FLUCTUATIONS IN THE FRAGMENTATION PROCESS*

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In this review, we present some general framework of sequential fragmentation, as provided by the newly proposed Fragmentation–Inactivation –Binary model, and to study briefly its basic and universal features. This model includes as particular cases most of the previous kinetic fragmentation models. In particular, we discuss how one arrives in this framework to the critical behaviour, called the shattering transition. This model is then compared to recent data on gold multifragmentation at 600 MeV/nucl.

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1. Introduction

For the description of many physical processes, involving complex systems, physicists try to use simple, averaged quantities. Indeed, in thermodynamics, fluctuations around the thermodynamic mean value obey generally the Gaussian law and they are small in large systems. In probability theory, the sum of large number of independent Gaussian variables $X_N = a_1 + a_2 + ... + a_N$, where $\langle a_j \rangle = a$ and $\langle a_j{}^2 \rangle = \sigma^2$ is itself distributed according to the Gaussian law and in the limit of large N, relative

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fluctuations in the distribution of X_N are small! Actually, non-Gaussian distributions are quite common in nature and, moreover, numerous examples exist showing that strong local fluctuations at various scales may exist even in large, macroscopic systems. These include hydrodynamic instabilities and turbulent flow in a classical incompressible fluid, distribution of luminous matter in the Universe, energy distribution of cosmic rays, multiparticle distributions in ultrarelativistic collisions of leptons, hadrons and nuclei, self-exciting magnetic fields in a conducting liquid, localization in disordered media and many others. In these and many other similar situations the average quantities, even though well defined, do not provide a sensible characterization of a studied system.

The structures in the random medium take the form of the "peaks and beach" structures in which the rare, high intensity peaks, localized randomly in space and time and concentrating the majority of the mass of the distribution are separated by large size, low intensity regions. The rare fluctuations (high intensity peaks) make then the main contribution to the mean moments of the distribution and the ratios of the moments over high and low intensity regions respectively, grow exponentially both in $\vec{x}$ and t. This peculiar behaviour of statistical moments in comparison with the Gaussian relation is called sometimes the *intermittency*. In terms of the Fourier analysis, intermittency is characterized by both a slow decrease of the contribution from higher Fourier harmonics and an anomalous phase relations among them which lead to the appearance of individual high peaks.

Recently, it was proposed to apply the intermittency analysis to the studies of the fragment-charge distribution following the decay of hot nuclear residuum and the scale-invariant fluctuations have been found in the collisions of heavy-nuclei at $E/A \simeq 1$ GeV [1]. The origin of these fluctuations in nuclear fragmentation are not yet clear but it was shown that they provide new observables which could help to distinguish between different fragmentation mechanisms. Below we discuss the possible origin of this phenomenon in cluster fragmentation processes using the newly proposed Fragmentation–Inactivation–Binary (FIB) model [2] which includes, as particular cases, most of the previous kinetic fragmentation models. In the latter model we give basic equations and discuss the phase diagram, in particular the critical behaviour and the critical fluctuations associated to it.

2. Time–evolution of a random quantity

Random quantities of the multiplicative type arise quite naturally in various evolutionary problems. Let us consider a typical evolutionary equa-

tion of the unstable type:

$$\frac{dX}{dt} = U(t;\tau)X\,, \tag{1}$$

where $U(t;\tau)$ is a Gaussian process ($\langle U\rangle_\tau = 0, \langle U^2\rangle_\tau = \sigma^2$. The solution of Eq. (1) is of the multiplicative type [3]:

$$X(t) = X(t_0)\exp\int\limits_{t_0}^{t} U(t')dt' \simeq_{t\gg\tau} \prod_{i=1}^{N=t/\tau}\left(\exp\left[\int\limits_{(i-1)\tau}^{i\tau} U(t')dt'\right]\right), \tag{2}$$

and

$$\xi_N \equiv \xi_t \equiv \ln X(t) \simeq \sum_{i=1}^{N=t/\tau}\left(\int\limits_{(i-1)\tau}^{i\tau} U(t')dt'\right) \equiv \sum_{i=1}^{N=t/\tau}\eta_i \equiv \eta_1+\eta_2+\ldots+\eta_N\,. \tag{3}$$

The CLT ensures that the distribution $P(\xi_N)$ is Gaussian. The general solution of the Eq. (2) has a form of a *product* of a large number of random quantities and is an *intermittent* quantity characterized by the moments:

$$\langle X^p(t)\rangle \simeq \langle X^p(t=0)\rangle \exp\left(\frac{p^2 t\sigma^2}{2}\right). \tag{4}$$

They grow exponentially with time and this growth rate is faster the higher is the rank of the moment. In other words, the relative fluctuations *grow* and may become *very large* at large times t. Moreover, $X(t)$ is distributed according to a log-normal distribution. Analogous features to those discussed above for the solutions of Eq. (2) can be found for the linear scalar equations with diffusion and a random breading which are often used to describe the kinetics of chemical and nuclear reactions [4].

Obviously, not all evolutionary equations yield as their solution the intermittent random quantity. For example, the evolutionary equation of the type: $dX/dt = U(t;\tau)$ gives the solution which in the limit of large t approaches the Gaussian random variable [3].

2.1. Dynamics of an incompressible fluid

Turbulent flows of the incompressible fluid are characterized by their intrinsic instability and unpredictability. Any small perturbation at a given instant of time leads to a strong distortion of the flow pattern. The peculiarity of the turbulent flows is in its ability to transfer momentum and heat

among extremely different scales. This process is associated with the large non-statistical fluctuations. In principle, the statistical mechanics of the incompressible flow in the turbulent regime, could be build up on the basis of the Navier–Stokes equations. Unfortunately, all analytical approaches encounter the closure problem, *i.e.* the dynamical equations for the fluid correlation functions of a given rank are coupled to the higher order correlation functions. Consequently, the closure of the infinite hierarchy of equations for the correlation functions at any given order introduce in the chaotic (turbulent) regime an uncontrollable error in the description of the flow and its properties. The real progress in understanding the mechanism of the chaotic energy (momentum) transfer could be achieved by noticing that the process deals with the hierarchy of structures ("eddies") at different scales [5]. In this picture, the turbulent energy transfer is given by a self-similar cascade process of "eddy" breaking down, in which there is a transfer of energy to smaller and smaller scales ("eddies") until the fragmentation is stopped by the dissipation. This semi-phenomenological approach, in which the energy cascade is given in terms of multiplicative processes with a phenomenological fragmentation function, turned out to be an extremely successfull model of the chaotic energy transfer among the "eddies" in the turbulent flow. It is obvious, that the fragmentation function in this theory should be related to the nature of singularities of the Navier–Stokes equations, developing at high Reynolds numbers. Unfortunately, nobody was able until now to explain this relation.

2.2. Instabilities in the motion of nucleon in the random medium

A standard tool in describing the nuclear dynamics in the low energy domain is the Boltzmann–Uehling–Uhlenbeck (BUU) model or the Boltzmann–Langevin (BL) model. Whereas the first model is well suited for the description of nuclear processes involving small fluctuations around the average, the latter type of models is in principle capable of describing large fluctuations as well.

One may try to justify BUU or BL equations by relating them to the BBGKY hierarchy of the kinetic equations for N-particle distribution functions. In this case, BUU and BL type of models are obtained by closing the hierarchy at the level of the two-particle correlation functions and by selecting a special form for the two-particle collision integral. In the low density limit and under the condition that the flow of the one-body density is laminar, one may reasonably hope that the restricted framework provided by BUU or BL transport equations, yields a correct approximation of the exact equations for the N-body density matrix. However, in the strongly chaotic domain the approximate equations may be insufficient as the error grows exponentially, and so is the loss of the information about the studied

system. Hence, in the chaotic regime the BBGKY kinetic equations, on one hand, *cannot be replaced* by the closed system of equations for the low rank correlation functions and, on the other hand, the reduced set of equations (BUU, BL, *etc.*) *cannot be solved* accurately.

Simulations of the BUU or BL equations for energetic heavy-ion collisions show that these equations develop various instabilities resembling those known before in the Fermi liquid theory (spinodal instability [6]) and in the theory of rotating uniformly charged liquid drop with a surface tension [7]. As a result, the complicated, unstable and non-analytical flow of the particle distribution appears. In BL approach these instabilities are further magnified by the coupling of the random fluctuating force to the particle trajectories which leads to the intermittent type of instabilities of the particle trajectory in the phase space. In the early stage of the collision when the particle density piles up, the probability for the three and more particle collisions are not negligible, and the foundation of the Boltzmann type of equations becomes questionable. This is the so-called transient regime of the collision [8, 9] when the large amplitude fluctuations in the phase-space develop and determine the "catastrophic" phenomena such as the copious fragment production. In this regime many of the quantitative aspects of the BUU and/or BL simulations are not reliable and one has to look for the semi-phenomenological approach, as in the case of the fluid dynamics in the chaotic regime. For the description of the nuclear multifragmentation such a semi-phenomenological approach, the Fragmentation–Inactivation–Binary (FIB) model [2], has been studied recently.

3. Fragmentation–Inactivation–Binary model

By "sequential fragmentation", we mean the disaggregation of an initially connected system *via* a sequence of individual break-up events. According to the considered case, this sequence will be either temporal, *i.e.* each individual event occurs at a definite time, or topological, which means an incomplete ordering of the set of events. Let us now consider a binary sequential fragmentation of the cluster labelled N. This fragmentation is drawn here (see Fig. 1) as a topological sequence and all the possible informations are not available. For instance, we know that i appears after N but we cannot decide if i has appeared before or after $(N-i)$. The associated temporal sequence should give the time t_1 when cluster $N \longrightarrow i + (N-i)$ as well as the times t_2, when $i \longrightarrow k + (i-k)$, and t_3 when $(N-i) \longrightarrow j + (N-i-j)$. Obviously, there is a larger amount of information in the temporal sequence. Contrary to the geometry of the clusters, the size-distribution provides a collective description of the system. In other words, one unique break-up event may be quite complicated and

specific of the microscopic physics governing this splitting, but after the system has undergone a large number of such breaks, most of the microscopic details are smoothed out.

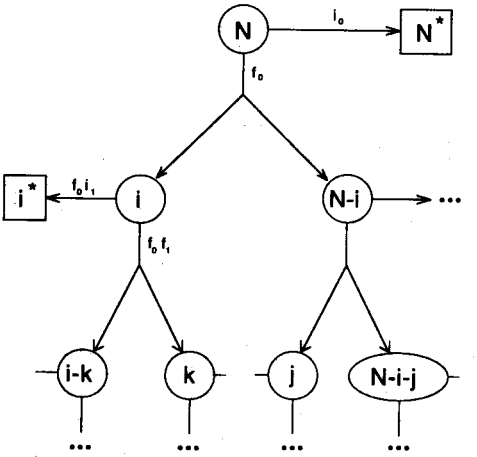

Fig. 1. Schematic representation of a topological sequence of the binary sequential fragmentation of the cluster labelled ○.

To put the above ideas into practice, some general assumptions must be done:

- There are several individual break-up events in each sequential fragmentation [1].
- Each individual break-up can be simulated by a "black-box".

This latter hypothesis is indeed non-trivial and we give now some details. One defines the fragmentation kernel F_{ij}, which is a positive probability function giving the mean probability of the break-up event: $(i+j) \longrightarrow (i)+(j)$ per unit of time. Note that the existence of such a probability function is a mean-field hypothesis. The notation (i) means one cluster of generalized size i, *i.e.* a vector with any conservative additive quantities (*e.g.* the mass, the momentum, the energy,...) as the components. To simplify the form of the subsequent equations, we restrict the basic process to a binary fragmentation.

Stop of the fragmentation occurs when all fragments are inactive. We assume that once a given cluster is inactive, it cannot break-up again. This is a low-density approximation in the sense that the activity cannot be brought from outside, *e.g.*by fragment-fragment collisions. One defines then

[1] This hypothesis is opposed to an instantaneous multifragmentation where all the final fragments are generated at the same time.

the inactivation kernel I_k ($I_k > 0$), which is the mean probability per unit of time of the occurence of the inactivation event: $(k) \longrightarrow (k)^*$, with $(k)^*$ being an inactive cluster of generalized size k. With these low-density mean-field assumptions, one can now write the basic equations describing the time evolution of the size distribution of the fragmenting system.

3.1. Basic equation

Let us define the state of the system by a set of two vectors: $\vec{m}, \vec{n}$ where $\vec{m} = (m_1, m_2, \ldots)$ is the list of the number of active monomers, dimers, *etc.*, and $\vec{n}$ is the same but for inactive clusters. At the beginning of the process, we have: $\vec{m} = \vec{u}_N$, where $\vec{u}_N$ has zero components, except 1 at the n^{th} place, and $\vec{n} = \vec{0}$. At the end of the same process, we have: $\vec{m} = \vec{0}$ and we wish to know what $\vec{n}$ is. The initial condition will be denoted by subscript N when needed.

The Marcus equation gives the change in the probability $w_N(\vec{m}, \vec{n}; t)$ to get one state $\vec{m}, \vec{n}$ at the time t, as:

$$\begin{aligned}\frac{\partial w_N}{\partial t}(\{\vec{m}, \vec{n}\}; t) = \\ \sum_{i,j} F_{ij}[(m_{i+j}+1) w_N(\{(\ldots, m_i - 1, \ldots, m_j - 1, \ldots, m_{i+j}+1, \ldots), \vec{n}\}; t) \\ - m_{i+j} w_N(\{\vec{m}, \vec{n}\}; t)] \\ + \sum_k I_k[(m_k+1) w_N(\{(\ldots, m_k+1, \ldots), (\ldots, n_k - 1, \ldots)\}; t) \\ - m_k w_N(\{\vec{m}, \vec{n}\}; t)] \qquad (5)\end{aligned}$$

In each bracket [ ], the first term corresponds to the gain term and the second one to the loss term, due to the events of the type either: $(i+j) \longrightarrow (i) + (j)$ or $(k) \longrightarrow (k)^*$. The probability, during the time dt, of such events is $F_{ij} m_{i+j} dt$ and $I_k m_k dt$, respectively. In this form, Eq. (1) is rather complicated, because of the explicit dependence on various states of the system. It can be rewritten in a more convenient form with the help of the generating function:

$$\tilde{w}_N(\vec{x}, \vec{y}; t) = \sum_{\vec{m}, \vec{n}} w_N(\{\vec{m}, \vec{n}\}; t) \vec{x}^{\vec{m}} \vec{y}^{\vec{n}}, \qquad (6)$$

where $\vec{x}^{\vec{m}}$ means the product: ${x_1}^{m_1} {x_2}^{m_2} \ldots$ and the sum runs over all the states. After some algebraic manipulations, one finds:

$$\frac{\partial \tilde{w}_N}{\partial t}(\vec{x}, \vec{y}; t) = \sum_k \left(\sum_{i=1}^{k-1} F_{i,k-i}(x_i x_{k-i} - x_k) + I_k(y_k - x_k) \right) \frac{\partial \tilde{w}_N}{\partial x_k}(\vec{x}, \vec{y}; t), \qquad (7)$$

which is now an ordinary first-order differential equation on a $2N+1$-dimensional space, with the boundary conditions $\tilde{w}_N(\vec{x},\vec{y};0) = x_N$. A similar equation for the aggregation problem has been derived by Marcus [10].

From Marcus equation, one may derive a set of equations for the average values:

$$\bar{m}_k(N;t) = \sum_{\vec{m},\vec{n}} w_N(\{\vec{m},\vec{n}\};t) m_k = \frac{\partial \tilde{w}_N}{\partial x_k}(\vec{1},\vec{1};t), \tag{8}$$

where $\bar{m}_k(N;t)$ denotes the average number of clusters of size k, for an initial condition N at time t. In this way one obtains the master equations:

$$\begin{aligned} \frac{d\bar{m}_k}{dt}(N;t) &= -\Phi_k \bar{m}_k(N;t) + 2 \sum_{i \geq k+1} F_{k,i-k} \bar{m}_i(N;t) \\ \frac{d\bar{n}_k}{dt}(N;t) &= I_k \bar{m}_k(N;t), \end{aligned} \tag{9}$$

where we have denoted:

$$\Phi_k = I_k + \sum_{i=1}^{k-1} F_{i,k-i}. \tag{10}$$

Per unit of time, one has in average $I_k \bar{m}_k$ clusters of the size k which become inactive, $\bar{m}_k$ clusters of size k which disappear by fragmentation and $2 \sum_{i \geq k+1} F_{k,i-k} \bar{m}_i$ clusters of size k which appear by fragmentation of larger clusters (the factor 2 comes from the distinction between F_{ij} and F_{ji} in the definition of the fragmentation kernel). Comparing with the Marcus equation, one has lost a considerable amount of informations since the master equations work with the average values. On the other hand, the master equations have the great advantage to be easy to solve. Because of their linearity, they can be solved numerically by downward iterative integrations for every choice of kernels and initial conditions.

Another equations can be written following a very different way, investigated by Esipov *et al.* [11]. Let us look at the first event of the sequence: it is either a fragmentation $(N) \longrightarrow (i) + (N-i)$ or an inactivation $(N) \longrightarrow (N)^*$. Suppose that this first event was a fragmentation at a time t_1. We may then consider one state $\{\vec{m},\vec{n}\}$ at a given time t for the initial condition i, and another state $\{\vec{m}',\vec{n}'\}$ at a time $t - t_1$ for the initial condition $N - i$, with the constraints: $\vec{m} = \vec{m}' + \vec{m}''$ and $\vec{n} = \vec{n}' + \vec{n}''$. This leads to the following equation:

$$w_N(\{\vec{m},\vec{n}\};t) = \delta(\vec{m}-\vec{u}_N)\delta(\vec{n})e^{-\Phi_N t}$$
$$+ I_N\delta(\vec{m})\delta(\vec{n}-\vec{u}_N)\frac{1-e^{-\Phi_N t}}{\Phi_N}$$
$$+ \sum_{i=1}^{N-1}\int_0^t F_{i,N-i}\Big[\sum_{\substack{\vec{m}'+\vec{m}''=\vec{m}\\ \vec{n}'+\vec{n}''=\vec{n}}}$$
$$\times\, w_i(\{\vec{m}',\vec{n}'\};t'-t)w_{N-i}(\{\vec{m}'',\vec{n}''\};t'-t)e^{-\Phi_N t'}\Big]dt' . \qquad (11)$$

It is interesting to notice that this fragmentation is a random multiplicative process, since the probability w_N on the rhs of Eq. (11) is equal to some linear functional of products like $w_i w_{N-i}$. These sorts of random multiplicative processes are widely studied because of their anomalous fluctuations [1, 12].

4. Scaling laws and the phase diagram

The above model without inactivation ($I_k \equiv 0$), which we shall call the standard model, has been studied extensively [13]. The main surprise has been the discovery that for certain fragmentation kernels, smaller clusters break up at increasingly rapid rates. This leads to an apparent loss of the total mass due to the formation of a "zero-size" particles phase as soon as $t > 0$. This "shattering" second-order phase transition we shall also recover in the FIB model.

In the standard model ($I_k \equiv 0$), there is no possibility to obtain non-trivial limit size-distribution as $t \longrightarrow \infty$. On the contrary, the FIB model leads to interesting asymptotic size-distribution. The master and cascade equations can be readily written in this case and their main properties have been given elsewhere [2]. Note just that the composed particle first moment $N_C = 1 - \bar{n}_1/N$ plays the role of an order parameter when the total mass becomes infinite. When the probability I_N/Φ_N that no event occurs tends to 1, then $N_C \longrightarrow 1$ and this is called the ∞-cluster phase since there remains a large cluster of size of order N. On the contrary, when this probability tends to 0, the parameter N_C is smaller than 1 and this means that a finite ratio of the total mass is converted into finite-size clusters: monomers, dimers, *etc.* . This is the shattered phase and the asymptotic ($N \longrightarrow \infty$) value of N_C coincides with the total mass as defined in the continuous model.

4.1. The infinite cluster

The average size of the largest fragment $\langle Z_{\max}\rangle_N$ in the system of size N is not easy to handle. However, the recurrent inequality can be derived by the following reasoning. The first event is either inactivation of the cluster of size N (*i.e.*$Z_{\max} = N$), or its fragmentation into clusters i and $N - i$. The cascade beginning with the cluster of size i leads to the largest cluster of the average size $\langle Z_{\max}\rangle_i$. Obviously, $\langle Z_{\max}\rangle_N \geq \langle Z_{\max}\rangle_i$, because there may be a larger cluster in the branch beginning with the $N-i$-cluster. The same reasoning can be done for the $N - i$-branch and we can write:

$$\langle Z_{\max}\rangle_N \geq N\frac{I_N}{\Phi_N} + \sum_{i=1}^{N-1} \mathrm{Max}[\langle Z_{\max}\rangle_i, \langle Z_{\max}\rangle_{N-i}]\frac{F_{i,N-i}}{\Phi_N} \tag{12}$$

with $\langle Z_{\max}\rangle_1 = 1$. Suppose now that the ratio $\langle Z_{\max}\rangle_N/N$ behaves like a positive constant σ when $N \longrightarrow \infty$. Replacing $\langle Z_{\max}\rangle_N$ by σN in Eq. (12) leads to the relation:

$$\sigma \geq \left[1 + \frac{2\sum_{i=1}^{N/2} iF_{i,N-i}}{NI_N}\right]^{-1}. \tag{13}$$

It is easy to show, that when α and β are not in the shattering phase then the rhs of Eq. (13) tends to 1. This means that, in the average, almost all the mass is concentrated in one big cluster. For this reason, we call this phase the ∞-cluster phase. In the shattered phase, $\sigma < 1$ because $\langle n_1\rangle_N + \langle Z_{\max}\rangle_N \leq N$, *i.e.* the ∞-cluster phase and the shattered phase cannot coexist. We have been unable to determine the upper bound for σ and values of σ in different phases must be found numerically. We have found numerically, that in the whole shattered phase $\sigma = 0$ and, more precisely, $\langle Z_{max}\rangle_N/N$ decreases in this phase as a power-law of N. A typical example is shown in Fig. 2.

4.2. Few remarks on the choice of the fragmentation kernel

Discussing the phase-diagram of FIB model, we have assumed the homogeneous fragmentation function. For some models, one has a precise idea of the form of this fragmentation kernel either by an exact derivation, or numerical simulations or yet by physical arguments. One canonical example is the percolation on the Bethe lattice for which one can deduce [15]: $F_{ij} \propto (ij)^{-3/2}$ with the proportionality constant being a function of $i + j$. Fragmentation of the percolation cluster in the ordinary 2D percolation at a critical point, has been investigated numerically [14] and the

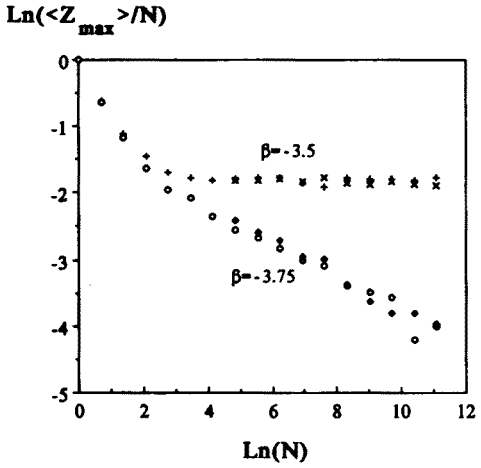

Fig. 2. The average size of the largest fragment $\langle Z_{\max} \rangle$ normalized by the initial size of the system N, is plotted *vs.* N for the FIB model at $\alpha = 2.5, \beta = -3.5$ (on the transition line) and $\alpha = -2.5, \beta = -3.75$ (in the shattered phase).

(i, j)-dependence of the fragmentation kernel deduced from these studies is similar to the solution for the Bethe lattice [15].

Let us now analyze the process of fragmentation due to the spinodal instabilities of the Fermi liquid associated with the density fluctuations. The dispersion relation for long-wavelength density fluctuations is $\omega^2 = {c_s}^2 q^2$, where c_s is the sound velocity and q is the wavenumber. For ${c_s}^2 < 0$, the modes have imaginary frequencies and their amplitudes grow exponentially with a growth rate $\Gamma = -i\omega$. The typical mass fragment produced by this instability is: $\tilde{n} \propto q^{-3}$ and hence the growth rate of an unstable mode is: $\Gamma \propto -i\tilde{n}^{-1/3}$. In a given volume there are $A/\tilde{n}$ of such instabilities and hence the effective growth rate is $\Gamma_{\text{eff}} \propto -i\tilde{n}^{-4/3}$. The factor by which a mode can grow during the time it stays in the unstable region is: $\gamma = \exp \int \Gamma_{\text{eff}} dt$ and typically the fluctuation requires a growth exponent $\gamma \simeq 3$ to disrupt the system [6]. Assuming that Γ_{eff} is approximately constant in time, the necessary time to disrupt the system is $t \propto {\Gamma_{\text{eff}}}^{-1}$. Hence, the fragmentation probability function for a binary process becomes: $F_{ij} \propto i^{2\alpha} + j^{2\alpha}$ with $\alpha = -2/3$, *i.e.* the homogeneous additive kernel. One should stress that the form of F_{ij} does not depend on whether we take collision limit or collisionless limit of the Fermi liquid [6].

4.3. The average mass distribution: The multiplicative fragmentation kernel

If both fragmentation and inactivation kernels are homogeneous functions of the sizes, one expects scaling laws under the critical conditions.

Let us assume the multiplicative fragmentation kernel, *e.g.*: $F_{ij} \propto (ij)^{\alpha}$ (see Fig. 3), and the inactivation kernel I_k in the form: $I_k = I_1 k^{\beta}$. The transition line is defined here as: $\beta = 2\alpha + 1$ if $\alpha > -2$ and $\beta = \alpha - 1$ if $\alpha < -2$. In fact, one finds that the size-distribution in the whole shattered phase behaves like a power law when the size of the clusters is intermediate ($1 \ll k \ll N$) [2]:

$$\frac{\bar{n}_k(N)}{N} \sim k^{\beta - 2\alpha - 3}. \tag{14}$$

Note that the exponent of this power law is independent of the strength I_1 of the inactivation and is always larger than 2. On the transition line, the size-distribution is still a power law:

$$\frac{\bar{n}_k(N)}{N} \sim \frac{N^{-\tau-2}}{k^{\tau}}, \tag{15}$$

but the exponent τ is the solution of the equation:

$$2B(\alpha + \tau, \alpha + 1) = I_1 + B(\alpha + 1, \alpha + 1), \tag{16}$$

and B is the Euler-beta function. This exponent is an explicit function of I_1 and is always smaller than 2. There is, of course, no such power law in the ∞-cluster phase.

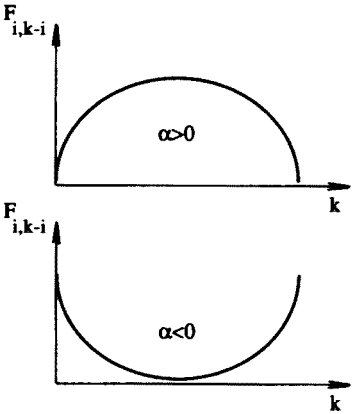

Fig. 3. The sketch of the fragmentation probability function F_{ij} of the FIB model.

4.4. The average mass distribution: The additive fragmentation kernel

If the fragmentation kernel is a homogeneous additive function of the size, *e.g.*: $F_{ij} \propto \frac{1}{2}(i^{2\alpha} + j^{2\alpha})$ and the inactivation kernel is $I_k = I_1 k^{\beta}$, then

the transition line is found at: $\beta = 2\alpha + 1$ if $2\alpha > -2$ and $\beta = \alpha - 1$ if $2\alpha < -2$. The size-distribution in the whole shattered phase behaves like a power law and the exponent of this power law is the solution of the equation:

$$\Gamma(\tau + \beta - 1) = \Gamma(\tau + \beta - 2\alpha - 2)\,\Gamma(2\alpha + 2)\,, \tag{17}$$

where Γ is the Euler–gamma function. On the transition line, the size-distribution is still a power law but the exponent τ is now the solution of the equation:

$$\frac{\Gamma(\tau)}{\Gamma(\tau + 2\alpha + 1)} = \frac{\tau - 1}{(\tau + 2\alpha)\Gamma(2\alpha + 2)} + \frac{I_1}{\Gamma(2\alpha + 1)}\,. \tag{18}$$

No such power law is found in the ∞-cluster phase. These results are of course very similar to the multiplicative case.

4.5. The mass–mass correlation functions and intermittency

Fluctuations in the fragment-size distribution can be studied using the scaled factorial moments (sfms) [1, 12]. In FIB model they can be conveniently defined using the derivatives of the genetrating function (6). The critical phenomenon of the shattering phase transition is accompanied by the intermittent fluctuations at the transition line [2] (see Fig. 4) [2]. One can argue, that everywhere outside the transition line the sfms should disappear. For that let us insert a trial function for probabilities w_N:

$$w_N(\vec{m}, \vec{n}; t) = f(t) \prod_k \left(\frac{\langle m_k \rangle^{m_k}}{m_k!} \frac{\langle n_k \rangle^{n_k}}{n_k!} \right) \tag{19}$$

into the Marcus Eq. (7). If m_k are close to the average value $\langle m_k \rangle$, then:

$$\frac{1}{f}\frac{\partial f}{\partial t} = -\sum_k \langle m_k \rangle \sum_{i+j=k} F_{ij}\,. \tag{20}$$

It is clear from this equation that whenever the system is away from a critical state then the probabilities $w_N(\vec{m}, \vec{n}; t)$ are given by a product of Poisson distributions and hence the sfms vanish.

In small systems, for which one cannot speak about a phase-transition in the usual sense, the intermittent pattern of fluctuations could be the only

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[2] The situation is somewhat less clear in the ∞-cluster phase. In this case, a large part of the total mass is concentrated in one big cluster and , hence, the sfms are calculated for very small effective systems.

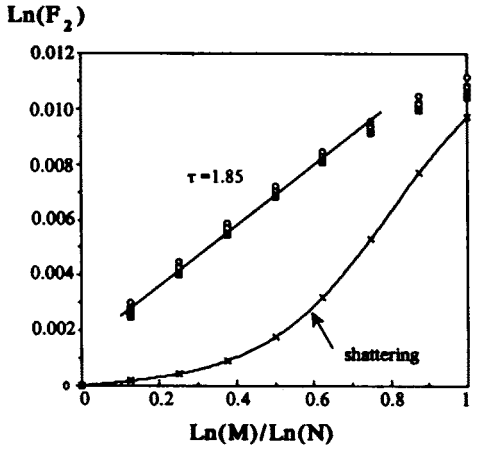

Fig. 4. The normalized scaled factorial moments $F_2(M)$ of the fragment size distribution are shown for selected exponents α and β at the transition line of the FIB model ($N = 256$): $\alpha = -1/2$, $\beta = 0, I_1 = 2\sqrt{\pi}\Gamma(27/20)/\Gamma(37/20)$ (open points), $\alpha = 0, \beta = 1, I_1 = 3/37$ (open squares), $\alpha = 1, \beta = 3, I_1 = 137/8778$ (black points) and $\alpha = 2, \beta = 5, I_1 = 28709/8738730$ (black squares). For all these cases $\tau = 1.85$. The case *alpha* $= 0, \beta = 0, I_1 = 1$ in the shattered phase is shown for comparison.

sign of a critical phenomenon. We have also studied the size-dependence of these fluctuations. For $\alpha = 0, \beta = 1, I_1 = 1$, the calculations yield in the large N limit: $F_2(M) \sim \log N - \frac{2}{3}\log\frac{N}{M}$, where N is the total mass of the system and M is the number of divisions of the total mass. Normalizing $F_2(M)$ by the value of $F_2(N) \sim \log N$ gives:

$$\frac{F_2(M)}{F_2(N)} \sim \frac{1}{3} + \frac{2}{3}\frac{\log M}{\log N}. \tag{21}$$

Hence, the spectrum of fluctuations as measured by sfms, is a power-law $F_p(M) \sim M^{f_p}$ with the intermittency exponent f_p which weakly (logarithmically) depends on the system size N. In other terms, one may conclude that the natural scaled variable appear here to be $\log M/\log N$ instead of $\log M$.

It is tempting to compare this result to Parisi–Frish multifractal model [16] of fully developed turbulence. In a fluid at high Reynolds numbers R, the scaled energy spectrum $\log\varepsilon/\log R$ is a universal function of the scaled wavelength $\log\lambda/\log R$ [16]. Since the Reynolds number is proportional to the total size L of the fluid, the good variable is $\log\lambda/\log L$, quite analogous to $\log M/\log N$ in the FIB model. Recently, a similar result was obtained by considering the dependence of the strength of the fluctuations on the low-density cut-off in the random cascading picture of the multiparticle production [17].

4.6. Fragment-charge correlation data of ALADIN

The Au-fragmentation data of ALADIN have been successfully reproduced by choosing the kernels at the critical line of the FIB model [18]. Different choice of parameters α, β, I_1, corresponding to different temporary evolutions, turned out to give the same results for the multiplicity spectra, providing the critical exponent τ ($\tau = 1.85$) of the power-law fit of the fragment-mass distribution is kept fixed [18]. In this case the fluctuations in the fragment-size distribution, as measured by sfms, exhibit an intermittent pattern for all those "equivalent" parameters α, β, I_1 .

In Fig. 5, we compare results of the FIB model with the ALADIN data for the collision of Au nuclei at 600 MeV/u on Cu target (triangles). In Fig. 5a, the average charge of the largest fragment $\langle Z_{\max} \rangle$ produced in the break-up process is plotted as a function of the composed particle first moment N_C. Fig. 5b shows the average multiplicity of the intermediate mass fragments of charge $3 \leq Z \leq 30$. Finally, Fig. 5c exhibits the average value of the quantity $\gamma_2 \equiv M_2 M_0 / {M_1}^2$, where M_i are the conditional moments: $M_i = \sum_i \tilde{n}_i {Z_i}^k$ and the sum runs over all fragments of charge Z_i and multiplicity $\tilde{n}_i$ except the largest fragment and the particles of charge $Z = 1$. The agreement between the results of FIB model and the data is good for small and intermediate values of N_C with some deviation for $N_C \longrightarrow 1$ which corresponds to quasi-fission or evaporation events typical for low excitation energies. To account for these low energy quasi-fission and evaporation processes, one would have to select another parameters of the kernels. However, in order to attain the simplicity of the model we restrict the fitting by taking one "universal" parametrisation of the kernels, independent of the value of N_C.

The fluctuation pattern changes with N_C, as it should, since splitting events in different classes in N_c for given exponents α, β, is equivalent to replacing original kernel functions by the effective ones which, in general, are not scale-invariant. Anyway, the experimental intermittency indices are very well reproduced by the FIB model, providing an additional indication for the shattering phase transition in the nuclear multifragmentation at $E/A = 600$ MeV [18].

4.7. The time behaviour

The existence of "equivalent" parametrisations of the fragmentation and inactivation kernels, for which the same asymptotic multiplicity distributions are obtained but which correspond to very different temporal evolutions, requires the investigation of the time-behaviour of the FIB model. The quantity Φ_k in Eq. (9) has a meaning of the inverse of a characteristic time. The time-evolution of the number of inactive fragments is given by

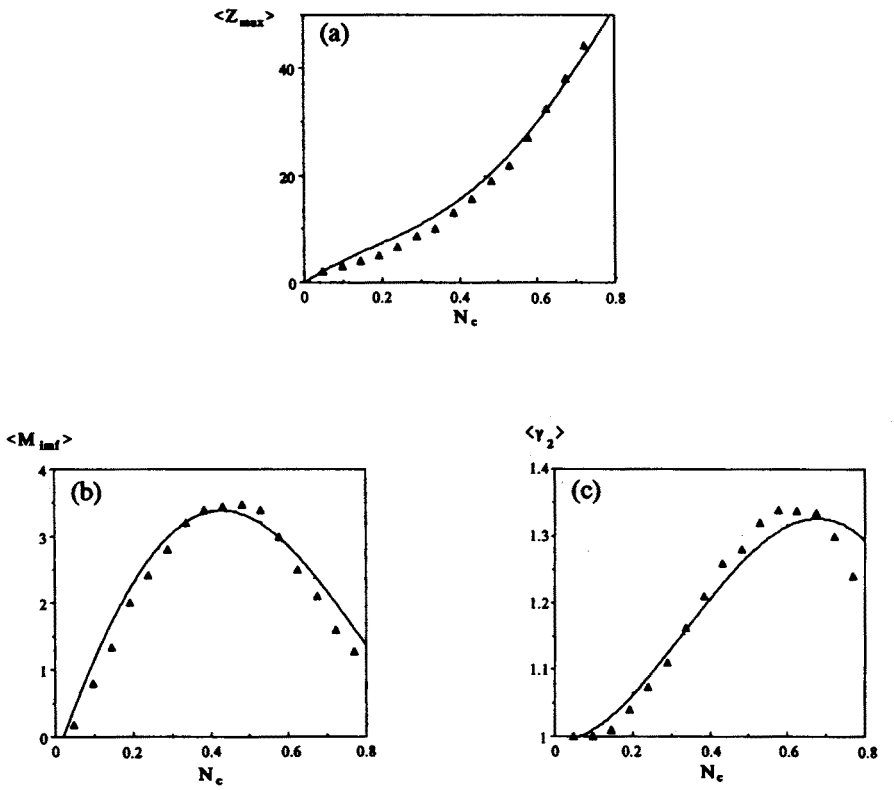

Fig. 5. The average charge of the largest fragment $\langle Z_{max} \rangle$ (a), the average multiplicity of the intermediate mass fragments $\langle M_{IMF} \rangle$ (b) and the average value of γ_2 (c) are plotted as a function of the composed particle moment N_C for the experimental data of ALADIN [20] (triangles) and for the FIB model (full line). For more details see the description in the text.

Eq. (9). Eliminating the number of active fragments, these equations can be written in the compact form:

$$\frac{d\bar{n}_k}{dt}(N;t) = -\Phi_k \bar{n}_k(N;t) + 2I_k \sum_{i=k+1}^{N} \frac{F_{k,i-k}}{I_i} \bar{n}_i(N;t) + I_N \delta(N-k) \quad (22)$$

with the initial condition: $\bar{n}_k(N;t) = 0$. Taking only the first term on the

rhs one obtains:

$$\bar{n}_k(N;t) = \sum_{j=k}^{N} I_k b_{j,k}(1 - \exp(-\Phi_j t)) \tag{23}$$

with the coefficients given by the backward recurrent relations:

$$b_{N,N} = 1/\Phi_N$$

$$b_{j,k} = \frac{2}{\Phi_k - \Phi_j} \sum_{i=k+1}^{j} F_{k,i-k} b_{j,i} \qquad \text{for } k < N \text{ and } j > k$$

$$b_{k,k} = \frac{2}{\Phi_k} \sum_{i=k+1}^{N} F_{k,i-k} \sum_{j=i}^{N} b_{j,i} - \sum_{j=k+1}^{N} b_{j,k} \qquad \text{for } k < N \tag{24}$$

Note that $b_{j,k}$'s depend on N. For our purpose, more interesting will be the evolution of the total mass of the inactive clusters:

$$\bar{N}_s(N;t) = \sum_{k=1}^{N} \left(\frac{k}{N}\right)^s \bar{n}_k(N;t)\,. \tag{25}$$

The temporal evolution of this quantity follows from Eq. (17) and equals:

$$\frac{d\bar{N}_s(N;t)}{dt} =$$

$$I_N + \sum_{k=1}^{N} \bar{n}_k(N;t)\left[\sum_{i=1}^{k-1} F_{i,k-i}\left(2\left(\frac{i}{N}\right)^s \frac{I_i}{I_k} - \left(\frac{k}{N}\right)^s\right) - \left(\frac{k}{N}\right)^s I_k\right]. \tag{26}$$

It is easy to see that the value of the derivative of $\bar{N}_c(N;t)$ at $t = 0$ equals I_N, *i.e.* the inverse of a characteristic time of the fragmentation process. The successive time derivatives at $t = 0$ correspond to characteristic values of the successive powers of the inverse time, namely:

$$1 = \sum_{j=1}^{N} \sum_{k=1}^{j} \left(\frac{k}{N}\right)^s I_k b_{j,k}\,, \tag{27}$$

$$I_N = \sum_{j=1}^{N} \Phi_j \sum_{k=1}^{j} \left(\frac{k}{N}\right)^s I_k b_{j,k}\,, \tag{28}$$

$$I_N{}^2 + \sum_{k=1}^{N-1} (I_N - 2\left(\frac{k}{N}\right)^s I_k) F_{k,N-k} = \sum_{j=1}^{N} \Phi_j{}^2 \sum_{k=1}^{j} \left(\frac{k}{N}\right)^s I_k b_{j,k}\,, \tag{29}$$

$$\ldots = \ldots . \tag{30}$$

Eq. (23) is particularly important as it gives a direct access to the inactivation kernel. This is important because for identical asymptotic distributions, one may be able to get an access to the time behaviour which is mainly governed by the inactivation kernel.

5. Conclusions

Strong random fluctuations exist in a number of physical problems concerning the evolution of systems. Origin of these fluctuations is not universal and the theory explaining their appearance is not yet completed. Nevertheless, in a number of transport problems, the origin of structures and strong fluctuations is ultimately related to the randomness of the environment which is the scene of the transport. We have shown that in this case the evolution of the system can be often described by the effective evolution equations of the unstable type. These stochastic equations have generally a multiplicative structure.

In this work, we have discussed the origin of intermittent pattern of fluctuations in the fragment-size distribution using the FIB random cascading model for homogeneous kernels at the transition line. The precise form of the fragmentation functions entering in FIB model have to be deduced from the experimental data and/or from the analysis of the non-analytical instabilties of the dynamical equations. The FIB model belongs to a different universality class than a 3D percolation as manifested by different values of the critical exponent τ. Nevertheless, the critical behaviour for both models is associated with the scale-invariant fluctuations which are well exposed by the scale dependence of sfms.

Scaling laws are rare and very usefull. They both provide a skeleton of a quantitative theory of the studied process and help to define appropriate global variables of this theory. The language of thermodynamical reduction of nuclear many-body problem, which is so customary to us, is doubtful for the description of dynamical formation of intermediate mass fragments in HI collisions.

In the above discussion of FIB Model, we have considered the mass fragmentation, but the extension to other variables such as the energy-momentum is possible. Similar evolution equations appear in various field theory models describing jet fragmentations before their transformation into the observed hadrons [19]. The possibility of applying FIB model to the description of spinodal instabilities remains exciting subject for future studies.

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