at finite temperature over the quark condensate at $T = 0$, since the symmetry breaking term of the $O(4)$ Lagrangian $\mathcal{L}_{\rm SB} = cn_0$ is identified with the symmetry breaking term $\mathcal{L}_{\rm SB} = -2m\bar{q}q$ in the QCD-Lagrangian. We also show the result of the linear σ-model SU(3)×SU(3), which are presented in the next section, and the results of chiral perturbation theory [2]. The result for $\langle\bar{q}q(T)\rangle/\langle\bar{q}q(T=0)\rangle$ is rather insensitive to the cut-off. It agrees well with the three loop calculation of Ref. [2]. Chiral symmetry is only very gradually restored. At low temperature the $\pi\pi$-interaction is weak and $\langle\bar{q}q\rangle$ does not change very much.

The linear σ-model: SU(3)×SU(3)

For a Euclidean metric the Lagrangian of the linear sigma-model is given as

$$\begin{aligned}\mathcal{L} =& \tfrac{1}{2}\partial_\mu\Phi\partial_\mu\Phi^+ - \tfrac{1}{2}\mu_0^2 \mathrm{tr}\Phi\Phi^+ + f_1\left(\mathrm{tr}\Phi\Phi^+\right)^2 + f_2 \mathrm{tr}\left(\Phi\Phi^+\right)^2 \\ &+ g\left(\det\Phi + \det\Phi^+\right) + \varepsilon_0\sigma_0 + \varepsilon_8\sigma_8 ,\end{aligned} \tag{12}$$

where the (3×3) matrix field $\Phi(x)$ is given in terms of Gell–Mann matrices λ_ℓ $(\ell = 0,\ldots,8)$ as

$$\Phi = \frac{1}{\sqrt{2}} \sum_{\ell=0}^{8} (\sigma_\ell + i\pi_\ell)\,\lambda_\ell . \tag{13}$$

Here σ_ℓ and π_ℓ denote the nonets of scalar and pseudoscalar mesons, respectively. As order parameters for the chiral transition we choose the meson condensates $\langle\sigma_0\rangle$ and $\langle\sigma_8\rangle$. The chiral symmetry of $\mathcal{L}$ is explicitly broken by the term $(-\varepsilon_0\sigma_0 - \varepsilon_8\sigma_8)$, corresponding to the finite quark mass term $2m_q\bar{q}q + m_s\bar{s}s$ on the quark level. The chiral limit is realized for vanishing external fields ε_0 and ε_8. Note that the action $S = \int d^3x d\tau \mathcal{L}$ with $\mathcal{L}$ of Eq. (12) may be regarded as an effective action for QCD, constructed in terms of an order parameter field Φ for the chiral transition. It plays a similar role to Landau's free energy functional for a scalar order parameter field for investigating the phase structure of a Φ^4-theory [4].

The six unknown couplings of the sigma-model (Eq. (12)) (μ_0^2, f_1, f_2, g, ε_0, ε_8) are assumed to be temperature independent and fitted to the pseudoscalar masses at zero temperature. Further experimental input parameters are the pion decay constant $f_\pi = 93$ MeV and a high lying (O^+) scalar mass $m_\sigma = 1.59$ GeV (*cf.* Table I). For the remaining scalar masses and the coupling constants we obtain the values of Table I.

TABLE I

Tree level parametrization of the SU(3)×SU(3) linear sigma model (input data taken from experiment).

Input

m_π [MeV]	m_K [MeV]	m_η [MeV]	$m_{\eta'}$ [MeV]	f_π [MeV]	$m_\sigma = m_{f_0}$ [MeV]
138.04	495.66	547.45	957.75	93	1590

Output

μ_0^2 [GeV^2]	f_1	f_2	g [GeV]	ε_0 [GeV^3]	ε_8 [GeV^3]
0.758	12.166	3.053	1.527	0.02656	-0.03449

m_{a_0} [MeV]	$m_{K_0^*}$ [MeV]	$m_{f_0'}$ [MeV]	f_K [MeV]
914.05	913.35	764.71	128.81

The interpretation of the observed scalar mesons is controversial. There are good reasons to interpret the (0^+) mesons at 980 MeV as meson bound states. The model underestimates the strange quark mass splitting in the scalar meson sector, the value for $m_{K_0^*}$ comes out too small.

The effective theory can be related to the underlying QCD Lagrangian by comparing the symmetry breaking terms in both Lagrangians and identifying terms with the same transformation behaviour under SU(3)×SU(3).

Taking expectation values in these equations we obtain the following relations between the light quark condensates, strange quark condensates and meson condensates

$$\varepsilon_0 \langle\sigma_0\rangle = -\tfrac{1}{3}\,(2\hat{m} + m_s)\,(2\,\langle\bar{q}q\rangle + \langle\bar{s}s\rangle)\ ,$$
$$\varepsilon_8 \langle\sigma_8\rangle = -\tfrac{2}{3}\,(\hat{m} - m_s)\,(2\,\langle\bar{q}q\rangle - \langle\bar{s}s\rangle)\ . \qquad (14)$$

We use $\hat{m} \equiv (m_u + m_d)/2 = (11.25 \pm 1.45)$ MeV and $m_s = (205 \pm 50)$ MeV for the light and strange quark masses at a scale $\Lambda = 1$ GeV [5]. From the scalar meson condensates at $T = 0$, $\langle\sigma_0\rangle = 0.144$ GeV and $\langle\sigma_8\rangle = -0.0415$ GeV we get

$$\langle\bar{q}q\rangle = -\,(235 \pm 60\mathrm{MeV})^3\ ,$$
$$\langle\bar{s}s\rangle = -\,(290 \pm 30\mathrm{MeV})^3\ , \qquad (15)$$

in accordance with values from PCAC relations [5] within the error bars. Since we treat the coefficients $\varepsilon_0, \varepsilon_8$ of $\langle\sigma_0\rangle$ and $\langle\sigma_8\rangle$, and $\hat{m}, m_s$ of $\langle\bar{q}q\rangle$ and $\langle\bar{s}s\rangle$ as temperature independent, we will use Eqs (14) for all temperatures to translate our results for meson condensates into quark condensates.

We also check that the pseudoscalar meson mass squares, in particular m_π^2 and m_K^2 are linear functions of the symmetry breaking parameters $\varepsilon_0, \varepsilon_8$. Varying $\varepsilon_0, \varepsilon_8$ while keeping the other couplings fixed we can simulate the sigma model at unphysical meson masses. Since the current quark masses are assumed to depend linearly on ε_0 and ε_8, an arbitrary meson mass set can be related to a mass point in the $(m_{u,d}, m_s)$-plane by specifying the choice of $(\varepsilon_0, \varepsilon_8)$. This may be useful in order to compare our results for meson (and quark) condensates with lattice simulations of the chiral transition.

The thermodynamics of the linear sigma model is determined by the partition function with the Lagrangian of Eq. (12)

$$Z = \int \mathcal{D}\Phi \exp\left\{ -\int_0^\beta d\tau \int d^3x \mathcal{L}(\Phi(\vec{x}, \tau)) \right\}\ . \qquad (16)$$

We will treat Z again in a saddle point approximation. As mentioned above, the saddle point approximation amounts to the leading order of a $1/N$-expansion. In this model $N = 2N_f^2 = 18$. Note that $\mathcal{L}$ of Eq. (12) would be $O(N)$-invariant, if $f_2 = 0$ and $g = 0$. Our input parameters lead to non-vanishing values of f_2 and g, therefore the $O(N)$-symmetry is only approximately realized.

We calculate the effective potential as a constrained free energy density $U_{\text{eff}}(\xi_0, \xi_8)$, that is the free energy density of the system under the constraint that the average values of σ_0 and σ_8 take some prescribed values ξ_0 and ξ_8. The values $\xi_{0_{\min}}$ and $\xi_{8_{\min}}$ that minimize U_{eff}, give the physically relevant, temperature dependent vacuum expectation values, *i.e.* $\langle\sigma_0\rangle = \xi_{0_{\min}}$, $\langle\sigma_8\rangle = \xi_{8_{\min}}$. Hence we start with the background field ansatz

$$\begin{aligned} \sigma_0 &= \xi_0 + \sigma_0' \,, \\ \sigma_8 &= \xi_8 + \sigma_8' \,, \end{aligned} \tag{17}$$

where σ_0' and σ_8' denote the fluctuations around the classical background fields ξ_0 and ξ_8. All other field components are assumed to have zero vacuum expectation value, *i.e.* $\sigma_\ell = \sigma_\ell'$ for $\ell = 1, \ldots, 7$ and $\pi_\ell = \pi_\ell'$ for $\ell = 0, \ldots, 8$. The relation between the effective potential U_{eff} and Z is given by

$$Z = \int d\xi_0 \int d\xi_8 \exp\left(-\beta V U_{\text{eff}}(\xi_0, \xi_8)\right) . \tag{18}$$

Next we insert the background field ansatz (17) in $\mathcal{L}$ and expand the Lagrangian in powers of $\Phi' = (1/\sqrt{2}) \sum_{\ell=0}^{8} (\sigma_\ell' + i\pi_\ell')\lambda_\ell$. The constant terms in Φ' lead to the classical part of the effective potential U_{class}. Linear terms in Φ_ℓ' vanish for all $\ell = 0, \ldots, 8$ due to the δ-constraints in Eq. (18). Quadratic terms in Φ' define the isospin multiplet masses m_Q^2, where $Q = 1, \ldots, 8$ labels the multiplets.

The cubic part in Φ' will be neglected, while the quartic term $\mathcal{L}^{(4)}(\Phi')$ is quadratized by introducing an auxiliary matrix field $\sum(\vec{x}, \tau)$. This is a matrix version of a Hubbard–Stratonovich transformation [6].

In the saddle point approximation we drop $\int \mathcal{D}\Sigma$ and use a SU(3)-symmetric ansatz with a diagonal matrix $\sum = \text{diag}(s, s, s)$. Hence the effect of the quadratization procedure is to induce an extra mass term $(s + \mu_0^2)$ and a contribution U_{saddle} to U_{eff}, which is independent of ξ_0 and ξ_8. This way we finally end up with the following expression for $\hat{Z}$

$$\begin{aligned} &\hat{Z}(\xi_0, \xi_8, s) = \exp\left(-\beta V (U_{\text{class}} + U_{\text{saddle}})\right) \\ &\times \int \prod_{Q=1}^{8} \mathcal{D}\varphi_Q' \exp\left(-\int_0^\beta d\tau \int d^3x \tfrac{1}{2} \sum_Q g(Q)(\partial_\mu \varphi_Q' \partial_\mu \varphi_Q'^\dagger + X_Q^2 \varphi_Q' \varphi_Q'^\dagger)\right) , \end{aligned} \tag{19}$$

where

$$X_Q^2 \equiv m_Q^2 + \mu_0^2 + s \,, \tag{20}$$

φ'_Q denotes σ'_Q for $Q = 1, \ldots, 4$ and π'_Q for $Q = 5, \ldots, 8$, $g(Q)$ is the multiplicity of the isospin multiplet. We have $g(1) = 3$ for the pions, $g(2) = 4$ for the kaons, $g(3) = 1 = g(4)$ for η, η', respectively. Correspondingly, the multiplicities for the scalar nonets are $g(5) = 3$, $g(6) = 4$, $g(7) = 1$, $g(8) = 1$ for the a_0, K_0^*, f_0, f_0'-mesons.

Thus we are left with an effectively free field theory. The only remnant of the interaction appears in the effective mass squared X_Q^2 via the auxiliary field s.

The choice of a self-consistent effective meson mass squared has been pursued already in Refs [7, 8]. This is an essentially new ingredient compared to earlier calculations of the chiral transition in the linear sigma model [9]. The positive contribution of s to the effective mass extends the temperature region, where imaginary parts in the effective potential can be avoided. In general, imaginary parts are encountered, when the effective mass arguments of logarithmic terms become negative. They are an artifact of the perturbative evaluation of the effective potential and of no physical significance, as long as the volume is infinite. In our application the optimized choice for s will increase as function of temperature and lead to positive X_Q^2 over a wide range of parameters.

Gaussian integration over the fluctuating fields Φ' in Eq. (19) gives

$$\hat{Z}(\xi_0, \xi_8, s) = \exp\Big\{-\beta V\Big[U_{\mathrm{class}} + U_{\mathrm{saddle}} + \frac{1}{2\beta}\sum_{Q=1}^{8} g(Q) \sum_{n\in Z} \int \frac{d^2k}{(2\pi)^3} \ln\left(\beta^2(\omega_n^2 + \omega_Q^2)\right)\Big]\Big\}\,, \tag{21}$$

where

$$\omega_Q^2 \equiv k^2 + X_Q^2\,, \tag{22}$$

and

$$\omega_n^2 \equiv \left(\frac{2\pi n}{T}\right)^2\,, \tag{23}$$

denote the Matsubara frequencies. In contrast to our former approach [8] we keep all Matsubara frequencies and evaluate $\sum_{n\in Z}$ in the standard way, see *e.g.* [10]. The result is

$$\hat{Z}(\xi_0, \xi_8; s) = \exp\left(-\beta V U_{\mathrm{eff}}(\xi_0, \xi_8; s)\right)\,, \tag{24}$$

$$U_{\mathrm{eff}}(\xi_0, \xi_8; s) = U_{\mathrm{class}} + U_{\mathrm{saddle}} + U_{\mathrm{th}} + U_0\,, \tag{25}$$

$$U_{\mathrm{th}} \equiv \frac{1}{\beta}\sum_{Q=1}^{8} g(Q) \int \frac{d^3k}{(2\pi)^3} \ln\left(1 - \mathrm{e}^{-\beta\omega_Q}\right)\,, \tag{26}$$

$$U_0 \equiv \tfrac{1}{2} \sum_{Q=1}^{8} g(Q) \int^{\Lambda} \frac{d^3k}{(2\pi)^3} \omega_Q \,. \tag{27}$$

Here we have indicated that $\hat{Z}$ and U_{eff} still depend explicitly on the auxiliary field s. The integral in Eq. (27) is regularized with a three-momentum cut-off Λ. The thermal contribution U_{th} vanishes at zero temperature and is finite for $T > 0$, while the zero point energy U_0 diverges as $\Lambda \to \infty$.

The linear sigma model is a renormalizable theory, and in principle the cut-off could be removed after a suitable renormalization prescription. We do not believe in this model as an effective description for QCD, when the momenta exceed a certain scale, say $\Lambda \approx 1 - 1.5$ GeV. Therefore we use a cut-off Λ. The necessity for renormalization arises, when we postulate a matching between the physical masses and condensates with the $T = 0$-values, and T approaches zero from above. Such a matching is guaranteed, if we impose the following subtractions on the zero point energy part

$$\begin{aligned} U_0^{\text{ren}}(X_Q^2(\xi_0,\xi_8);\Lambda) &:= U_0(X_Q^2) - \{U_0(m_{\text{phys}}^2) \\ &+ \frac{\partial U_0(m^2)}{\partial m^2}\Big|_{m^2_{\text{phys}}} \cdot (X_Q^2 - m^2_{\text{phys}}) + \tfrac{1}{2}\frac{\partial^2 U_0(m^2)}{\partial (m^2)^2}\Big|_{m^2_{\text{phys}}} \cdot (X_Q^2 - m^2_{\text{phys}})^2\} \,. \end{aligned} \tag{28}$$

Here m^2_{phys} is given by m_Q^2 evaluated at $\xi_0 = \langle \xi_0 \rangle$ and $\xi_8 = \langle \xi_8 \rangle$, *i.e.* for physical condensate values. The optimal choice s^* for the auxiliary field s is then determined by

$$\frac{\partial U_{\text{eff}}^{\text{ren}}}{\partial s}\Big|_{s^*} = 0 \,, \tag{29}$$

where $U_{\text{eff}}^{\text{ren}}$ equals U_{eff} of Eq. (25) with U_0 replaced by U_0^{ren} of Eq. (28).

Upon using Eq. (29) it is easily verified that the meson condensates $\langle \xi_0 \rangle$ and $\langle \xi_8 \rangle$ are free of extra contributions from the zero point energies at $T = 0$ if $\ln Z = -\beta V U_{\text{eff}}^{\text{ren}}(\xi_0, \xi_8; s^*)$. Thus a matching with $\langle \xi_i \rangle_T = 0$ is ensured. Similarly we find for the effective masses

$$X_Q^2|_{T=0} = m_Q^2 + s + \mu_0^2 = m^2_{\text{phys}} \,, \tag{30}$$

for $\xi_0 = \langle \xi_0 \rangle$, $\xi_8 = \langle \xi_8 \rangle$, since $s = -\mu_0^2$ at $T = 0$.

Note that the sensitivity to the cut-off in Eq. (28) is reduced from a Λ^4-dependence to a $1/\Lambda^2$-dependence. This is a desirable feature in view of the uncertainties in a suitable choice for Λ. We have taken $\Lambda = 1.5$ GeV and kept the cut-off finite throughout the calculations.

Now we are prepared to determine the temperature dependence of the order parameters $\langle\xi_0\rangle(T)$, $\langle\xi_8\rangle(T)$ from the minima of $U_{\rm eff}^{\rm ren}(\xi_0,\xi_8;s^*)$. Thermodynamic quantities like energy densities, entropy densities and pressure can be derived from Z in the standard way, if Z is approximated as

$$\hat{Z}^{\rm ren} \equiv \exp\left(-\beta V U_{\rm eff}^{\rm ren}(\xi_0,\xi_8;s^*)\right) . \tag{31}$$

For the parameters of Table I we vary the temperature and determine for each T the extremum of $U_{\rm eff}$ as a function of ξ_0, ξ_8 and s. The extremum is a minimum with respect to ξ_0 and ξ_8 and a maximum with respect to s.

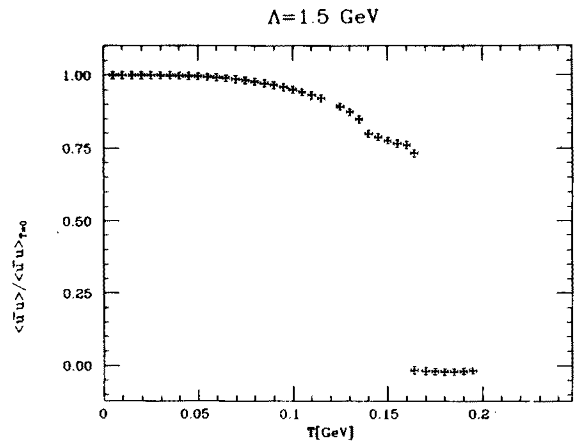

Fig. 2. Normalized light quark condensate $\langle\bar{q}q\rangle/\langle\bar{q}q\rangle_{T=0}$ vs temperature.

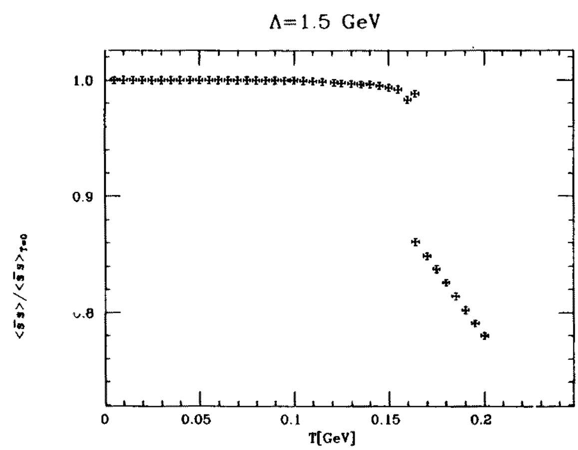

Fig. 3. Normalized strange quark condensate $\langle\bar{s}s\rangle/\langle\bar{s}s\rangle_{T=0}$ vs temperature.

In Figs 2 and 3 we show the variations of $\langle\bar{q}q\rangle(T)/\langle\bar{q}q\rangle_{T=0}$ and $\langle\bar{s}s\rangle(T)/\langle\bar{s}s\rangle_{T=0}$ as a function of temperature obtained from $\langle\xi_0\rangle(T)$ and

$\langle \xi_8 \rangle (T)$ with the help of Eq. (14). We observe a gradual decrease of the light quark condensate, whereas the strange quark condensate stays almost constant. A first order transition occurs at $T_c = 164$ MeV for $\Lambda = 1.5$ GeV.

At T_c only the SU(2)×SU(2) part of the chiral symmetry is restored, the strange quark condensate does not drop to zero.

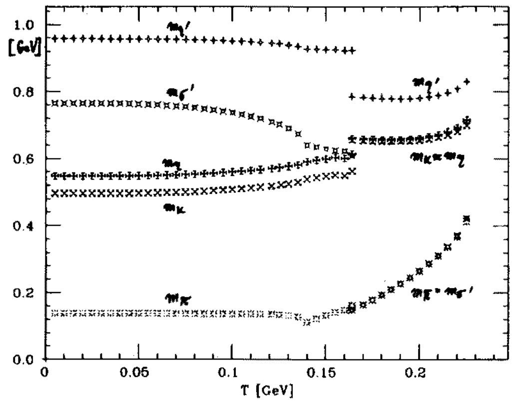

Fig. 4. Meson masses in GeV as a function of temperature T.

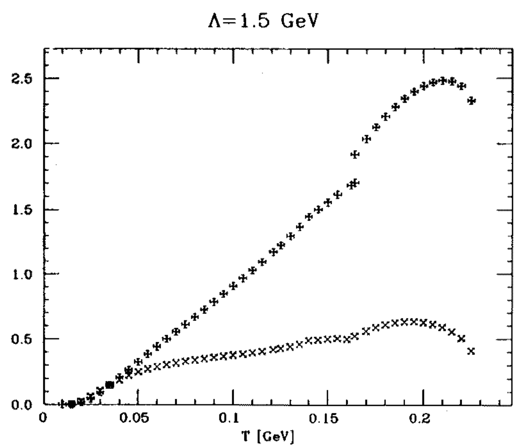

Fig. 5. Normalized energy density ε/T^4 and pressure p/T^4 vs temperature. The decrease of these quantities above $T \approx 200$ MeV as a function of T indicates the breakdown of our approximation scheme.

In our lowest order calculation the temperature dependence of these masses is determined by the temperature dependence of the condensates (*cf*. Fig. 4). The masses m_π^2 and $m_{\sigma'}^2$ become degenerate at T_c, $m_\pi^2 \approx m_{\sigma'}^2 \approx (147\mathrm{MeV})^2$. Below T_c the meson masses stay remarkably constant.

Above T_c, the $\pi - K$-splitting is increased rather than reduced. Accordingly the strange meson contribution to the energy density in this temperature region is reduced compared to the low-temperature hadron gas.

In Fig. 5 we give the energy density ε/T^4 and pressure p/T^4 as function of temperature for a cut-off $\Lambda = 1.5$ GeV. The gap in the energy densities at T_c, which is a measure for the latent heat, is obviously rather small, about 10% of ε at T_c. Sizeable contributions to ε come mainly from 8 degrees of freedom, the pions, the kaons and the f_0 meson.

4. Discussion of the results

For low temperatures the physics of the nonlinear SU(2)×SU(2) and linear SU(3)×SU(3) σ-model are identical. In Fig. 1 we show the light quark condensates calculated in both models. Above $T \approx 120$ MeV the extra degrees of freedom in the SU(3)×SU(3) calculation become important. At higher temperatures $T > T_c$ also the linear sigma model will certainly fail as an effective model for QCD due to the lack of quark–gluon degrees of freedom. Nevertheless it would be interesting to study, at what temperature the full SU(3)×SU(3) symmetry is restored. At very high temperatures the effective potential becomes proportional to $\sum_Q X^2(Q)T^2$, the linear terms proportional to σ_0 and σ_8 in the masses of O^+ and O^- mesons cancel and temperature tries to fully restore the broken symmetry.

Finally we remark that our value for T_c is rather close to the lower limit of the Hagedorn temperature $T_H (T_H \sim 160$ MeV). This may not be entirely accidental. In our model the $1/N$-expansion means a large number of flavours, since $N = 2 \cdot N_f^2$. In order to keep QCD an asymptotically free theory also the number of colours N_c has to increase. Correspondingly our approximation is similar to Hagedorn's description of the hadron gas as a resonance gas. We expect that corrections from subleading terms in our $1/N_f$-expansion will implicitly amount to corrections also to the large N_c-limit. The chiral transition for unphysical values of strange quark and light quark masses will be investigated in a future publication.

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