

ON MEAN FREE PATH STATISTICS OF RELATIVISTIC HEAVY IONS IN NUCLEAR EMULSION

H.C. Rosu

Instituto de Física, Universidad de Guanajuato
Apartado Postal E-143, 37150 León, Gto, México
e-mail address: rosu@ifug.ugto.mx

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We survey at the general level, and in a rather wide and historical context, some less known mathematical details of various mean free path statistics in nuclear emulsion as applied to relativistic projectile fragmentations. A number of comments are provided on Feller's paradox, ordered, censored, and truncated statistics, Erlang and Poisson observers of nuclear emulsion. All these are related to the statistical approaches used about a decade ago at Berkeley Bevalac for *anomalous*, some of which should be considered as standard methods for mean free paths' estimators of relativistic heavy ions in emulsion.

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The mean free path λ is a statistical concept introduced by Clausius in 1858 that appears in physics in two situations:

- (i) In the qualitative evaluation of kinetic phenomena through the following approximation of the collision term in Boltzmann equation

$$\text{Coll}(f) \approx -(f - f_0)/\tau \approx -\frac{\bar{v}}{\lambda}(f - f_0). \quad (1)$$

- (ii) In detection problems, where λ is the parameter of the exponential distribution of interacting distances

$$f(x)dx = \exp(-x/\lambda)dx/\lambda. \quad (2)$$

In the following we shall be interested in determining the mean free paths of relativistic heavy ions in nuclear emulsion. The first tracks of

relativistic heavy ions in emulsion stacks were obtained in 1948 in cosmic ray experiments by Bradt and Peters [1]. Shortly after, in the period 1954-1957, well-known authors in cosmic ray physics reported mean free paths smaller than expected for some isolated long heavy-ion fragmentation cascades but this was considered as being due to the highly biased sample of events recorded by a detector of limited dimensions like an emulsion stack [2].

The mean free path and the interaction cross-section are connected by $\lambda = (n\sigma)^{-1}$ where n is the concentration of the scattering centres. This simply corresponds to the case of one molecule moving through the rest of the molecules taken as a fixed background in molecular physics. Since emulsion is a heterogeneous medium, one should make use of the more general formula $\lambda = \sum (n_i \sigma_i)^{-1}$, where n_i corresponds to the composition of the emulsion in use. For σ_i one can consider the geometrical cross section from molecular physics with the difference that the nuclear interaction is present only when a certain overlap of the cross sections occurs (Bradt-Peters formula)

$$\sigma_{BP} = \pi(R_p^2 + R_t^2 - \delta) = \pi r_0^2 (A_p^{1/3} + A_t^{1/3} - b)^2. \quad (3)$$

The indices p and t stand for projectile and target. The overlap parameter b can be related to the radius parameter r_0 . In nuclear physics the radius parameter is in the range $r_0 = (1.15 \div 1.45) \cdot 10^{-13}$ cm. In this way, one can obtain a theoretical (Bradt-Peters) mean free path and compare it with the experimental (statistical) one. Geometrical arguments are very powerful as far as quantum mechanics does not come into play too much, as it is the case of projectile fragmentations in nuclear emulsion. However we face the problem of the precise experimental determination of the mean free path. One can think of two methods for that:

- (i) For a big number of events, one can plot the exponential distribution of the interaction paths and determine the mean free path graphically. This method is too empirical.
- (ii) A very general method is certainly based on the maximum likelihood function in its various forms.

Thirty years ago it was in vogue [3] to use the so-called Bartlett functions S and S' [4]. In this procedure, the Bartlett function S' is plotted against $1/\lambda$. This function is normally distributed with zero mean and dispersion equal to one. The maximum likelihood value of the mean free path is given by the equation $S'(\lambda^*)$, that is the intersection with the $1/\lambda$ axis. Moreover, the errors (one standard deviation) are given by $S'(\lambda) = \pm 1$. Unfortunately the expression for $S'(\lambda)$ is very complicated. Some time ago, Yost [5] generalized the results of Bartlett to the case when the observations are affected by experimental errors. These are important in the time

of flight measurements, but for nuclear emulsions the errors of the measured distances are negligible.

New statistical methods have been introduced in 1980, when Friedländer *et al.* [6] presented their interesting statistical treatment of Bevalac projectile fragmentations in Ilford G5 emulsions. Despite their conclusions concerning the *anomalons*, the Berkeley works have been a natural continuation of the cosmic ray works in the fifties with the essential difference that the emulsion was irradiated with relativistic heavy ion beams from Berkeley Bevalac. Moreover, emulsion pellicles were exposed to 200 A GeV O¹⁶ and S³² ion beams at CERN SPS [7] and it is to be expected that emulsion groups will take their part in future heavy ion relativistic experiments. Some of the statistical methods of Friedländer *et al.* should be considered as standard ones for mean free paths estimators of heavy ions in emulsion at all these energies.

The first systematic studies of the heavy-ion mean free paths in nuclear emulsion were done by Judek [8] who reported anomalies in the case of projectile fragments (Judek effect/*anomalons*). The method used by Judek and later by Friedländer *et al.* to obtain the mean free path is based on the formula

$$\lambda^* = T/n, \quad (4)$$

where λ^* is the maximum likelihood estimate, T is the total length to be watched until one gets n interactions (emulsion stars) on both interacting and non-interacting tracks. Usually T has been taken in bins of 1 cm starting from the mother stars and in the first 2-3 cm a distance dependent mean free path for projectile fragments was reported, which has been fitted by means of *anomalons*.

We pass now to the scope of the present work which is to sketch some less known mathematical details of the mean free path statistics of relativistic heavy ions in nuclear emulsion.

The first thing one should realize from the mathematical point of view is that the propagation of relativistic heavy ion beams in emulsion is a Poisson stochastic point process along the direction of propagation [9]. More exactly, due to some lost interactions it is a rarefied or damaged Poisson process [10]. This is directly suggested by the visual aspect of the fragmentation processes, especially for projectile fragmentations, as they are seen by means of a microscope. At relativistic energies the ionization loss of heavy ions is much less than their kinetic energies, and what one may see are rather long ionization tracks interrupted occasionally by some stars, representing the nuclear interactions/collisions with the emulsion nuclei. Considering these stars as random points along the propagation/ionization lines one naturally

arrives at the idea of a random stochastic process in the space variable. Since the distances between points are exponentially distributed, the stochastic process is by definition a Poisson point process.

We would like first to warn on the presence of paradoxes. Stochastic processes as the entire theory of probabilities are affected by paradoxes generated by the intricacies of the statistical way of thinking. For Poisson processes, Zernike discussed already in 1928 the so-called "*Weglängenparadoxon*" [11]. A form of Zernike's paradox is well-known in solid state physics with regard to the scattering of electrons in crystals [12]. It refers to the mean time interval between the last and next collision for the Poisson electron process. This is about twice the normal average time between collisions. On the other hand in the well-known book of Feller [13] one can find the *waiting time paradox*. Previously we applied the *waiting time paradox* (or more appropriately for emulsion — *waiting distance paradox*) to the anomalous artifact [9]. This paradox shows up whenever the measurements are performed from the very beginning of the Poisson process, which consequently is not yet stationary (ergodic). It happens that this is just the case of projectile fragments in emulsion within the statistical treatment of Judek and later of Friedländer *et al.* When this paradox is taken into account, there comes out that near the origin of the Poisson process, the mean free path is distance dependent in the following way

$$\lambda_w = \lambda[1 - \frac{1}{2} \exp(-x/\lambda)] \quad (5)$$

for $x \ll \lambda$. Hence, whenever one is trying to detect the very first collision will enter the paradox and will find out a much smaller mean free path. For the particular case $x = 0$, the mean free path will be only half of the normal one. Indeed the Berkeley group noticed that a mean free path twice smaller than the normal one could be accommodated by the data assuming that *all* projectile fragments are born anomalous and shortly after disappear. So, this is just the statement of the *waiting distance paradox* [9, 13]. Why does not the paradox show up for the primaries? In this case one may invoke the loss of memory property of Poisson processes, and so to say, is old enough to be stationary. The point is that for the projectile fragments the 'mother stars' play the role of the origin of the stochastic process, and therefore one should be cautious with the non-stationarity property. A complication arises due to the fact that there is not one stochastic process, but several ones corresponding to the different charge values [9].

The most important statistical test used by the Berkeley group, considered to be independent of parametrization ($\lambda_Z \approx AZ^{-b}$) is an F test in the disguised form of the integral probability, which is known to be uniformly distributed between 0 and 1 by a textbook theorem [13]. However, the F test, as a variance ratio test, will check only the normality of the maximum

likelihood estimates around the true mean free path. These estimates are not ideal gaussian variables in the case of small samples like those ones of fixed Z and at small distances from the origin. Besides, the F test is known not to be a very robust test against deviations from normality [14]. This non-robustness of the F test is clearly demonstrated by employing other tests. For example, the null hypothesis of no anomalous is not rejected at a confidence level of 0.95 by a more robust test, like the Kolmogorov-Smirnov test which is based on the largest deviation of the empirical distribution from the expected cumulative one [15]. Moreover, the K-S test has also the advantage of being independent of the theoretical distribution.

Pshenin and Voinov [16] have shown that for small samples, the maximum likelihood estimate for λ at fixed number of primary tracks (say N) and variable number of interactions (say n) is biased to lower values, according to the following formula

$$B = \lambda^* - \lambda = -D/[1 - \exp(-D/\lambda)] + ND < 1/n >, \quad (6)$$

where B is the bias and D is the censoring distance. This result has been known since a long time in statistics [17, 18]. The exact formula for the mean $< 1/n >$ is not very simple but one can use the approximation [19]

$$\langle 1/n \rangle = [N(1 - \exp(-D/\lambda)) - \exp(-D/\lambda)]^{-1} \quad (7)$$

working very well for values $X = N(1 - \exp(-D/\lambda)) > 10$. An even more accurate result in the range $X > 5$ is given in [18]

$$\langle 1/n \rangle = \frac{N-2}{N} [(N-1)(1 - \exp(-D/\lambda))]^{-1}. \quad (8)$$

All these results do apply only for very small samples.

Garpman *et al.* [20] proposed as the maximum likelihood estimate

$$\lambda^* = T/(n+1). \quad (9)$$

The reason is that this estimate is finite also for $n = 0$ and the Monte Carlo simulation is better. This is just like saying that if you did not find any interaction the mean free path is equal to the measured distance.

We now go on with a short presentation of censored and truncated distributions of the emulsion tracks. The censored distribution is the distribution of both interacting and noninteracting tracks, while the truncated one is based only on the interacting tracks included in the censoring/truncating distance D . One might think that the censoring distribution carries more information. However this is not so. Both the censored and truncated distributions lead to the same maximum likelihood estimate. This can be shown by means of results from the ordered statistics [14].

Suppose there are k interacting tracks from a total of N tracks within the truncation distance and we order them from smaller to larger ones: $0 < x_1 < x_2 < \dots < x_k < D$. Then the maximum likelihood function constructed from these distances only is [14]

$$L = \frac{n!}{(n-k)!} \lambda^{-k} \exp(-y_k/\lambda), \quad (10)$$

where

$$y_k = \sum_{j=1}^{k-1} x_j + (n-k+1)x_k. \quad (11)$$

The resulting ordered maximum likelihood estimate will be

$$\lambda_o^* = k^{-1} \sum_{j=1}^{k-1} x_j + k^{-1}(n-k-1)x_k = k^{-1} \sum_{j=1}^k x_j + k^{-1}(n-k)x_k. \quad (12)$$

This estimate is practically the same as the censored one which is given by

$$\lambda_c^* = k^{-1} \sum_{j=1}^k x_j + k^{-1}(n-k)D \quad (13)$$

for the same sample of k events. Actually censoring can be done in a number of ways [21]. In censoring of type I, the censoring distance is fixed and the number of events is a random variable. In censoring of type II, the number of events k is fixed and the censoring distances x_k become random variables. The ordered estimate can be considered as a censored one of type II, whereas the Berkeley estimate is of type I. A third type of censoring, in which the censoring distance is different from track to track (so-called random censoring used in life test data [22]) gives no new result for the estimation problem. There are only minor differences in the estimate variances for all types of censoring [21].

Finally, we want to comment on the type of observers paying attention to a stochastic process. This problem is discussed by Egon [23] in connection with the theory of signal detection. In the case of emulsion detectors, the common observers are gamma (Erlang) observers. They scan the Poisson process until a certain number of events are registered. For them the distances between the events are important in doing their statistics. Poisson observers instead are looking for a preassigned number of events in a given distance interval. However, the estimation problem gives the same result as for the Erlang observers. This is so because in both cases actually what we have is the same family of exponential distributions.

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