APPLICATIONS OF CHAOTIC ADIABATIC DYNAMICS: STATISTICAL FLUCTUATIONS IN ONE–BODY DISSIPATION*

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Statistical fluctuations in one-body dissipation are considered. A method for calculating the size of such fluctuations is presented, then applied to a specific example. Quantal corrections are discussed.

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1. Introduction

In this seminar I will present results concerning fluctuations in one-body dissipation during a dynamical nuclear process.

The study of one-body dissipation begins with the question, Given a gas of independent particles moving about (chaotically) inside a slowly time-dependent potential well, how does the total energy of this gas evolve with time?

In the case of a hard-walled infinite potential well, this question is answered by the Wall Formula, developed by Błocki, Świątecki, and others [1], which gives $\dot{E}_T$, the rate of change of the total energy of the gas, in terms of the changing shape of the container and the average speed of the particles. For a more general potential well, Koonin and Randrup [2] have used linear-response theory to obtain an analogous expression for the rate of energy dissipation. In both cases, no knowledge of initial conditions beyond the initial distribution of energies is assumed. Now, since an exact evaluation of $\dot{E}_T$ as a function of time (t) can be obtained only by following

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the individual trajectories of the particles, the results of Refs. [1, 2] necessarily represent an *average* rate of dissipation; for a particular set of initial conditions, $\dot{E}_T(t)$ will fluctuate around this predicted average, resulting in a final change in total energy, ΔE_T, somewhat different from the predicted value. These are the "statistical fluctuations" on which I will focus.

2. Calculation of fluctuations

For a given time-dependent potential well, and a given set of particle energies at $t = 0$, let $\langle \Delta E_T \rangle$ denote the average amount by which the total energy of the gas has increased, after a time τ. This average is taken over all possible realizations of initial conditions consistent with the known initial energies. Let σ_T denote the standard deviation in ΔE_T, again with respect to all acceptable realizations of initial conditions; this gives the typical accumulated effect of the above-mentioned statistical fluctuations in $\dot{E}_T(t)$. The ratio of σ_T to $\langle \Delta E_T \rangle$ reveals to what extent such fluctuations are important: if $\sigma_T / \langle \Delta E_T \rangle \ll 1$, then the approximation

$$\Delta E_T \approx \langle \Delta E_T \rangle \, , \tag{1}$$

is valid; otherwise, the total change in the energy of the gas for a particular set of initial conditions can differ significantly from that predicted by the formulae of Refs. [1, 2].

Since the particles in the gas are independent of one another, computing σ_T reduces to computing separately the standard deviations in the changes in the individual particle energies:

$$\Delta E_T = \sum_{j=1}^{N} \Delta E_j \, , \qquad \sigma_T^2 = \sum_{j=1}^{N} \sigma_j^2 \, . \tag{2}$$

Here N denotes the number of particles, ΔE_j is the change in the energy of the jth particle (for a particular realization of initial conditions), and σ_j is the standard deviation in ΔE_j (over all realizations). Defining $P(E, \tau; E^0, 0)$ to be the probability distribution for finding a particle to have energy E at time τ, given that at $t = 0$ its energy was known to be E^0, we have

$$\begin{aligned} \sigma_j^2 &= \int dE (E - \langle E_j \rangle)^2 P(E, \tau; E_j^0, 0) \\ \langle E_j \rangle &= \int dE \, E \, P(E, \tau; E_j^0, 0) \, , \end{aligned} \tag{3}$$

where E_j^0 is the known initial energy of the j-th particle. Our task therefore is to solve for P, for N different values of the initial energy: $E^0 = E_1^0, \ldots E_N^0$.

The central result of Ref. [3] gives the following evolution equation for P:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial E}(g_1 P) + \frac{1}{2}\frac{\partial^2}{\partial E^2}(g_2 P). \tag{4}$$

This is a Fokker-Planck equation; explicit expressions for the drift and diffusion coefficients g_1 and g_2, both functions of energy and time, are given in Ref. [3], and are not reproduced here. My point is that we may use this equation, along with the initial condition

$$P(E, 0; E^0, 0) = \delta(E - E^0), \tag{5}$$

to solve for $P(E, \tau; E_j^0, 0)$, $j = 1, \ldots, N$, from which we obtain the different σ_j^2's, and ultimately σ_T, the size of the statistical fluctuations in the total energy absorbed by our gas of particles.

3. A specific example

Ordinarily, the method outlined above for obtaining σ_T must be implemented numerically. An exception is the case of a hard-walled infinite potential well, *i.e.* a gas of particles inside a closed cavity whose shape changes slowly with time. While not an astonishingly realistic nuclear mean-field potential, this serves well as an instructive first step toward understanding one-body dissipation. [1] As shown in Ref. [4], if one ignores correlations between bounces in the motion of particles inside the cavity, then the coefficients g_1 and g_2 in Eq. (4) are greatly simplified, and one obtains:

$$\frac{\partial P}{\partial t} = \frac{(2m)^{1/2}}{2V} \oint da\dot{n}^2 \frac{\partial}{\partial E}\left[E^2 \frac{\partial}{\partial E}\left(E^{-1/2}P\right)\right]. \tag{6}$$

Here, m denotes the particle mass, V the constant volume of the cavity, and $\oint da\,\dot{n}^2$ is the integral, over the entire surface of the cavity, of the normal outward wall velocity squared (as in the Wall Formula).

From Eq. (6), it is straightforward to obtain, to leading order in the wall velocity $\dot{n}$,

$$\langle \Delta E_T \rangle = \frac{(2m)^{1/2}}{V} \int_0^\tau dt \oint da\,\dot{n}^2 \sum_{j=1}^{N} (E_j^0)^{1/2} \tag{7}$$

$$\sigma_T^2 = \frac{(2m)^{1/2}}{V} \int_0^\tau dt \oint da\dot{n}^2 \sum_{j=1}^{N} (E_j^0)^{3/2}. \tag{8}$$

The first of these equations is the Wall Formula prediction for the amount of energy dissipated (confirming that the present approach agrees with the approach originally taken in deriving the Wall Formula; see Ref. [5].)

If we further take the distribution of initial energies to correspond to one particle per each of the lowest N energy levels, and use a semi-classical approximation for the density of energy levels, then after some algebra we arrive at

$$\frac{\sigma_T}{\langle \Delta E_T \rangle} = \left(\frac{10}{9} \frac{1}{Nf} \right)^{1/2}, \tag{9}$$

where $f = \langle \Delta E_T \rangle / E_T^0$, and E_T^0 is the total initial energy of the gas. (f is the average *fractional* increase in the energy of the gas.) Eq. (9) provides an immediate estimate of the size of statistical fluctuations — around an average given by the Wall Formula — in the energy absorbed by a gas of particles inside a slowly time-dependent cavity.

4. Quantal aspects

We now discuss the question of classical–quantal correspondence: to what extent do our classical predictions apply when we consider this problem quantally?

In the quantal formulation, one has a many-body wave function Ψ evolving under a Hamiltonian $\hat{H}(t)$ describing N independent particles in a time-dependent potential well. The quantities of interest are then

$$\langle \Delta E_T \rangle = \langle \Psi | \hat{H} | \Psi \rangle - \langle \Psi_0 | \hat{H} | \Psi_0 \rangle \tag{10}$$

$$\sigma_T^2 = \langle \Psi | \left(\hat{H} - \langle \Psi | \hat{H} | \Psi \rangle \right)^2 | \Psi \rangle, \tag{11}$$

where Ψ_0 is the N-particle wave function at $t = 0$.

Consider first the case in which we ignore spin statistics, and write Ψ_0 as a simple product of N single-particle wave functions. Then the particles are truly independent of one another, and the question of classical-quantal correspondence reduces to that of whether or not we are in the semi-classical regime, *i.e.* whether or not the *single*-particle wave functions evolve so that expectation values of operators follow the microcanonical averages of corresponding classical observables. This question has been considered in detail by Wilkinson and Austin [6]. If one is indeed in the semiclassical regime, then Eqs (7) and (8) provide decent substitutes for Eqs (10) and (11).

Now consider the case in which Ψ_0 is a Slater determinant (Fermi statistics). A straightforward evaluation of Eqs (10), (11) yields the comparison

$$\langle \Delta E_T \rangle_{\rm Sl} = \langle \Delta E_T \rangle_{\rm sp}$$
$$\sigma^2_{T,{\rm Sl}} = \sigma^2_{T,{\rm sp}} - \sum_{i \neq j} \left| \langle \phi_i | \hat{u}^{-1} \hat{h} \hat{u} | \phi_j \rangle \right|^2 . \tag{12}$$

Here, the subscripts "Sl" and "sp" denote evaluation using Ψ_0 a Slater determinant, and Ψ_0 a simple product of single-particle states, respectively. The ϕ's represent the single-particle states initially filled $(i, j = 1, \cdots, N)$, $\hat{h}(t)$ is the single-particle Hamiltonian, and $\hat{u}$ is the single-particle time-evolution operator. We see that anti-symmetrization of the wave function does not alter the expression for $\langle \Delta E_T \rangle$, and so in the appropriate limit the classical predictions for the average energy dissipated apply. On the other hand, anti-symmetrization introduces an extra, non-positive term to the expression for σ^2_T. Until a way of easily estimating this term is found, our classical results for σ_T (*e.g.* Eq. (9)) provide only an upper limit on the size of statistical fluctuations in dissipated energy, for a system obeying Fermi statistics.

5. Summary

To summarize, one may use a general result from chaotic adiabatic dynamics (Eq. (4)) to predict the size of statistical fluctuations in the amount of energy lost to one-body dissipation. For the instructive case of a hard-walled potential, this yields a simple formula for estimating the importance of such fluctuations (Eq. (9)). The applicability of such classical predictions to a quantal gas (within the independent-particle approximation) hinges on two questions. First, are we in the semi-classical regime? Second, by how much does the Exclusion Principle suppress these fluctuations?

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