STRUCTURE OF PARTICLES CONTAINING HEAVY QUARKS*†

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Selected results concerning the spectroscopy of $b\bar{b}$ quarkonia and the semileptonic and leptonic decays of hadrons containing b-quarks are reviewed. Problems concerning the inclusion of relativistic corrections and the interpretation of wave functions are pointed out.

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1. Introduction

Known particles containing heavy quarks can be divided into two large groups: particles containing a heavy quark and a heavy antiquark, and particles containing only one heavy (anti)quark. Baryons containing two, or three heavy quarks are expected, but have not yet been seen. We shall discuss some of the theoretical problems resulting from the comparison of theoretical models with the data.

The quarkonia, *i.e.* mesons consisting of a heavy quark and a heavy antiquark, have been successfully described in terms of fairly simple potential models. We shall limit our discussion to the case of the $b\bar{b}$ resonances, because the internal motion there is to a good approximation nonrelativistic. Consequently, they seem to be the best understood family of heavy quark systems. A very good recent review can be found in Ref. [1]. In the following section we shall summarize the theoretical ideas concerning the $b\bar{b}$ quarkonia

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and stress that their success is due to a strict adherence to the suggestions of QCD. Models contradicting QCD find it difficult to explain the available data. The remaining problem is that the results seem too good for the nonrelativistic potential models being used. Thus, behind the successful phenomenology there may be something deeper not yet understood.

The particles consisting of one heavy quark and some light stuff are more difficult to describe. In the valence approximation the light stuff reduces to an antiquark in the case of mesons and to two quarks, or a diquark, in the case of baryons. The nonrelativistic quark model has had some success also for such particles [2]; since the resulting velocities of the light quarks come out relativistic, however, it is not considered very respectable. In Section 3 we shall show that relativistic effects are crucial for the understanding of the semileptonic decays of such systems [3], even when the recoil velocity is nonrelativistic. In Section 4 we shall present a discussion of purely leptonic decays, which suggests that something is missing in our understanding of the wave functions of the heavy quark–light antiquark systems [4].

Thus we begin with a case, where the simple wave function picture starting with the Schrödinger equation is very satisfactory, we go over to a case where the relativistic boost is very important and conclude with the discussion of a case, where the wave function picture requires a modification, but what modification is not yet clear.

2. $b\bar{b}$ quarkonia

2.1. Introduction

The first quarkonia — bound $c\bar{c}$ systems — were discovered about twenty years ago [5, 6]. Almost immediately it was realized that such systems can be described using the Schrödinger equation with simple, more or less QCD motivated, potentials. A good example is the Cornell potential [7]

$$V(\vec{r}) = -\frac{\alpha}{r} + k\,r + \text{const}\,. \tag{1}$$

At short distances this potential is Coulombic, as expected from one gluon exchange. At large distances it is linear, as expected for two quarks connected by a colour string. The parameters α and k obtained from fits to the experimental data are in reasonable agreement with $\frac{4}{3}\alpha_s$, expected for one gluon exchange, and with the string tension. The trouble is that velocities are large

$$\langle \vec{v}^2 \rangle \approx \tfrac{1}{4}c^2\,, \tag{2}$$

where c is the velocity of light. Thus the Schrödinger description is somewhat doubtful. On the other hand, relativistic descriptions of two-body interacting systems are controversial. In particular there is no reference frame having all the most useful properties of the nonrelativistic centre of mass frame.

The $b\bar{b}$ quarkonia discovered three years later [8, 9] consist of heavier partons, which consequently have smaller velocities. For potentials of the shape used for quarkonia (approximately logarithmic in the relevant range) the average $\langle \vec{v}^2 \rangle$ is roughly proportional to the inverse of the quark mass. Thus for the $b\bar{b}$ quarkonia it is about three times smaller than for the $c\bar{c}$ quarkonia, which makes the nonrelativistic approximation quite reasonable. Since we want to begin with the simplest possible case, we shall concentrate on these $b\bar{b}$ systems. For many years people were writing that the $t\bar{t}$ systems will be the best nonrelativistic case, but today we know [10] that the t-quark is sufficiently heavy to decay into a real W boson and a b-quark, which makes it so short-lived that it has no time to form quarkonia. The potentials proposed for the $c\bar{c}$ quarkonia were found to work well also for the $b\bar{b}$ quarkonia, which was evidence for the flavour independence of the forces between quarks. This independence is an important confirmation of QCD.

Let us now list the observables for $b\bar{b}$ systems (further referred to as quarkonia), which should be reproduced by the theory. We consider only the simplest quantities, where the predictions require few phenomenological assumptions.

The quarkonia with masses above twice the mass of the B meson are broad and in order to obtain their mass distributions one has to perform uncertain coupled-channel calculations. Therefore, we shall limit our discussion to the quarkonia with masses below 10.557 GeV. In this mass range there are three S-wave vector particles denoted $\Upsilon(1S)$, $\Upsilon(2S)$ and $\Upsilon(3S)$ and two P-wave multiplets denoted $\chi(1P)$ and $\chi(2P)$. A nonrelativistic calculation cannot reproduce the splittings of the P-wave states and such calculations give only the centres of gravity of the multiplets. Several other particles are expected [11]: the pseudoscalar partners of the Υ particles (denoted $\eta_b(nS)$), the $\chi(3P)$ multiplet, the ${}^1P_1$ partners of the χ multiplets, two D-wave multiplets and one F-wave multiplet. Each of the D and F multiplets should consist of two submultiplets differing, just like for the S and P multiplets, by the relative orientation of the spins of the quark and the antiquark.

The wave functions are not directly measurable, but some information about them can be obtained.

- For the Υ particles the wave function at the origin, usually given as $|\psi(0)|^2$, can be obtained from the total width of the leptonic decay $\Gamma_{\Upsilon \to e^+e^-}$, or of the hadronic decays $\Gamma_{\Upsilon \to 3\text{gluons}}$, or of the radiative decays $\Gamma_{\Upsilon \to \gamma + 2\text{gluons}}$. As we shall discuss further each of these methods has its difficulties, but some information can be obtained.

- For the χ states by definition $\psi(0) = 0$, but information about the derivative $|\psi'(0)|$ can in principle be obtained from the decay width into hadrons (the decay into e^+e^- is forbidden). In practice, since for dimensional reasons the decay width is suppressed by a large factor proportional to M_χ^{-2}, the decays necessary for this analysis have not yet been observed experimentally.
- The matrix elements of the type $\langle\psi_\chi|\vec{r}|\psi_\Upsilon\rangle$, can be deduced from the measured probabilities of electric dipole transitions between the χ states and the Υ states.

There were hopes to extract useful information from the study of the strong decays of the type $\Upsilon' \to \Upsilon''(\pi\pi)$, but for the moment these decays seem to be too complicated [1] and the effort is rather to predict their probabilities. The discovery of the η_b particles should make it possible to measure the matrix elements for the magnetic dipole transitions between the η_b-s and the Υ-s, which would supply additional valuable information about the wave functions.

2.2. Masses

In order to characterize the success of various potentials in predicting the mass spectrum of the $b\bar{b}$ quarkonia it is useful to use the parameter [1]

$$\sigma = \sqrt{\overline{[(M_{\text{exp}} - M_{\text{th}}) - \overline{(M_{\text{exp}} - M_{\text{th}})}]^2}}\,, \tag{3}$$

where the horizontal lines denote averaging over the five data points. From the theoretical point of view, using this quantity one eliminates the uninteresting constant term in the potential. From the experimental point of view, one eliminates the overall scale error and thus significantly reduces the experimental uncertainties. In Ref. [1] the values of σ for twenty-seven models are compared. They range from 2.3 MeV for the most successful model of Fulcher [12] to 27.5 MeV for the least successful model. Since the masses of the quarkonia are about 10 GeV, the relative errors are between 0.02% and 0.27%. The agreement is very good. The relativistic corrections to the kinetic energy are expected to be of the order of $\langle\vec{v}^2\rangle^2/(8c^4) \approx 0.1\%$, thus the errors are smaller than expected. There may be something interesting to discover behind this observation.

The potential used by Fulcher is

$$V(r) = 0.159\,r - \frac{1}{r}f(0.431r)\,, \tag{4}$$

where 0.159 and 0.431 are constants chosen so as to get the best fit and function $f(t)$ is defined by

$$f(t) = 1 - 4 \int_0^\infty \frac{e^{-yt}\, d\ln y}{[\ln(y^2-1)]^2 + \pi^2}\,. \tag{5}$$

This is a simple modification of the potential proposed by Richardson [13], who, however, postulated an additional relation between the two constants of the model. For large values of r this potential is linear with a reasonable value of the string tension, just like the Cornell potential, but for small distances

$$f(t) \approx -\frac{1}{\ln t}\,. \tag{6}$$

Thus the potential approximately takes into account the running of the strong coupling constant with $\Lambda_{\rm QCD} = 0.431$. The effect of the running of α_s is not very strong. Substituting for illustration the averages $\sqrt{\langle r^2\rangle}$ calculated for the various quarkonia by Buchmüller and Tye [14] one finds for $\Upsilon(1S)$: $\alpha_s \approx 0.61$, for $\chi(1P)$: $\alpha_s \approx 0.70$ and then a slow increase to $\alpha_s \approx 0.73$ for $\Upsilon(3S)$. Nevertheless, this small effect improves the fit. This is an observation, which we shall make repeatedly: following the suggestions of QCD improves the fits!

3. Wave functions at the origin

It might seem that the value of the wave function at the origin, we shall use $|\psi(0)|^2$ to characterize it, can be modified by changing the potential for small values of r and thus can be easily changed in any model. Actually this is not true. As shown long ago by Schwinger [15] for a system of two bodies of masses m_1 and m_2, described by the Schrödinger equation with an interaction potential $V(r)$

$$|\psi(0)|^2 = \frac{1}{4\pi}\frac{m_1 m_2}{m_1+m_2}\int_0^\infty d^3r\,|\psi(r)|^2 \frac{dV(r)}{dr}\,. \tag{7}$$

Thus the value of the wave function at the origin depends on the potential and on the wave function in all space.

The electronic width of an Υ particle is given by the formula

$$\Gamma_{\Upsilon\to e^+e^-} = \frac{4\alpha^2}{9M_\Upsilon}\,|\psi(0)|^2\left[1 - \frac{16\alpha_s}{3\pi} + O(\alpha_s^2)\right]\,. \tag{8}$$

The problem with this formula is that the first order radiative correction, let us denote it $-a$, is large. For a typical value $\alpha_s = {}^1/_4$ one finds $a \approx 0.40$. Thus the correction factor given by the square bracket is about 0.60. The higher order corrections, however, are unknown. Thus the correction factor could be just as well $\sqrt{1-2a} \approx 0.45$ or $1/(1+a) \approx 0.71$. In order to avoid this uncertainty one can [1] compare with experiment the ratios

$$R_{ee}(n) = \frac{\Gamma_{\Upsilon(nS)\to e^+e^-}}{\Gamma_{\Upsilon(1S)\to e^+e^-}} . \tag{9}$$

The radiative corrections are believed to depend little on the excitation of the Υ particle and it is assumed that they cancel in the ratio. This approach, of course, can give only the ratios of the values of the wave functions at the origin. Besson and Skwarnicki [1] have compared the predictions of seventeen models with experiment. For the other ten models from their list the predictions have not been given by the authors. Out of the ten models giving the best mass spectra seven give predictions for the ratios R. The average error

$$\overline{\frac{|R_{\rm th} - R_{\rm exp}|}{R_{\rm exp}}} \approx 4.7\% \tag{10}$$

is close to the experimental uncertainty.

The simple perturbative description of the decays $\Upsilon \to \gamma + 2$ gluons is a reasonable first approximation, but it has been known to overestimate the momenta of the photon and to require values of α_s inconsistent with those obtained from other sources [16]. The second objection may be controversial, because in order to eliminate the factor $|\psi(0)|^2$ occurring in the formulae one has to use either data on electronic decays or data on nonleptonic decays. In both cases the relation between the data and the theoretical formula used is uncertain. A possible way out of the difficulty has been recently suggested by Consoli and Field [17], who propose to ascribe to the two gluons masses (1.17 ± 0.08) GeV. This number is chosen so as to get a good average energy of the photon and thus the data could be used to determine other quantities, *e.g.* the constant α_s.

The perturbative description of the decays $\Upsilon \to 3$ gluons also seems to require somewhat unorthodox values of the constant α_s. According to a recent suggestion of Chiang Hüfner and Pirner [18] this can be remedied by correcting for the fact that the three gluons are emitted at different points.

To summarize: the determination of the values of the wave functions at the origin is in principle possible, but the corresponding formulae have large uncertainties. The best we have for the moment are the ratios of the wave functions determined from the electronic decays.

3.1. Matrix elements $\langle\chi|\vec{r}|\Upsilon\rangle$

The widths of the electric dipole transitions $\Upsilon(3S) \to \gamma + \chi(2P)$ and $\Upsilon(2S) \to \gamma + \chi(1S)$, *i.e.* between quarkonium states, which would be degenerate, if the potential were purely coulombic, are well reproduced by eleven out of the twelve models given in the compilation [1]. The only failure is of interest here. It is by factors of about four, for a potential of the form $V(r) \sim r^{-\beta}$, where β is a small positive number. Such potentials [20] and closely related logarithmic potentials [21] have been successful in calculations of mass spectra and of electronic decay widths of quarkonia. On the other hand they contradict QCD, which suggests potentials singular at $r = 0$. The present result shows that they are inconsistent with experiment, which is another argument in favour of QCD. Four of the papers consider relativistic corrections. For the $\Upsilon(3S) \to \chi(2P)$ transitions these corrections are small, and since the nonrelativistic predictions almost exactly coincide with the central value of the experimental result, do not affect the agreement with experiment. For the $\Upsilon(2S) \to \chi(1P)$ transitions they are of similar size, but since the nonrelativistic predictions are systematically below the experimental central value and the corrections are negative, agreement with experiment slightly deteriorates.

For the $\chi \to \gamma + \Upsilon$ transitions the difficulty is that experimentally only the branching ratios have been measured, while the theoretical predictions are for the partial widths. The best one can do for the moment, is to compare with theory only the ratio

$$\frac{\Gamma(\chi(2P) \to \Upsilon(1S))}{\Gamma(\chi(2P) \to \Upsilon(2S))}. \tag{11}$$

This ratio is equal to the ratio of the corresponding branching ratios and consequently is measurable. All the eleven models successful in the previous cases give results higher than the experimental one. The factor is from about 1.2, which is within the experimental errors, to about 2.2. The relativistic corrections are rather large and strongly model dependent. Even their sign is controversial.

The transition $\Upsilon(3S) \to \gamma + \chi(1P)$ is the most difficult to describe, because the wave function of the $\Upsilon(3S)$ changes sign at two values of r, while the radial wave function of the $\chi(1P)$ has the same sign in all space. Consequently, the value of the radial integral is a result of cancellations between numbers larger than the final result. As expected, the predictions do not agree with each other. The ratio of the largest to the smallest predicted width is about 120. One of the predictions agrees with experiment, but it is difficult to take that very seriously. The relativistic corrections are very large and very controversial.

3.2. Relativistic corrections

From present experience the relativistic corrections in the physics of the $b\bar{b}$ quarkonia seem to be usually small and controversial. They are probably numerically important for at least some of the electromagnetic dipole transitions. There are, however, problems, where relativistic effects, in spite of their small absolute magnitude, are the main factor. An important case is the problem of mass splittings in the χ multiplets. For each χ multiplet one writes

$$M_2 = M_{cg} + \alpha_{LS} - 0.4\alpha_T\,, \tag{12}$$

$$M_1 = M_{cg} - \alpha_{LS} + 2\alpha_T\,, \tag{13}$$

$$M_0 = M_{cg} - 2\alpha_{LS} - 4\alpha_T\,, \tag{14}$$

where the subscript on the left hand side denotes the spin J of the χ-particle. Instead of the spin orbit term α_{LS} and the tensor term α_T one often discusses the parameter

$$r = \frac{M_2 - M_1}{M_1 - M_0} = 2\frac{\alpha_{LS} - 1.2\alpha_T}{\alpha_{LS} + 6\alpha_T}\,, \tag{15}$$

which has smaller experimental uncertainties. Experimentally $\alpha_{LS} \gg \alpha_T > 0$ for both χ multiplets, while $r(\chi(1P)) = 0.65 \pm 0.03$ and $r(\chi(2P)) = 0.58 \pm 0.01$ [1].

Nonrelativistically one has $\alpha_{LS} = \alpha_T = 0$. Thus in order to get any results at all, one has to include relativistic effects. When going over to the relativistic theory several problems arise.

- A fundamental theory of two-body interacting, relativistic systems does not exist. Therefore, one has to introduce some phenomenology. The question is how much and how.
- The nonrelativistic potential, even in the simplest versions of the relativistic theory, must be replaced by a vector piece and a scalar piece. The question is how to perform this splitting.
- New spin dependent interactions appear. The question is how to choose them. In phenomenological models they are usually somehow related to the spin independent part known to some extent from the nonrelativistic limit.

A generally accepted set of answers to these questions is not known, thus we shall limit our discussion to a few remarks.

The gauge potential is of the vector type, therefore, terms like the Coulomb term in the Cornell potential are ascribed to the vector part of the interaction. Assuming that also the confining linear part of the potential is of the vector type, one finds that there are no bound states at all [22, 23]. Therefore, the confinig part of the nonrelativistic potential is put into the scalar interaction. In the purely Coulomb case $r = 0.8$. Inclusion of scalar confinig interactions usually reduces r, but by itself gives a larger value of r for the $\chi(2P)$ states than for the $\chi(1P)$ states. The fact that

$$r_{\chi(2P)} < r_{\chi(1P)} < 0.8 \tag{16}$$

suggests the presence of a scalar short range interaction, so that the smaller $\chi(1P)$ states are more strongly affected than the larger $\chi(2P)$ states. This is indeed what is predicted by QCD, when many gloun exchanges are included [24].

4. Semileptonic $b \to c$ decays

4.1. Introduction

Let us consider the decays $H_b^0 \to H_c^+ l\overline{\nu}$, where H_Q stands for a hadron containing one heavy quark of type Q. Examples are $\overline{B}^0 \to D^+ l\overline{\nu}$, $\overline{B}^0 \to D^{*+} l\overline{\nu}$, $\Lambda_b^0 \to \Lambda_c^+ l\overline{\nu}$ and $\overline{B}_s^0 \to D_s^{*+} l\overline{\nu}$. In the heavy quark picture, the hadron H_b behaves as a hydrogen atom behaves, when the proton absorbs a 1 MeV photon. In hydrogen at this energy scale the binding energy is irrelevant and the proton recoils as if it were free. This is calculable very simply. If, however, we are interested in the probability of scattering the whole hydrogen atom in its ground state, we must multiply the transition amplitude for the free proton by the probability amplitude that the electron cloud follows the recoiling proton. For large recoils this is a small number. In general it is equal to the overlap $\langle \lambda, v | \lambda, v' \rangle$, where $|\lambda, v\rangle$ denotes the state vector of an electron in the ground state of a Hydrogen atom moving with velocity v. λ is the helicity of the electron state (not just of the electron) defined as the projection of the total angular momentum of the electron on the direction $\vec{v}' - \vec{v}$. It does not change under the boost converting $|\lambda, v\rangle$ into $|\lambda, v'\rangle$. This reasoning is well-known in quantum mechanics and leads to the sudden approximation. Further we shall exploit this analogy to find the decay amplitude of H_b.

For simplicity we neglect the lepton mass. Then the kinematical limits on the square of the mass of the lepton pair, further denoted q^2, are

$$0 \le q^2 \le (M_{Hb} - M_{Hc})^2 \approx (m_b - m_c)^2 \,. \tag{17}$$

The differential decay width is

$$\frac{d\Gamma}{dq^2} = \ldots \overline{|\mathcal{M}|^2} \,. \tag{18}$$

In his formula the dots denote known kinematical factors and the overline refers to the averaging of the square of the absolute value of the decay amplitude $\mathcal{M}$ over the angular distributions: of hadron H_c in the H_b rest frame and of lepton l in the rest frame of the lepton pair. The decay amplitude besides the known factor $G_F V_{cb}/\sqrt{2}$ is the product of two four-vectors: of the leptonic current and the hadronic current. The leptonic current $\overline{u}_l \gamma^\mu (1-\gamma^5) v_\nu$ is well known. In the rest frame of the lepton pair its components t, x, y, z are

$$-2\sqrt{q^2}(0,\ i\sin\phi + \cos\theta\cos\phi,\ i\cos\phi + \cos\theta\sin\phi,\ -\sin\theta)\,, \tag{19}$$

where the spherical angles refer to the lepton momentum in a reference frame, where the z axis is antiparallel to the momentum of H_b. Thus the problem reduces to a determination of the hadron current. The matrix elements of this current depend on the recoil velocity of H_c. It is convenient to use as parameter

$$\omega = v^\mu v'_\mu\,. \tag{20}$$

Kinematically, ω is the Lorentz factor of the recoiling hadron H_c as seen in the centre of mass system of the decay, or equivalently in the rest frame of H_b. Thus it is equal one for zero recoil and increases with increasing recoil. The kinematical limits are

$$1 \le \omega \le \frac{M_{Hb}^2 + M_{Hc}^2}{2M_{Hb}M_{Hc}}\,. \tag{21}$$

The decay process can be analyzed into several steps. Hadron H_b with spin J_{Hb} and helicity λ_{Hb} results from the coupling of the b-quark with spin $1/2$ and helicity λ_b with some light stuff with total angular momentum J_L and helicity λ. This introduces the Clebsch–Gordan coefficient

$$\langle J_L, \lambda; \tfrac{1}{2}, \lambda_b | J_{Hb}, \lambda_{Hb}\rangle\,. \tag{22}$$

The b-quark decays as if it were free — here we use the analogy with the hydrogen case discussed above — this introduces the factor

$$J_Q^\mu = \overline{u}_c \gamma^\mu (1-\gamma^5) u_b\,.$$

The free Dirac bispinors u_c and u_b are well-defined, because: in the heavy quark limit the velocity of each of the heavy quarks is equal to the velocity of the (heavy-light) hadron containing it and therefore known; the parity is purely plus (no antiparticle admixture), because the gap between the positive and negative energy states is very large in the heavy quark limit; the helicity is known, because we have made the Clebsch–Gordan expansion of the wave function of H_b and we ask for the final states with well defined helicities of quark c. The light stuff has to be boosted from a system moving with velocity v to a system moving with velocity v'. The overlap of the state before the boost and the state after the boost is not known. It is only known that since no interaction is involved in this boost, various conservation laws hold. In particular the helicity λ must be conserved. We shall denote this overlap by

$$\langle \lambda, \vec{v} | \lambda, \vec{v}' \rangle = \sqrt{\frac{\omega + 1}{2}} \xi_\lambda(\omega) \tag{24}$$

and call the functions $\xi_\lambda(\omega)$: Isgur–Wise functions. This is the generally accepted terminology for meson decays. For baryon decays the kinematical factor is sometimes omitted. The Isgur–Wise functions for different decays are, of course, in general different. In the last step, the boosted light stuff and the c quark recouple to form particle H_c with spin J_{Hc} and helicity λ_{Hc}. This introduces the factor

$$\langle J_L, \lambda; \tfrac{1}{2}, \lambda_c | J_{Hc}, \lambda_{Hc} \rangle . \tag{25}$$

The complete decay amplitude $\mathcal{M}$ can be expressed as a sum of products of the factors presented above. Out of these only the Isgur–Wise functions are unknown. Thus, the differential decay width can be in each case written in terms of known terms consisting of Clebsch–Gordan coefficients and kinematical factors, and of the Isgur–Wise functions. All the nontrivial part of the decay dynamics is contained in the Isgur-Wise functions. Since the spin projections of H_c are under control, one can also calculate asymmetries, ratios of the longitudinal to transverse decays *etc.* We stress that this discussion refers to the leading order calculation in the framework of the heavy quark approximation. When corrections for finite masses of the heavy quarks and the QCD corrections are introduced, the problem becomes much more complicated. For our purpose, however, which is to show the importance of the relativistic effects for the calculation of the Isgur–Wise functions, the leading order calculation is enough.

4.2. Isgur–Wise functions

The number of independent Isgur–Wise functions is reduced by symmetries. Parity conservation for the overlap implies [25] that for a given decay

$$\xi_{-\lambda}(\omega) = \eta_{Hb}\eta_{Hc}(-1)^{J_{Hc}-J_{Hb}}\xi_\lambda(\omega) , \tag{26}$$

where η_H denotes the internal parity of hadron H. This relation implies in particular that a single Isgur-Wise function is enough to describe not only the decays of Λ_b, where $J_L = 0$, but also the decays of the pseudoscalar mesons, where $J_L = {}^1/_2$. Hyperfine effects are proportional to the magnetic moment, *i.e.* inversely proportional to the mass, of the heavy quark. Therefore, in the heavy quark limit they are negligible and in particular the same Isgur-Wise function describes the decays $\overline{B} \to Dl\overline{\nu}$ and $\overline{B} \to D^*l\overline{\nu}$. The static colour field does not depend on the flavour of the source. Therefore, the state vectors of the light stuff in hadrons H_b are identical to those of the corresponding hadrons H_c [26]. In particular, in the case of zero recoil the overlaps corresponding to the decays $\overline{B} \to D$, $\overline{B} \to D^*$ and all the other quoted before as examples coincide with the normalization integrals and equal one. By a similar argument, for pairs of particles like $\overline{B}$ and D^{**}, where D^{**} denotes a P-wave excitation of the D system, the Isgur–Wise function vanishes at zero recoil. Thus, the value $\xi_\lambda(1)$ is known in many cases.

There is also some general information about the ω-dependence of the Isgur–Wise functions. Since

$$\xi_\lambda(\omega) = \sqrt{\frac{2}{\omega+1}}\langle\lambda\vec{v}|\lambda,\vec{v}'\rangle\,, \tag{27}$$

and the overlap cannot exceed one, we find [27]

$$\xi_\lambda(\omega) \le \sqrt{\frac{2}{\omega+1}}\,. \tag{28}$$

There is also a lower bound, but this involves coupling constants, which are not directly measurable, though they can be indirectly estimated [28].

For the cases, where $\xi_\lambda(1) = 1$, one often discusses the real positive parameter ρ^2 defined by

$$\xi_\lambda(\omega) = 1 - \rho^2(\omega-1) + O[(\omega-1)^2]\,. \tag{29}$$

From the inequality quoted above $\rho^2 \ge {}^1/_4$. This is known as the Bjorken bound. Experimentally ρ^2 is very difficult to measure. One reason is that there is no phase space for zero recoil decays, but in practice much more important is a purely experimental factor: the most accessible are the decays $\overline{B} \to D^*l\nu$; the decaying $\overline{B}$ mesons are usually decay products of the $\Upsilon(4S)$ resonances produced in e^+e^- collisions almost at rest; therefore, at small recoils the D^*-s are also almost at rest; the D^*-s are identified by their decays into $D\pi$ with the pion having a momentum of 40 MeV in the rest frame of the D^*; thus the experimental problem is to measure very slow pions and this is very hard. Great progress is expected, when some asymmetric B factory producing $\Upsilon(4S)$ resonances with a known, reasonable large momentum becomes operational.

Let us concentrate on the parameter ρ^2 for the $\overline{B} \to D^*$ decays. Present experimental estimates give typically values of ρ^2 between one and two [29]. A sum rule analysis [28] has given $\rho^2 \approx 1.0$ and a lattice calculation [30] yields $\rho^2 \approx 1.2$. These two calculations refer to first principles. Any model, however, which gives the state of the light stuff can be used to find the state vector of the light stuff and consequently the overlap and the Isgur-Wise function. A successful model could be expected to give also here a good prediction. Therefore, it came as a surprise that the nonrelativistic quark model, otherwise rather successful, gives [2] $\rho^2 \approx 0.6$. The question arises: what is wrong with this particular application of the model. Today it seems clear that the failure is due to the use of the Galilean boost when calculating the overlap of the wave function for the $\overline{B}$ meson at rest and of the recoiling D^*. The wave functions of the mesons found in their rest frames certainly are not exact, but by themselves would not produce an error of this size. When the boost is made more relativistic, much better results are obtained. Details of the model do not seem to be crucial here. We shall describe a calculation [3] based on the MIT bag model modified by fixing the heavy quark in the centre of the bag. Similar results can be obtained by using the nonrelativistic quark model modified by including some relativistic kinematics [31].

4.3. Modified MIT bag model

In the standard MIT bag model it is assumed that the colour field of the valence quarks in a hadron produces a bubble (bag) of perturbative vacuum in the much more complicated physical vacuum filling empty space. The bubble is assumed to be spherical, with its size determined by the equilibrium of the external pressure of the physical vacuum and of the pressure from inside exerted by the valence quarks. The valence quarks move freely, or almost, within the bag. For mesons containing a heavy quark, which moves slowly, a necessary modification is to fix the heavy quark at the centre of the bag [32, 33], because the bubble has ample time to follow the motion of the heavy quark. This model is certainly grossly oversimplified, but it reproduces the spectroscopy of the particles with heavy quarks [34] and it describes relativistically the motion of the light stuff — in this case of one, or two, light valence quarks.

In spite of the relativistic description of the quark motion the model is not explicitly covariant. For instance it is not clear how to boost a bag. For a boost from rest to uniform motion with velocity $\vec{v}$ parallel to the z-axis, we include three factors.

- The Lorentz contraction of the bag, with the contraction along the z axis by a factor γ^{-1}, where γ is the Lorentz factor corresponding to velocity $\vec{v}$.

- The inverse Lorentz transformation of the space-time points

$$L^{-1}(\vec{v})(t,\, x,\, y,\, z) = (-vz,\, x,\, y,\, \gamma z)\,, \tag{30}$$

- The boost matrix for a Dirac bispinor. This is a Hermitian 4×4 matrix $S(\vec{v})$.

The effect of the first two items is that the wave function that fills the spherical bag at rest, also fills the contracted moving bag. The effect of the last two items has an analogue in the rigid rotation around the z-axis of a vector in the (x, y) plane. Consider, for example, a vector $(a_x, 0)$ at a point with polar coordinates (r, ϕ). After rotation by an angle α, the vector at $(r, \phi + \alpha)$ will be the transform of this vector. This corresponds to the transformation of the coordinates. The transformed vector has components $(a_x \cos\alpha, a_x \sin\alpha)$. This transformation corresponds to the multiplication by the boost matrix S. The rotation, if any, contained in the action of the boost matrix is known as the Wigner rotation.

The normalization of the wave function is unchanged by this boost. The contraction of the integration volume, identical with the volume of the bag, is exactly compensated by the boost factor. The overlap function for two bags moving one with respect to the other depends on the reference frame in which it is calculated. Following Lie-Svendsen and Høgaasen [35] we choose the modified Breit frame *i.e.* the frame, where the velocities of the two bags are equal and opposite (in the ordinary Breit frame the momenta are equal and opposite). This leads to particularly simple formulae, because the two bags are contracted equally. We checked that choosing the rest frame of one of the mesons changes little the numerical results. In the modified Breit frame the integrand of the overlap integral is

$$\Phi^{\dagger}(L^{-1}(-\vec{v})(0,\vec{x}))S^{\dagger}(-\vec{v})S(\vec{v})\Phi(L^{-1}(\vec{v})(0,\vec{x}))\,. \tag{31}$$

Here $\vec{v}$ is the velocity of the initial meson in the modified Breit frame. The first observation is that

$$S^{\dagger}(-\vec{v}) = S(-\vec{v}) = S^{-1}(\vec{v})\,. \tag{32}$$

Thus the two boost factors cancel and there is nothing to compensate for the contraction of the bag. This is one relativistic effect making the Isgur–Wise function steeper. The second effect is that, while for $\vec{v} = 0$ the integrand is positive definite, for $\vec{v} \neq 0$, the time is no longer the same for both bags and in all the points within the integration region; consequently, the two factors get z-dependent phases, which, moreover, do not cancel each other. This gives a further reduction of the integral. A change of variables $z' = \gamma z$ converts the integral over the contracted bag into an integral over the sphere

B corresponding to the volume of the uncontracted bag and one finds for the overlap

$$\langle\lambda, -\vec{v}|\lambda, \vec{v}\rangle = \frac{1}{\gamma}\int_B d^3x\, \Phi^\dagger(0,\vec{x})\Phi(0,\vec{x}) \exp\left(2i\frac{v}{\gamma}E_q z\right). \tag{33}$$

Here E_q is the energy of the light antiquark in the centre of mass frame of the meson.

4.4. Discussion

The nonrelativistic formula for the overlap is [2]

$$\langle\lambda, -\vec{v}|\lambda, \vec{v}\rangle = \int d^3p\, \Phi^*(\vec{p} - m_q\vec{v}_c)\Phi(\vec{p}), \tag{34}$$

where $\vec{v}_c$ denotes the recoil velocity of the D^* meson in the $\overline{B}$ rest frame. One recognizes the Galilean boost: the velocity $(\vec{p}/m_q)$ of the light antiquark in the rest frame of the recoiling meson is the difference of the velocity in the rest frame of the decaying meson and the velocity of the meson. Expanding in a power series in the only non-zero component of the velocity v_{cz}; taking into account the normalization of the function Φ and its spherical symmetry, as well as the fact that in momentum representation $i\hbar(\partial/\partial p_z) = z$; we find, using the relation between the overlap and the Isgur–Wise function and the definition of the parameter ρ^2:

$$\rho^2 = \tfrac{1}{4} + \frac{m_q^2\langle r^2\rangle}{3}. \tag{35}$$

Substituting typical values [2] $m_q = 0.33$ GeV and $\langle r^2\rangle = (3\text{GeV}^{-1})^2$, one gets $\rho^2 \approx 0.6$. The calculation was performed in the $\overline{B}$ meson rest frame, but calculations in the modified Breit frame, or in the rest frame of the D^* meson, give the same result. Let us see now how the relativistic boost changes this result.

The formula for the parameter ρ^2 is similarly obtained, except that the expansion of the integrand is in powers of z. The result is

$$\rho^2 = \tfrac{1}{2} + \frac{E_q^2\langle r^2\rangle}{3}. \tag{36}$$

The additional $^1/_4$ results from the Lorentz contraction of the bag and the change of m_q into E_q is typical for relativistic calculations. Substituting a typical value $E_q = 0.5$ GeV and keeping the previous estimate of $\langle r^2\rangle$ one finds the estimate $\rho^2 \approx 1.25$. Substituting the values from the modified MIT bag model, the result is $\rho^2 \approx 1.24$ [3].

This explains qualitatively the origin of the difficulty encountered in the purely nonrelativistic quark model. The quantitative results of the bag model [3] are also of interest. One finds that for the $\overline{B}_s \to D_s^*$ transitions $\rho^2 \approx 1.625$. This results from a competition between two factors: the B_s and D_s^* are somewhat smaller than the B and D, which reduces ρ^2. On the other hand, however, the energy E_s is bigger than the energy E_d, because the s-quark is heavier. This increases ρ^2 and is more important than the previous effect. Our predictions contradicted a prediction obtained using chiral arguments [36], but has recently been confirmed by lattice calculations [37]. For $\Lambda_b \to \Lambda_c$ transitions the prediction is $\rho^2 \approx 2.23$. This large increase, as compared to the meson case, results from the fact that for baryons the overlap is, roughly, the product of overlaps for the two light quarks. One can also calculate numerically the overlaps and consequently the Isgur–Wise functions for finite recoils. In all the cases studied the numerical result could be very well approximated by

$$\xi(\omega) = \left(\frac{2}{\omega + 1}\right)^{a+\frac{b}{\omega}} . \tag{37}$$

For $\overline{B} \to D^*$ decays: $a = 2$, $b = 0.6$. The resulting function agrees with all the available experimental data.

A more fundamental problem is the estimate of the parameter

$$\overline{\Lambda} = M_{HQ} - M_Q , \tag{38}$$

which is very important in heavy quark effective theory. The value of this parameter is poorly known. For B and D mesons it is usually chosen in the range [0.3 GeV, 0.7 GeV]. In the modified MIT bag model, the mass of the meson is the sum of the rest mass of the heavy quark and of the energy E_q of the light quark. Thus $E_q = \overline{\Lambda}$ and formula (36), or in fact any model of this type, can be used to estimate $\overline{\Lambda}$. In particular putting $\langle r^2 \rangle = (3\ \mathrm{GeV}^{-1})^2$ and $\rho^2 = 1.2$ one gets $\overline{\Lambda} = 0.5$ GeV, which is very reasonable. An interesting question is, how reliable is this result? As seen from the comparison with the nonrelativistic calculation, the crucial point is: is the boost performed correctly? In the calculation all the binding energy was ascribed to the light quark. This is at best an approximation. One would like to know, how to boost a system of two interacting quarks. This, however, is a very difficult question. The majority view is that even its analogue for the positronium is unsolved. Perhaps the fact that one of the quarks is much heavier than the other, makes the special case of interest here more tractable.

5. Decay constants

5.1. Introduction

Let us consider the decay of a pseudoscalar meson $P = (Q\overline{q})$, (or $P = (\overline{Q}q)$, further we omit similar remarks) of mass M_P into a lepton of mass m_l and a massless antineutrino. Since we are interested in mesons containing heavy quarks, our examples will be B^- and D^-. A similar discussion could be given for the mesons B_s^- and D_s^-. Since meson P has zero spin and the antineutrino is right-handed, the lepton must be right handed and its angular distribution in the P rest frame must be spherically symmetric. For a given direction of the lepton momentum, the conservation of energy and momentum fixes the momenta of both the decay products. Thus, all the dynamics of the decay is contained in one constant, the decay constant f_P, which fixes the leptonic decay width. The corresponding formula is

$$\Gamma_{P\to l\overline{\nu}} = \frac{G_F^2}{8\pi} M_P m_l^2 \left(1 - \frac{m_l^2}{M_P^2}\right)^2 |V_{qQ}|^2 f_P^2 \,. \tag{39}$$

Here V_{qQ} is the element of the Cabibbo-Kobayashi-Masakawa matrix. This formula corresponds to the conventions, where $f_\pi \approx 131$ MeV. Caution is necessary, because conventions, where f_π is larger, or smaller, by a factor $\sqrt{2}$ are also occasionally used. The decay constant is closely related to the matrix elements of the hadronic weak current:

$$\langle 0|J_\mu|P\rangle = i f_P p^\mu \,, \tag{40}$$

where p^μ is the four-momentum of P.

The leptonic decay is imagined as a two-step process. First the two valence partons of the parent must meet. The probability for that to happen is proportional to the square of the absolute value of the wave function of the $(Q\overline{q})$ pair at zero separation $|\psi(0)|^2$. The second step is the annihilation of the Q and $\overline{q}$ sitting on top of each other. Thus

$$\Gamma_{P\to l\overline{\nu}} \sim |\psi(0)|^2 P_{\mathrm{ann}} \,, \tag{41}$$

where the annihilation probability P_{ann} may depend on the partons Q and $\overline{q}$, but not on the structure of the meson. The structure of the meson affects the decay probability only *via* the factor $|\psi(0)|^2$. The decay constant is proportional to the square root of the leptonic decay width and, therefore, to $|\psi(0)|$. The usual formula [38] is

$$f_P = \sqrt{12}\frac{|\psi(0)|}{\sqrt{M_P}} \,. \tag{42}$$

The numerical coefficient follows from a specific calculation, but the proportionality to $1/\sqrt{M_P}$ seems difficult to avoid: Since the dimension of the decay constant is MeV and that of the wave function is $\mathrm{MeV}^{3/2}$, a mass is necessary to get the correct dimension of the right hand side. A substitution of the heavy quark mass instead of the meson mass changes little in the heavy quark limit. A substitution of the light quark mass is contraindicated, because the decay constant should remain finite, when the mass of the light quark tends to zero.

On the other hand the prediction

$$f_B = \sqrt{\frac{M_D}{M_B}} f_D \approx 0.6 f_D \tag{43}$$

is very unlikely to be correct. Experimental data are not yet available, except for the rather weak upper bound $f_D < 290$ MeV following from the limit for the decay width $\Gamma_{D\to\mu\bar{\mu}}$ given in [41], but calculations using QCD sum rules [39, 28], as well as calculations on lattices [42, 43], suggest that

$$f_B \approx f_D \,. \tag{44}$$

More references concerning sum rule calculations can be found in Ref. [4] and concerning lattice calculations in Ref. [44]. We shall comment on the sum rule calculations latter. Concerning the lattice calculations let us note that they are somewhat indirect [44]. Using the static approximation as starting point one can (formally, *i.e.* forgetting about decays $Q \to Wq$ *etc.*) calculate the decay constants for very heavy mesons, much heavier than the B. One finds the expected dependence $f_P \sim 1/\sqrt{M_P}$, but the extrapolation down to the masses of B and D mesons gives unacceptably high values. On the other hand, one can calculate the decay constants for fairly light mesons with masses up to, perhaps, the mass of the D meson. There the coupling constant increases with the meson mass. Using both results one can postulate an interpolation formula

$$f_P \approx \frac{0.6\mathrm{GeV}}{\sqrt{M_P}}\left(1 - \frac{0.8\mathrm{GeV}}{M_P}\right) \tag{45}$$

and use that to obtain f_B. One could, perhaps, find this theoretical evidence noncompelling, but a similar problem occurs for light mesons, where there are experimental data. Van Royen and Weisskopf [38] found it impossible to explain, without *ad hoc* assumptions, the experimental result $f_K > f_\pi$. This became known as the Van Royen–Weisskopf paradox. Their solution was to postulate that $|\psi(0)| \sim \sqrt{M_P}$. Today, however, we have the heavy quark effective theory, which implies that the wave functions of the light stuff in the D meson and in the B meson are nearly the same. This is being used when proving that the Isgur–Wise function equals one at the no recoil point, and thus is confirmed by the successful determination of the absolute value of the element V_{cb} of the Cabibbio–Kobayashi–Masakawa matrix [45]. Consequently, the Van Royen–Weisskopf paradox remains a paradox, probably closely related to the effect for the B and D mesons we are interested in. It may happen that this problem is easier to solve for the heavy mesons than for the light ones. For instance, another old observation that the masses of the corresponding vector and pseudoscalar mesons satisfy the relation $M_V^2 - M_P^2 = \text{const}$, whatever the flavours of the quark and antiquark constituting the mesons, is easily understood in the heavy quark effective theory, where the hyperfine splitting $M_V - M_P$ is an effect of order m_Q^{-1}.

To summarize: it is very likely that estimate (44) is better than the "obvious" estimate (43). In the language of lattice calculations the question is: why the coefficient of the $1/M_P$ term is so large? In the language of wave functions a proposal was the Van Royen–Weisskopf Ansatz; this, however, seems inconsistent with the heavy quark effective theory. In the following section we will show that the problem disappears, if one gives up the notion of wave functions in favour of duality arguments. This is not a full solution of the problem, but it is instructive.

5.2. *Use of semilocal duality*

The intermediate boson W on its mass shell is a purely vector particle and consequently has three components. As a virtual particle mediating weak decays in the GeV region, however, the W boson has a fourth, scalar components, which we shall denote W_0. Let us consider the decay width Γ_{hadr} for the weak decays of low mass W_0 bosons into hadrons containing the heavy quark Q, as a function of the square of the four-momentum q^2 of the W_0. For definiteness we shall put b for the Q quark, but analogous arguments apply to the c quark. For $q^2 < M_B^2$ this function vanishes, because there is nothing to decay into. At $q^2 = M_B^2$ there is a sharp peak and again the function drops to zero. For W_0 masses above $M_B + M_\pi$ there is a slowly rising continuum contribution and peaks corresponding to excited B states. A very different picture corresponds to the q^2 dependence of the decay width Γ_{part} of the W_0 of mass $\sqrt{q^2}$ into a $b\overline{u}$ pair. This is not something really measurable. We interpret it as the result of a calculation including perturbative effects and the nonperturbative corrections as suggested by the operator product expansion. The nonzero values start at the

W_0 mass equal $m_b + m_u$, *i.e.* below[1] the B mass, and from this point on the function increases monotonically. Semilocal duality is the claim that the two functions $\Gamma_{\mathrm{hadr}}(q^2)$ and $\Gamma_{\mathrm{part}}(q^2)$, though locally quite different, become similar, when suitably averaged over the W_0 mass.

This claim, if justified, has important implications: Γ_{hadr} contains a contribution from the decay $W_0 \to B$, which integrated over a small mass range around $q^2 = M_B^2$ is proportional to the square of the decay constant f_B. Since Γ_{part} does not involve f_B, equating the two (averaged) decay widths gives an equation for the decay constant. In this approach the use of wave functions is eliminated. It is possible to calculate and compare the two decay constants f_B and f_D. We shall show now that this program realized in a very simpleminded way [4] yields a result consistent with (44). In the following section we will briefly describe the much more sophisticated, but corresponding to the same physical idea, way of handling of this problem in the framework of sum rules.

5.3. *Simple calculation*

In order to use semilocal duality, one has to know the quark masses and to choose an averaging prescription. Following Ref. [4] we choose

$$m_c = 1.47 \text{ GeV}, \qquad m_b = 4.6 \text{ GeV}. \tag{46}$$

When calculating the decay constant, the necessary averaging should be in the region $q^2 \approx M_b^2$. We choose the simplest prescription and integrate the decay widths $\sqrt{q^2}\,\Gamma_{\mathrm{hadr}}(q^2)$ and $\sqrt{q^2}\,\Gamma_{\mathrm{part}}(q^2)$ from the threshold m_b^2 ($m_u \approx 0$) to $(m_b + E_c)^2$. The parameter E_c should be chosen so large that semilocal duality is applicable and so small that the integral of Γ_{hadr} contains only the contribution from the B peak. The mass of the first radial excitation of the B meson is not yet known, but a rough estimate is possible. According to HQET, the mass gap between the first radial excitation of the B meson and the ground state B meson should be close to the mass gap between the first radial excitation of the D meson and the ground state D meson. The mass of the radial excitation of the D meson is not known either, but it is probably close to the masses of the known P-wave excitations D_1 and D_2, *i.e.* to 2.45 GeV. Since the mass of the c quark has been chosen 1.47 GeV, this corresponds to $E_c \approx 1$ GeV. Further we shall introduce better estimates, but this one is roughly correct. The factor $\sqrt{q^2}$ in the prescription for averaging is not very important numerically, but it has been included, because it simplifies the formulae.

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[1] In the sum rule approach the low quark masses corresponding to current quarks are used.

The hadronic integral does not depend on the parameter E_c. A standard calculation gives

$$I_{\text{hadr}} = C\pi f_B^2 M_B^4 , \tag{47}$$

where C is a known constant. For the partonic integral, we found the approximation

$$I_{\text{part}} = \tfrac{2}{3} C m_b^3 E_c^3 \left[1 + \tfrac{3}{44} \left(\frac{E_c}{m_b} \right)^2 - \pi^2 \frac{\langle \bar{q} q \rangle}{E_c^3} \right] . \tag{48}$$

The constant C is the same as in I_{hadr}. The last term in the bracket is the contribution of the quark condensate, which according to our present understanding fils the physical vacuum. For a typical value

$$\langle \bar{q} q \rangle = -(0.23 \text{ GeV})^3 , \tag{49}$$

this is a 10% correction for $Q = c$ and a 5% correction for $Q = b$. Equating the two averages, reintroducing the general notation P and Q and rearranging, one finds that the equation for the decay constant with better than one per cent accuracy can be written in the form

$$\sqrt{M_P} f_P = \frac{1}{\pi} E_c^{3/2} \left(\frac{m_Q}{M_P} \right)^{3/2} \left[1 - \frac{\pi^2}{2} \frac{\langle \bar{q} q \rangle}{E_c^2} + \frac{3}{88} \left(\frac{E_c}{m_Q} \right)^2 \right] . \tag{50}$$

In order to estimate the parameter E_c necessary to find the ratio f_B/f_D, we shall use two methods. Let us consider first the static limit $m_Q \to \infty$. One finds

$$\left(\sqrt{M_P} f_P \right)_{\infty} = \frac{1}{\pi} (E_c^{\infty})^{3/2} . \tag{51}$$

Lattice calculations yield for the right hand side $(0.6 - 0.7)$ GeV, which implies $E_c^{\infty} = (1.5 - 1.7)$ GeV. This does not contradict our previous estimate, because the parameter E_c depends on the energy scale. Since in the high mass limit the ratio $m_Q/M_P \approx 1$, we find

$$f_B = 1.0\, f_D . \tag{52}$$

As another estimate we use the values of the parameters found in sum rule calculation at energy scales corresponding to the physical meson masses. They are

$$E_c^D \approx 1.08 \text{ GeV} , \tag{53}$$

$$E_c^B \approx 1.30 \text{ GeV} , \tag{54}$$

$$\overline{\Lambda} \equiv M_P - m_Q \approx 0.70 \text{ GeV} . \tag{55}$$

Substituting these numbers one finds

$$f_B \approx 0.9\, f_D\,. \tag{56}$$

In the following section we present a short introduction to the method of sum rules, which yields similar results from similar assumptions, but in a much more respectable way.

5.4. More advanced calculation

In the sum rule approach to decay constants of pseudosclar mesons the basic concept is the correlator

$$\Pi_5(q^2) = i\int d^4x e^{iqx}\langle 0|TA_5(x)A_5^\dagger(0)|0\rangle\,. \tag{57}$$

where

$$A_5(x) = im_Q\bar{q}\gamma_5 Q \tag{58}$$

is the pseudosclar weak current.

Further steps are analogues of the steps described in the previous section. One has to calculate the hadronic and the partonic version of the correlator, perform an analogue of averaging, compare the two expressions and solve for the decay constant. The analogue of averaging is the use of dispersion relations for $q^2 \to -\infty$. For the hadronic version one obtains

$$\Pi_5^{\mathrm{hadr}}(q^2) = \frac{f_P^2 M_P^4}{M_P^2 - q_2 - i\epsilon} + \frac{1}{\pi}\int\limits_{s_c}^{\infty} ds\, \frac{\mathrm{Im}\,\Pi_5^{\mathrm{pert}}(s)}{s - q^2 - i\epsilon} - \mathrm{Sub}\,. \tag{59}$$

Note that the hadronic spectrum is much simplified here. The first term is the contribution from the P pole and all the rest is approximated by a perturbative expression from some threshold at s_c to infinity. The position of the threshold s_c is a parameter to be adjusted. Note also the subtraction terms denoted Sub, which are necessary to make the integral convergent. The partonic correlator is written in th form

$$\Pi_5^{\mathrm{part}}(q^2) = \frac{1}{\pi}\int\limits_{m_Q^2}^{\infty} ds\, \frac{\mathrm{Im}\,\Pi_5^{\mathrm{pert}}(s)}{s - q^2 - i\epsilon} - \mathrm{Sub} + \Pi_5^{\mathrm{cond}}(q^2)\,. \tag{60}$$

The last term contains the condensates with perturbatively calculable coefficients. Each condensate has dimension energy to some power. The higher this power the smaller (probably!) the contribution. The first three condensates are: the $\langle\bar{q}q\rangle$ condensate of dimension 3, which has already been introduced, and the condensates of dimension 4 and 5:

$$\langle\alpha_s G_{\mu\nu^a} G^{a\mu\nu}\rangle \approx (0.441\ \text{GeV})^4, \tag{61}$$

$$g_s\langle\bar{q}\sigma_{\mu\nu}G^{\mu\nu}q\rangle \approx 0.8\ \text{GeV}\langle\bar{q}q\rangle\,. \tag{62}$$

In these formulae G denotes the gluon field. We shall not discuss the condensates any further.

Before equating the hadronic and the partonic expressions for the correlator one usually takes their Borel transforms, *i.e.* acts on both expressions with the operator

$$\hat{B}_M = \lim_{n\to\infty} \frac{(q^2)^n}{\Gamma(n)}\left(-\frac{d}{dq^2}\right)^n, \tag{63}$$

where the parameter

$$M^2 = -\frac{q^2}{n} \tag{64}$$

is held fixed. The subtraction terms, which are polynomials in q^2, give zero under this transformation and the equation for the decay constant takes the form

$$\frac{f_P^2 M_P^4}{M^2}\exp\left(\frac{-M_P^2}{M^2}\right) = \frac{1}{\pi M_2}\int_{m_Q^2}^{s_c} ds\, \text{Im}\,\Pi_5^{\text{pert}}(s)\exp\left(\frac{-s}{M^2}\right) + \hat{B}_M\,\text{Cond}\,, \tag{65}$$

where Cond denotes the contributions from the condensates. We do not write here the final equation obtained after substituting explicitly the perturbative terms and the condensates, because it is rather long [39, 40]. A careful analysis of the dependence of the various terms on the energy scale and on the renormalization scheme is also crucial here. The estimate of errors is very uncertain. Moreover, there are two parameters: M and s_c, which have to be estimated from arguments based on common sense and on the assumption that the solution should be stable, *i.e.* depend little on these parameters. The result of all this analysis is that $f_B \approx f_D$. The reason should be clear: semilocal duality is being used instead of the wave functions of $Q\bar{q}$ bound states. Typical results are $f_D \approx 170$ MeV, $f_B \approx 190$ MeV.

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