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STRING DESCRIPTION OF QUARKS DEGREES OF FREEDOM*

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This work presents a simple way of incorporating quark degrees of freedom (spin, charge and colour) into the classical string model. We introduce the model and derive from it the classical equations of motion.

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1. Introduction

In our previous paper [1] we have proposed the way one can take into account quark degrees of freedom (spin and colour) when constructing string description of a meson. We have rewritten the action describing propagation of a classical particle with spin and colour degrees of freedom in terms of fields living on the "worldsheet" bounded by particle's trajectory.

Although from technical point of view presented ideas were almost trivial (we have just used Stokes' theorem to convert line integral into the surface integral) let us stress, that the possibility of describing the *whole* meson dynamics in the string language does not seem to be trivial at all.

The present paper is organized as follows. First we extend the results of our previous work [1] to the case of the charged particles with colour in the external non–Abelian as well as electromagnetic fields. Then we check the consistency of our model by deriving the equations of motion from obtained action. We end our paper with the remark concerning the possibility of taking into account the non–zero quark mass by adding to the action the term depending on the internal (Gaussian) curvature of the string worldsheet.

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2. String in the external fields

Let us start with the action describing propagation of a classical particle in the external, non–Abelian gauge field A^{α}_{μ} [2–4]. We choose to work in Minkowski space–time because it allows to interpret closed particle trajectory as a process consisting of creation of particle and antiparticle (quark and aniquark), their propagation through space–time and finally their annihilation. The action reads:

$$S = \int dt(L_1 + L_2)\,, \tag{1}$$

where

$$\begin{aligned} L_1 &= -\tfrac{1}{2}(e^{-1}\dot{x}^2 + em^2)\,,\\ L_2 &= \tfrac{i}{2}(\psi^{\mu}\dot{\psi}_{\mu} + e^{-1}\chi\psi^{\mu}\dot{x}_{\mu})\tfrac{i}{2}(\psi_5\dot{\psi}_5 + m\chi\psi_5)\\ &\quad + \tfrac{i}{2}\theta^{a}\dot{\theta}^{a} - gI^{\alpha}(A^{\alpha}_{\mu}\dot{x}^{\mu} - \tfrac{i}{2}e\psi^{\mu}F^{\alpha}_{\mu\nu}\psi^{\nu})\,. \end{aligned} \tag{2}$$

Here, correspondingly, x^{μ} and e are bosonic (commuting) variables, the position four-vector of particle's trajectory and the square root of one-dimensional metric on the worldline and ψ_{μ}, ψ_5, χ and θ^{a} are fermionic (anticommuting) ones, introduced to describe intrinsic degrees of freedom (spin and colour). For definiteness we have chosen θ^{a} to be real Grassmann variables (the other alternatives are equivalent, see [2]) and defined the isospin of a particle by the following formula:

$$I^{\alpha} = \tfrac{1}{2}\theta^{a}T^{\alpha}_{ab}\theta^{b}\,,$$

where T^{α} are generators of gauge group under consideration.

The action above is invariant under group of reparametrizations $t \to f(t)$ with $\dot{f}(t) > 0$, accompanied by the transformations:

$$\begin{aligned} e &\to e' = e\dot{f}\,,\\ \chi &\to \chi' = \chi\dot{f}\,, \end{aligned} \tag{3}$$

with $x^{\mu}, \psi^{\mu}, \psi_5$ and θ^{a} unchanged.

The action is also invariant under supersymmetry transformations whose infinitesimal form generated by the anticommuting variable $\alpha(t)$ reads:

$$\begin{aligned} \delta x^{\mu} &= i\alpha\psi^{\mu}\,,\\ \delta\psi^{\mu} &= -\alpha(\dot{x}^{\mu} - \tfrac{i}{2}\chi\psi^{\mu})/e\,,\\ \delta e &= -i\alpha\chi\,,\\ \delta\chi &= 2\dot{\alpha}\,,\\ \delta\psi_5 &= m\alpha\,,\\ \delta\theta^{a} &= g\alpha\psi^{\mu}A^{\alpha}_{\mu}T^{\alpha}_{ab}\theta^{b}\,, \end{aligned} \tag{4}$$

and finally it does not change under the action of the local gauge group:

$$A_\mu \equiv A^\alpha_\mu T^\alpha \to \omega A_\mu \omega^{-1} + \tfrac{i}{g}\partial_\mu \omega\omega^{-1}\,,$$
$$\theta \equiv (\theta^a) \to \omega\theta\,, \tag{5}$$

where ω is an arbitrary group element.

It is straightforward to add an interaction with an external electromagnetic field in the form:

$$L_3 = -q(C_\mu \dot{x}^\mu - \tfrac{i}{2}e\psi^\mu G_{\mu\nu}\psi^\nu)\,, \tag{6}$$

where q is the electromagnetic charge of the particle, C_μ is the external electromagnetic field and $G_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$. The complete action is thus of the form:

$$S = \int dt(L_1 + L_2 + L_3)\,. \tag{7}$$

The action above is invariant under (3), (4) and (5) as well as under Abelian gauge transformation:

$$C_\mu \to C_\mu + \partial_\mu \eta\,, \tag{8}$$

where $\eta(x)$ is an arbitrary function.

Let us make a comment on the appearance of the second, non-minimal term in the Lagrangian (6). It has to be present in the action integral in order to insure the closure of Poisson bracket (or rather — because of presence of constraints — Dirac brackets) algebra. It also appears naturally in the superspace formulation of presented theory.

Let us now take a compact surface Σ in the space–time and consider its boundary $\partial\Sigma$ as a trajectory of the $Q\bar{Q}$ pair (to have such an interpretation we have to restrict ourselves to surfaces, such that vectors tangent to their boundaries are always time-like). Next we parametrize $\partial\Sigma$ with t and continue the fields $x, e, \psi_\mu, \psi_5, \chi$, and θ^a that appear in the action (7) in a smooth way over the whole surface Σ.

To simplify our notation we define:

$$D^{(k)}_{ab} := \delta_{ab}\partial_k + ig\partial_k x^\mu A^\alpha_\mu T^\alpha_{ab}\,,$$
$$f(t) := m\chi\psi_5 + \psi^\mu(gI^\alpha F^\alpha_{\mu\nu} + qG_{\mu\nu})\psi^\nu\,. \tag{9}$$

Let us now parametrize Σ by $\xi^k, k = 0, 1$. Points that belong to $\partial\Sigma$ are thus parametrized by $\xi^k_1(t)$ and $\xi^k_2(t)$ where the lower index serves to distinguish between the quark and the aniquark trajectory.

It is now straightforward to use Stokes' theorem and rewrite (7) in the form:

$$S = \int_\Sigma d^2\xi[\tfrac{i}{2}\varepsilon^{kl}\{\partial_k\psi^\mu\partial_l\psi_\mu + \partial_k\psi_5\partial_l\psi_5 + \partial_k(e^{-1}\chi\psi_\mu)\partial_l x^\mu + \partial_k(\theta^a D^{(k)}_{ab}\theta^b)\}$$
$$+ \partial_k C_\mu \partial_l x^\mu + \tfrac{i}{2}div\vec{h}] - \tfrac{1}{2}\int_{\partial\Sigma} dt(e^{-1}\dot{x}^2 + em^2)\,, \tag{10}$$

where $\vec{h}(\xi)$ is an arbitrary function which has only to satisfy the condition $\vec{h} \cdot \vec{n} = f$ for ξ that belong to $\partial\Sigma$, $\vec{n}$ being the unit vector tangent to Σ and normal to $\partial\Sigma$ (in the embedding space metric). For n to be defined on the whole boundary $\partial\Sigma$, we have to smooth $\partial\Sigma$ in the points of creation and annihilation of the quark-antiquark pair. It results in the failing of the time-likeness of $\partial\Sigma$ in the neighbourhood of those points and spoils there our $Q\bar{Q}$ interpretation of $\partial\Sigma$. Nevertheless, it doesn't cause any serious problems, because quarks creation and annihilation are of pure quantum nature and we cannot insist on having the correct classical description of the creation and annihilation processes within the region of the size of the quarks Compton wavelength.

We have separated the mass term from the above expression, as it is rather unnatural to rewrite it in the proposed way and, moreover, as we discuss in the last section of this paper, seems to exist another, more convenient way of generating the mass term in the string description of mesons.

It is clear from the construction that the above action is invariant under gauge transformations (5), (8); it is also invariant under reparametrizations of the "worldsheet" Σ and SUSY transformations generated by the infinitesimal Grassmann variable that now depends on ξ^k [1].

We can thus say that we have managed to describe quark's degrees of freedom in the language of string variables.

3. Equations of motion

Before presenting equations of motion derived from (10) let us pause for a moment and remind some facts connected with the variational principle in the presence of Grassmann variables [5]. As the action describing them is of the first degree in time derivatives, we are not allowed to fix their values both in the initial and final times. On the other hand, if we fix only their initial value and consider the final one to be arbitrary, we will exclude some "physical" solutions of equations of motion.

Let us illustrate this point on the simplest possible example, namely on the case of the free evolution of a Grassmann variable described by the action:

$$S = \int dt \theta^a \dot{\theta}^a \,. \tag{11}$$

If we compute the equation of motion following from the action above under assumption $\delta\theta^a(t_1) = 0$, $\delta\theta^a(t_2)$ – arbitrary, we arrive at the equation:

$$\dot{\theta}^a(t) = 0 \,,$$

together with the boundary condition $\theta^a(t_2) = 0$. Thus the only possible solution is $\theta^a(t) = 0$, and we loose all the solutions of the form $\theta^a(t) = \theta^a_0$ for θ^a_0 being constant, non–zero Grassmann number.

To remedy this we have to change our variational principle. If we add to the action the "boundary" term in the form:

$$\theta^a(t_1)\theta^a(t_2)$$

and impose on variations of θ in the initial and final times the conditions:

$$\delta\theta^a(t_1) + \delta\theta^a(t_2) = 0\,,$$

we are left only with the differential equation $\dot{\theta}^a(t) = 0$ without any further conditions.

In the same way one can proceed also in the case of surface integrals involving Grassmann numbers. By the appropriate modification of the action principle we can get rid of unwanted boundary terms with support on initial and final times, and then we are left only with the "bulk" equations of motion and boundary conditions from the integrals over time-like boundaries.

Let us now add the action (10) to the action describing the simplest bosonic string, *i.e.* the Nambu–Goto string:

$$S_{NG} = -\gamma \int_{\Sigma} d^2\xi \sqrt{-g}\,, \tag{12}$$

where g stands for determinant of the induced metric $g_{ij} = \partial_i x^\mu \partial_j x_\mu$.

In the conformal gauge $(\partial_0 x^\mu \pm \partial_1 x^\mu)^2 = 0$ one obtains for the interior of the string the d'Alembert equation:

$$(\partial_0^2 - \partial_1^2)x^\mu(\xi) = 0\,, \tag{13}$$

together with a set of boundary conditions. To simplify their form let us denote ξ^0 by τ and ξ^1 by σ and choose them in such a way, that the string ends are described by $\sigma = 0$ and $\sigma = \pi$ respectively.

First, upon variation of e, χ and ψ_5 one gets the equations:

$$\begin{aligned}(\dot{x}^2 - i\chi\dot{x}^\mu\psi_\mu)/e^2 - m^2 + i\psi^\mu(gI^\alpha F^\alpha_{\mu\nu} + qG_{\mu\nu})\psi^\nu = 0\,,\\ (\dot{x}^\mu\psi_\mu + m\psi_5)/e = 0\,,\\ 2\dot{\psi}_5 - m\chi = 0\,,\end{aligned} \tag{14}$$

where dot means differentiation with respect to τ for $\sigma = 0, \pi$.

Using the reparametrization invariance (3) we can put e to be equal to $\frac{1}{m}$. Next we use SUSY transformation (4) to set $\chi = 0$. Then we get from (14) $\dot{\psi}_5 = 0$ and we can eliminate ψ_5 by constant SUSY transformation. The two remaining equations of (14) and the rest of our boundary conditions now take the form:

$$
\begin{aligned}
m\dot{x}^2 - m^2 + i\psi^\mu(gI^\alpha F^\alpha_{\mu\nu} + qG_{\mu\nu})\psi^\nu &= 0\,, \\
\dot{x}^\mu \psi_\mu &= 0\,, \\
\dot{\psi}_\mu - \tfrac{i}{m}(gI^\alpha_{\mu\nu} + qG_{\mu\nu})\psi^\nu &= 0\,, \\
\dot{\theta}^a + ig(A^\alpha_\mu \dot{x}^\mu - \tfrac{i}{2m}\psi^\mu F^\alpha_{\mu\nu}\psi^\nu)T^\alpha_{ab}\theta^b &= 0\,, \\
m\ddot{x}_\mu - \gamma x'_\mu - gI^\alpha(\partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\nu)\dot{x}^\nu + gA^\alpha_\mu \dot{I}^\alpha - qG_{\mu\nu}\dot{x}^\nu & \\
-e\psi^\rho(gI^\alpha \partial_\mu F^\alpha_{\rho\lambda} + q\partial_\mu G_{\rho\lambda})\psi^\lambda] &= 0\,. \qquad (15)
\end{aligned}
$$

Here prime denotes differentiation with respect to σ.

We see that the last equation in (15) has the form of the Newton equations for the string ends. Various components of the acting force can be interpreted as coming from the string tension (the first one), from interaction of colour and electric charges with external fields (the second a and the third ones) and from the interaction of the particle's spin (let us remind that the Pauli–Lubański four-vector is constructed in the pseudoclassical description from the fields ψ_μ) with their gradients (the last two terms).

The equations of motion for the string in the external electromagnetic field alone (which is the special case of the one above) were obtained and analysed in details in [6].

4. Concluding remarks

To conclude, we have finally managed to describe classical (or rather — because of the inclusion of a Grassmann numbers — pseudoclassical) dynamics of a meson ($Q\bar{Q}$ pair) in the language of "string" variables. This description not only preserves all the symmetries of the classical particle's action, but also indicates the possibility of some extended symmetry of the system, that can be helpful for instance in the quantization procedure [1]. The coincidence of the equations (15) with the ones obtained from the standard particle's action [2] indicates self-consistency of the presented procedure.

Let us end this paper with the remark concerning the mass term in the string action. It was shown in [7], that the only *boundary* terms one can construct from the fields $x^\mu(\xi)$ and their derivatives up to second order, and which fulfill the requirement of being both reparametrization and Poincaré invariant, are — in four space-time dimensions — the term proportional to the curvature scalar R and the term that gives the number of self-intersections of the worldsheet. Their addition to the Nambu–Goto action forces the string ends to move with the speed less then the speed of light. So they effectively act as the mass term for the string ends, though in rather unusual way, because the equations one obtains in this way do not have the form of the Newton's equations. So the next task one should take would be to investigate the dynamics of a system described by the action (10) with the mass term replaced by the term with the curvature scalar. Work in this direction is in progress.

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