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PERTURBATION OF INITIAL DATA FOR SPHERICALLY SYMMETRIC CHARGED BLACK HOLE AND PENROSE CONJECTURE

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A new description of unconstrained degrees of freedom for the gravitational field introduced in [6] has been applied to the description of perturbed initial data from the Reissner–Nordström solution. When this solution describes a black hole ($m > q$) we can perform a generic fully nonlinear perturbation of initial data outside an apparent horizon in such a way that this horizon is an extremal two-surface in four-dimensional spacetime. A simple consequence of the considerations in: J. Jezierski, *Classical Quantum Gravity* **11**, 1055 (1994) leads to the following theorem: Let Σ be an asymptotically flat spacelike surface which is nonsingular outside an extremal two-surface $S_{\min}$ (which is extremal as two-surface imbedded in spacetime) then $16\pi m_{\rm ADM}^2 \geq A$, where $m_{\rm ADM}$ denotes the A.D.M. mass of perturbed data (around Reissner–Nordström solution) on Σ and A is an area of the apparent horizon $S_{\min}$.

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1. Introduction

We have shown in [6] that small (but fully nonlinear) perturbation (in the asymptotic region) of initial data (Einstein + Maxwell) from Reissner–Nordström solution leads to the growth of the A.D.M. mass. It is worth to notice that the result allows to formulate a theorem related with Penrose conjecture and cosmic censorship hypothesis (see for example [3]). The theorem says that a generic perturbation of the initial data (outside the horizon) leads to the inequality between the mass of the perturbed black hole and the area of the apparent horizon which in this case coincides with a minimal surface. The methods used for the proof have been initiated in [4], the first application to black hole situation has been published in [5] and this result is an important improvement of the previous one.

Let V be a compact, smooth tree-dimensional manifold with boundary ∂V. Take as an example the volume V diffeomorphic to $K(0, r_0, r_1)$ (a piece of $\mathbb{R}^3$ between the two spheres of radius r_0 and r_1, respectively). Limiting case $r_1 \to \infty$ will be considered.

Let $T = \partial V \times \mathbb{R}^1$. The space $M = V \times \mathbb{R}^1$ will be the interior of our spacetime "tube" and the boundary $T = \partial M$ will be a one-time-like and two-space-like surface in our spacetime.

Let us fix a coordinate chart (x^μ) on M such that (x^1, x^2) is a coordinate chart on ∂V (*e.g.* spherical angle θ and φ), $x^3 = r$ is any third coordinate on V which is constant on ∂V. Moreover, x^0 denotes the time coordinate. So we have

$$V_t := \{x \in M : x^0 = t\} = \bigcup_{r \in [r_0, r_1]} S(r) \text{ where } S(r) := \{x \in V : x^3 = r\},$$

$$T = \{x \in M : x^3 = r_0\} \cup \{x \in M : x^3 = r_1\}.$$

We use the following convention for indices: greek indices $\mu, \nu, \ldots$ run from 0 to 3; $k, l, \ldots$ are coordinates on V and run from 1 to 3; $A, B, \ldots$ are coordinates on ∂V and run from 1 to 2.

Let (g_{kl}, P^{kl}) be the Cauchy data for Einstein equations in a three-dimensional bounded volume V with boundary ∂V. This means that g_{kl} is a Riemanian metric on V and P^{kl} is a symmetric tensor density which we identify with the A.D.M. momentum [1], *i.e.*

$$P^{kl} = \sqrt{\det g_{mn}}(g^{kl} \operatorname{Tr} K - K^{kl}),$$

where K is the second fundamental form (external curvature) of the imbedding of V into the spacetime M.

The 12 functions (g_{kl}, P^{kl}) must fulfill four Gauss–Codazzi constraints:

$$P^l_{i|l} = 8\pi \sqrt{\det g_{mn}} T_{i\mu} n^\mu = -2(\det g_{mn})^{-\frac{1}{2}} \varepsilon_{ijk} \mathcal{E}^j \mathcal{B}^k, \tag{1}$$

$$\begin{aligned}(\det g_{mn})\mathcal{R} - P^{kl}P_{kl} + \tfrac{1}{2}(P^{kl}g_{kl})^2 &= 16\pi(\det g_{mn})T_{\mu\nu}n^\mu n^\nu \\ &= 2(\mathcal{E}^k\mathcal{E}^l + \mathcal{B}^k\mathcal{B}^l)g_{kl},\end{aligned} \tag{2}$$

where $\mathcal{E}^k$, $\mathcal{B}^l$ are vector densities:

$$\mathcal{E}^k = \sqrt{\det g_{ij}} E^k \quad \mathcal{B}^l = \sqrt{\det g_{ij}} B^l,$$

E^k and B^l are electric and magnetic field, respectively, fulfilling Maxwell equations. By $\mathcal{R}$ we denote the (three-dimensional) scalar curvature of g_{kl}, n^μ is a future time-like four-vector normal to hypersurface V, $T_{\mu\nu}$ is an energy-momentum tensor of Maxwell field, ε_{ijk} is a completely antisymmetric tensor and the calculations have been made with respect to the three-metric g_{kl} ("|" denotes covariant derivative, indices are raised and lowered *etc.*).

Einstein equations and the definition of the metric connection imply the first order (in time) differential equations for g_{kl} and P^{kl} (see Ref. [2], p. 525). The equations contain the lapse function N and the shift vector N^k as parameters.

To describe effectively the reduced phase space (the space of classes of gauge equivalent pairs (g_{kl}, P^{kl})) we can impose four gauge conditions which enable us to pick up a single representative within each gauge-equivalence class. The conditions are the following:

$$\frac{P^{33}}{\sqrt{g^{33}}} = 0 \quad \frac{k}{\sqrt{g^{33}}} = -\frac{2}{x^3}, \tag{3}$$

$$\frac{g_{AB}}{\lambda} = \frac{\sigma_{AB}}{\sigma}, \tag{4}$$

where σ_{AB} is the standard metric on a unit sphere S^2, ($\sigma_{AB}dx^A \otimes dx^B = d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes d\varphi$), $\sigma = \sqrt{\det \sigma_{AB}} (= \sin\theta)$ is a volume element on the unit sphere, $\lambda = \sqrt{\det g_{AB}}$ is a two-dimensional volume element on $S(r)$ and k is an extrinsic curvature of $S(r)$ with respect to the three-metric g_{kl}.

Conditions (3) are describing the two-dimensional surface of prescribed extrinsic curvature as a surface embedded in four-dimensional Einstein manifold (two-dimensional space of normal directions). They correspond to the choice of two-parameter family of surfaces (topologically spheres) described by the conditions $t = x^0 = \text{const.}$, $r = x^3 = \text{const.}$ and in general they lead to the nonlinear parabolic problems for functions t, r. The second part of gauge condition (3) is very similar to the one used in [7] (but not the same).

Let us notice that the condition for an apparent horizon in terms of the initial data simply says:

$$\frac{P^{33}}{\sqrt{g^{33}}} + \lambda k = 0,$$

and because of the first gauge condition both components of the above sum has to vanish and this simply means that the apparent horizon is an extremal surface ("saddle point") as two-surface imbedded in four-dimensional spacetime.

2. Basic inequality

Theorem:

Let Σ be an asymptotically flat spacelike surface which is nonsingular outside an extremal two-surface S_{min} (extremal as two-surface imbedded in spacetime M). Let $(g_{kl}, P^{kl}, E^k, B^l)$ be the perturbed initial data around the Reissner–Nordström solution such that:

1. *gauge conditions (3) and (4) hold,*
2. *for $r \leq r_0$ the initial data coincide with the background (Reissner–Nordström) ($g_{kl} = \eta_{kl},\ P^{kl} = 0,\ B^l = 0,\ E_A = 0,\ \sqrt{\det \eta_{kl}}E^3 = \sigma q$),*
3. *for $r > r_0$ we assume that perturbed data are sufficiently small and fulfill appropriate asymptotic conditions at spatial infinity such that constraints (1) and (2) possess appropriate solutions (see [6] for details) then*

$$16\pi m_{\mathrm{ADM}}^2 \geq A\,,$$

where m_{ADM} denotes the A.D.M. mass of perturbed data and A is an area of the apparent horizon.

Let us now integrate over V the scalar constraint in the same way as in [6]. We obtain the following inequality:

$$\begin{aligned} &-2\lim_{r\to\infty}\int\limits_{S(r)} r\sigma\Big[\frac{\lambda}{r^2\sigma}g^{33} - 1 - \frac{q^2}{r^2}\Big] + 2\int\limits_{S(r_0)} r\sigma\Big[\frac{\lambda}{r^2\sigma}g^{33} - 1 - \frac{q^2}{r^2}\Big] \\ &= 16\pi m_{\mathrm{ADM}} + 2\int\limits_{S(r_0)} r\sigma\Big(\eta^{33} - 1 - \frac{q^2}{r^2}\Big) \geq 0\,. \end{aligned} \tag{5}$$

If we take $S(r_0) = S_{\min}$ we get:

$$m_{\mathrm{ADM}} \geq \frac{1}{8\pi}\int\limits_{S_{\min}} r\sigma\Big(1 + \frac{q^2}{r^2} - \eta^{33}\Big) \geq \frac{1}{8\pi}\int\limits_{S_{\min}} r\sigma\,,$$

and the last inequality holds because $\eta^{33}|_{S_{\min}} = 0$ and $q^2 \geq 0$. It is easy to rewrite the result in a final form:

$$16\pi m_{\mathrm{ADM}}^2 \geq \frac{1}{4\pi}\Big(\int\limits_{S_{\min}} r\sigma\Big)^2 = 4\pi r_0^2 = A\,.$$

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