SKYRMIONS AND THEIR PION MULTIPOLE MOMENTS *

N.S. Manton

Department of Applied Mathematics and Theoretical Physics
University of Cambridge
Silver Street, Cambridge CB3 9EW, UK

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The pion multipoles of static Skyrmions with arbitrary baryon number are discussed. It is shown that a monopole moment cannot occur. The forces between separated Skyrmions are dominated by the dipole-dipole interaction, in general, and this gives some understanding of the structure of Skyrmions with baryon number up to four.

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Skyrme proposed a version of the non-linear sigma model to model both pions and nucleons [1]. The pions are the quantized waves of the perturbative theory and the nucleons are (spinning) solitons. The basic Skyrme model has an SU(2)-valued scalar field $U(t, \boldsymbol{x})$, and the action is constructed from the (right) SU(2) Lie algebra-valued current $R_\mu = (\partial_\mu U)U^{-1}$. The action (appropriately scaled) is

$$S = \int d^4x \left\{ \tfrac{1}{2}\,\mathrm{Tr}(R_\mu R^\mu) + \tfrac{1}{16}\,\mathrm{Tr}([R_\mu, R_\nu][R^\mu, R^\nu]) \right\}, \tag{1}$$

where the metric has signature $(- + ++)$.

There is chiral symmetry $U \to U_0\; U\; U_1$ where $U_0, U_1 \in \mathrm{SU}(2)$, but this is spontaneously broken since one must choose a vacuum field $U = \mathrm{const.}$ By convention, $U = 1$ is chosen as the vacuum, and this is invariant under isospin transformations $U \to U_0\; U\; U_0^{-1}$. The Goldstone bosons of the broken chiral symmetry are three in number – they are identified with the pions. More precisely, one writes the Skyrme field as

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$$U = \sigma + i\,\boldsymbol{\pi}\cdot\boldsymbol{\tau} \tag{2}$$

where $\boldsymbol{\pi}$ denotes the pion fields, $\boldsymbol{\tau}$ are Pauli matrices, and σ is constrained by $\sigma^2 = 1 - \boldsymbol{\pi}\cdot\boldsymbol{\pi}$. The vacuum is $\sigma = 1, \boldsymbol{\pi} = 0$. If the $\boldsymbol{\pi}$ fields are small and $\sigma \simeq 1$, then the action simplifies to $S = -\int d^4x\ \partial_\mu\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi}$, so the pions are massless.

The classical field equation obtained from the action S is

$$\partial_\mu\left(R^\mu + \tfrac{1}{4}[R_\nu,[R^\nu,R^\mu]]\right) = 0\,. \tag{3}$$

This states that the current $\tilde{R}^\mu \equiv R^\mu + \frac{1}{4}[R_\nu,\,[R^\nu,R^\mu]]$ is conserved. There is a further topological current

$$B^\mu = -\frac{1}{24\pi^2}\epsilon^{\mu\nu\sigma\tau}\ \mathrm{Tr}(R_\nu R_\sigma R_\tau)\,, \tag{4}$$

where $\epsilon^{\mu\nu\sigma\tau}$ is totally antisymmetric, and $\epsilon^{0123} = 1$. This is kinematically conserved since $\partial_\mu R_\nu = (\partial_\mu\partial_\nu U)U^{-1} - R_\nu R_\mu$, hence

$$\partial_\mu B^\mu = -\frac{1}{24\pi^2}\epsilon^{\mu\nu\sigma\tau}\ \mathrm{Tr}(R_\mu R_\nu R_\sigma R_\tau) = 0\,.$$

The last equality holds because of the cyclicity of the trace and the antisymmetry of the ϵ-tensor. The conserved charge associated with the current B^μ is

$$B = \int d^3x\ B^0 \tag{5}$$

and it is in fact an integer if U satisfies the boundary condition $\lim_{r\to\infty} U = 1$. (Here and below, r denotes $|\boldsymbol{x}|$.) B is identified with the baryon number of the field U.

For each integer B one can seek the static solution of the field equation (3) of lowest energy. This is a classical model of a nucleus of baryon number B, and the starting point for a quantized model. For static fields, the energy is

$$E = \int d^3x\left\{-\tfrac{1}{2}\,\mathrm{Tr}(R_iR_i) - \tfrac{1}{16}\,\mathrm{Tr}([R_i,R_j][R_i,R_j])\right\}, \tag{6}$$

and the field equation is

$$\partial_i\left(R_i + \tfrac{1}{4}[R_j,[R_j,R_i]]\right) = 0\,, \tag{7}$$

where latin indices run from 1 to 3. We shall refer to the minimal energy solutions, for $B \neq 0$, as Skyrmions (traditionally only the lowest energy solution with $B = 1$ was called a Skyrmion). They are known from numerical calculations for all $|B| \leq 6$ [2]. The solutions for negative B are obtained from those for positive B by replacing U by U^{-1}; for $B = 0$ it is the vacuum. The solutions have remarkable symmetry properties. The $B = 1$ Skyrmion is spherically symmetric; it is of the hedgehog form $U = \cos f(r) + i \sin f(r)\, \hat{\boldsymbol{x}} \cdot \boldsymbol{\tau}$. The $B = 2$ Skyrmion is axially symmetric, the $B = 3$ Skyrmion has tetrahedral symmetry, and the $B = 4$ Skyrmion has cubic symmetry; the $B = 5$ and $B = 6$ Skyrmions have low symmetry.

Let E_B denote the energy of the Skyrmion of baryon number B. It is believed that for all B, $E_B < E_{B'} + E_{B''}$ where B', B'' are any integers such that $B' + B'' = B$. A Skyrmion cannot therefore break up into smaller Skyrmions without an input of energy. Theoretical and numerical evidence does not contradict this inequality, although there is no proof. (There is a fairly convincing unpublished argument, due to Castillejo and Kugler, but it has a loophole).

The asymptotic field of a Skyrmion has $\sigma \simeq 1$ and $\boldsymbol{\pi}$ small. The field equation linearises and reduces to $\nabla^2 \boldsymbol{\pi} = 0$. Solutions of this Laplace equation have a multipole expansion, but the non-linear nature of the full field equation implies that only the leading multipole, and possibly a small number of subleading multipoles, are meaningful.

One might expect a Skyrmion could have a leading monopole term in its asymptotic field

$$\boldsymbol{\pi} \sim \frac{\boldsymbol{A}}{4\pi r}\,. \tag{8}$$

Certainly there is no problem constructing a finite energy Skyrme field with this asymptotic behaviour, although it cannot be very symmetric. However the field equation forbids this asymptotic behaviour. To see this, consider the integral

$$\int_{S^2(L)} \tilde{R}_i \, dS_i \tag{9}$$

over a sphere of radius L, with $L \to \infty$. The field equation (7) implies that this integral vanishes, but the asymptotic behaviour (8) implies that $\tilde{R}_i \sim -\boldsymbol{A} \cdot \boldsymbol{\tau}\, x_i / 4\pi r^3$, making the integral $-\boldsymbol{A} \cdot \boldsymbol{\tau}$. So $\boldsymbol{A} = 0$. (We have not seen it pointed out before that a Skyrmion can have no monopole moment.)

A Skyrmion generically has a dipole as its leading multipole

$$\pi^a \sim \frac{C_i^a x_i}{4\pi r^3}\,, \tag{10}$$

with C_i^a the dipole moment matrix. We can use the symmetry of the model to put this in standard form. By acting with a rotation and an isospin rotation, we can reduce C_i^a to a diagonal matrix

$$C_i^a = \begin{pmatrix} c_1 & & \\ & c_2 & \\ & & c_3 \end{pmatrix}, \tag{11}$$

since any 3×3 matrix can be expressed as $O_1 \; D \; O_2$, where O_1 and O_2 are rotation matrices and D is diagonal. In this standard form, the three components of the asymptotic pion field are those of three mutually orthogonal scalar dipoles along the three Cartesian axes. Any hedgehog solution of the field equation has $c_1 = c_2 = c_3$. We denote these diagonal entries for the $B = 1$ Skyrmion by c, whose numerical value is $c = 27.0$. The $B = 2$ Skyrmion has $c_1 = c_2 = 0$ and $c_3 \neq 0$, where the 3rd axis is the axis of symmetry. The $B = 3$ Skyrmion is not everywhere of hedgehog form, but it is asymptotically. The dipole moment matrix has the opposite sign, but similar magnitude, to that of the $B = 1$ Skyrmion. From afar, therefore, the $B = 3$ Skyrmion looks like a $B = -1$ Skyrmion (the anti-Skyrmion). The $B = 4$ Skyrmion, by virtue of the way its cubic symmetry is realised, has no asymptotic pion dipole. Its leading pion multipole is a quadrupole. Suitably orientated this is

$$\begin{aligned} \pi^1 &\sim \frac{d\left(x_2^2 - x_3^2\right)}{r^5}, \\ \pi^2 &\sim \frac{d\left(x_3^2 - x_1^2\right)}{r^5}, \\ \pi^3 &\sim \frac{d\left(x_1^2 - x_2^2\right)}{r^5} \end{aligned} \tag{12}$$

for some constant d.

The interaction energy of two well-separated Skyrmions is determined by their leading pion multipole moments. This in turn determines the forces and torques the Skyrmions exert on each other. The formula for the interaction energy is the same as in electrostatics, except that it has the opposite sign because the pion field is a scalar. Suppose two Skyrmions have non-vanishing dipole moments C_i^a and $\tilde{C}_i^a$, and their centres are separated by the vector $\boldsymbol{X}$. Their interaction energy is [3]

$$V = -\frac{1}{4\pi|\boldsymbol{X}|^5}\left(C_i^a \, \tilde{C}_i^a \, |\boldsymbol{X}|^2 - 3C_i^a\tilde{C}_j^a \; X_i \; X_j\right). \tag{13}$$

This formula implies that for two $B = 1$ Skyrmions at fixed separation $\boldsymbol{X}$, the interaction energy is most negative if the second Skyrmion is rotated relative to the first by π about an axis orthogonal to $\boldsymbol{X}$. With this relative orientation the energy becomes more negative as the Skyrmions approach each other, and there is maximal attraction.

It is possible to arrange four $B = 1$ Skyrmions so that the force between any pair is maximally attractive. One must locate the Skyrmions at the vertices of a regular tetrahedron, say at

$$\begin{aligned} {}^{(1)}\boldsymbol{X} &= (L, L, L)\,, & {}^{(2)}\boldsymbol{X} &= (-L, -L, L)\,, \\ {}^{(3)}\boldsymbol{X} &= (-L, L, -L)\,, & {}^{(4)}\boldsymbol{X} &= (L, -L, -L) \end{aligned} \tag{14}$$

and give them orientations so that the dipole moment matrices are, say

$$\begin{aligned} {}^{(1)}C &= \begin{pmatrix} c & & \\ & c & \\ & & c \end{pmatrix}, & {}^{(2)}C &= \begin{pmatrix} -c & & \\ & -c & \\ & & c \end{pmatrix}, \\ {}^{(3)}C &= \begin{pmatrix} -c & & \\ & c & \\ & & -c \end{pmatrix}, & {}^{(4)}C &= \begin{pmatrix} c & & \\ & -c & \\ & & -c \end{pmatrix}. \end{aligned} \tag{15}$$

The first and second Skyrmions, for example, are separated by the vector $(-2L, -2L, 0)$, and the orientation of the second is obtained from that of the first by rotating by π about the x_3-axis. They therefore maximally attract. A maximally attractive configuration of three $B = 1$ Skyrmions at the vertices of an equilateral triangle is obtained simply by removing one Skyrmion from the configuration above.

If a maximally attractive configuration of $B = 1$ Skyrmions relaxes, by allowing the Skyrmions to approach each other, it will settle at the minimal energy Skyrmion. In the process the net dipole moment doesn't change much. This helps us to understand the dipole moments of the Skyrmions for $B \leq 4$. For notice that the configuration of four $B = 1$ Skyrmions with positions and dipole moments as in (14) and (15) has total dipole moment zero, since ${}^{(1)}C + {}^{(2)}C + {}^{(3)}C + {}^{(4)}C = 0$. The net dipole moment remains zero as the configuration relaxes to the $B = 4$ Skyrmion with cubic symmetry. Similarly, if the first Skyrmion is removed, the three remaining Skyrmions have net dipole moment

$${}^{(2)}C + {}^{(3)}C + {}^{(4)}C = \begin{pmatrix} -c & & \\ & -c & \\ & & -c \end{pmatrix}. \tag{16}$$

The configuration relaxes to the $B = 3$ Skyrmion whose dipole moment is

$$\begin{pmatrix} -c' & & \\ & -c' & \\ & & -c' \end{pmatrix} \tag{17}$$

where c' and c are numerically similar. Finally, if the first and second Skyrmions are removed, the configuration has net dipole moment

$${}^{(3)}C + {}^{(4)}C = \begin{pmatrix} 0 & & \\ & 0 & \\ & & -2c \end{pmatrix}. \tag{18}$$

It relaxes to the $B = 2$ Skyrmion whose dipole moment is of the form

$$\begin{pmatrix} 0 & & \\ & 0 & \\ & & -2c'' \end{pmatrix} \tag{19}$$

where, again, c'' and c are numerically similar.

There is a rather simple-minded argument, which also helps us to understand how to orient $B = 1$ Skyrmions so as to minimize their energy. Consider the asymptotic dipole field of N such Skyrmions. Let the i-th Skyrmion be rotated by ${}^{(i)}R \in SO(3)$ relative to the standard hedgehog orientation, so that its dipole matrix is $c\,{}^{(i)}R$. The total dipole moment is $c\left({}^{(1)}R + {}^{(2)}R + \ldots + {}^{(N)}R\right)$. The asymptotic pion field far from all the Skyrmions is proportional to this, and the energy in the far field (integrated over the region outside a large sphere) is proportional to

$$c^2\ \mathrm{Tr}\left({}^{(1)}R + {}^{(2)}R + \ldots + {}^{(N)}R\right)\left({}^{(1)}R + {}^{(2)}R + \ldots + {}^{(N)}R\right)^T. \tag{20}$$

This quantity can be expressed as

$$c^2\left(3N^2 - \frac{1}{2}\sum_{i \neq j} \mathrm{Tr}\left({}^{(i)}R - {}^{(j)}R\right)\left({}^{(i)}R - {}^{(j)}R\right)^T\right) \tag{21}$$

using the fact that $\mathrm{Tr}(RR^T) = 3$ for any $R \in SO(3)$. To minimise (21) we should attempt to maximise each term in the sum. It is easy to check that if ${}^{(i)}R\,{}^{(j)}R^T$ is a rotation matrix for a rotation by θ about any axis, then $\mathrm{Tr}\left({}^{(i)}R - {}^{(j)}R\right)\left({}^{(i)}R - {}^{(j)}R\right)^T = 4 - 4\cos\theta$, and is maximized when $\theta = \pi$. For $N \leq 4$ it is possible to arrange $\theta = \pi$ for all terms in the sum. This is achieved with $c\,{}^{(i)}R$ given by the four matrices in (15). The quantity

(21) takes values $c^2 N(4-N)$. For $N > 4$ it is possible to make the quantity (21) equal to $c^2 \tilde{N}(4-\tilde{N})$ where $\tilde{N} = N \bmod 4$, by arranging groups of four $B = 1$ Skyrmions to have no net dipole moment, but we do not know if this result is optimal.

We conclude with some remarks on instanton-generated Skyrme fields. We recall that from any Yang-Mills instanton in $\mathbb{R}^4$, one may generate a Skyrme field in $\mathbb{R}^3$ by calculating the holonomy of the gauge potential along all lines parallel to the (Euclidean) time axis [4]. A Skyrme field of baryon number B is generated from an instanton of charge B. For more details and examples, see Refs [5, 6]. An instanton-generated Skyrme field does not satisfy the static field equation of the Skyrme model exactly (although it may do so approximately), so we cannot use our earlier argument to rule out a monopole moment; nevertheless, it has no such moment, as the following argument shows. We first invert the instanton in the unit sphere in $\mathbb{R}^4$. Since an instanton can be extended to a regular gauge field on S^4, the inverted instanton, in a suitable gauge, has a finite field tensor at the origin and is smooth there. After inverting the instanton, the Skyrme field is given by the holonomy along circles passing through the origin and tangent to a fixed line (the image of the time axis). The Skyrme field at large r is given by the holonomy on a small circle, since the line through $(0, \boldsymbol{x})$ is mapped to a circle of radius $1/2r$. The asymptotic Skyrme field depends on the area of this circle and on the component of the field tensor in the plane of this circle, which is approximately constant. In fact, $\sigma \sim 1$ and $\pi^a(\boldsymbol{x}) \sim \pi E_i^a x_i / 4r^3$. Here E_i^a is the electric part of the field tensor of the inverted instanton at the origin, $E_i^a x_i / r$ is the component in the plane of the circle and $\pi/4r^2$ is the area of the circle. The asymptotic pion field has a dipole form, with dipole moment matrix $C_i^a = \pi^2 E_i^a$. For the special case of a Jackiw, Nohl, Rebbi instanton, this dipole moment has been calculated in terms of the data defining the instanton. The formula is in the Appendix of Ref. [6].

The instanton-generated Skyrme fields which are closest in energy to the Skyrmions for $B \leq 4$ are now known [6], and their symmetries and asymptotic fields are analogous. However, the numerical values of the dipole moments differ somewhat, except for $B = 4$, where they vanish anyway.

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