ON REAL MASSLESS SCALAR FIELD IN TWO-DIMENSIONAL DE SITTER SPACE-TIME

G. WRÓBEL

Institute of Physics, Jagellonian University
Reymonta 4, 30-059 Cracow, Poland
e-mail address: wrobel@ztc386a.if.uj.edu.pl

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Quantum field theory of real massless scalar field in two-dimensional de Sitter space-time is considered. The scalar product in the subspace of pure Coulomb states is decomposed into irreducible unitary representations of the three-dimensional proper ortochronous Lorentz group. It is shown that the Coulomb field contains representations from the main series if the "fine structure constant" (defined in the text) $\alpha > 1$. If $0 < \alpha < 1$, there is additionally a representation from the supplementary series. The eigenvalue of the Casimir operator for this representation is $^1/_4\alpha(2-\alpha)$.

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1. Introduction

Staruszkiewicz [1] has shown that the electric part of the infrared electromagnetic field can be described as a real massless scalar field S "living" in the three-dimensional de Sitter space-time formed by the set of all unit space-like vectors. Upon quantization this theory allows to make some a priori statements on the numerical value of the fine structure constant $\alpha = e^2/\hbar c$ [1]. The theory can be summarized in the expressions for the total action

$$\mathcal{S} = \frac{1}{8\pi\alpha}\int dS \wedge {}^*dS$$

and the total electric charge

$$\frac{Q}{e} = -\frac{1}{4\pi\alpha}\int {}^*dS\,.$$

Here e is the elementary charge, $\alpha = e^2/\hbar c$ is the fine structure constant, S is the phase which is canonically conjugated with the total charge:

$$\left[\frac{Q}{e}, S\right] = i\,.$$

In the action integral one integrates over the whole de Sitter space-time while in the charge integral one integrates over an arbitrary Cauchy surface. * is the usual star operator defined *e.g.* by Thirring [2]. The numerical value of the elementary charge e adjusts mechanical and electrical units, like c — units of time and length. Thus one can put $e = 1$ for short.

The main goal of the present paper is to investigate the analogous theory in the two-dimensional de Sitter space-time. Our theory is not based on any physical model. Thus the constant in the action can be chosen arbitrarily. A natural convention in our case is to take the constant in the action integral equal to 4π (the double volume of the one-dimensional sphere):

$$\mathcal{S} = \frac{1}{4\pi\alpha}\int dS \wedge {}^*dS\,.$$

Analogously, the total charge is defined by an integral over a Cauchy surface in the de Sitter space-time

$$Q = -\frac{1}{2\pi\alpha}\int {}^*dS\,.$$

The above expressions set up completely the classical model and define the "fine structure constant" α. Variation of the action with respect to S yields the equation of motion

$$\Delta S = 0\,,$$

where Δ denotes the Laplace-Beltrami operator *i.e.* the d'Alembert operator in the two-dimensional de Sitter hyperboloid:

$$\Delta = \partial_\psi^2 + \tanh\psi\,\partial_\psi - (\cosh\psi)^{-2}\,\partial_\varphi^2\,,$$

φ and ψ being the elliptic and hyperbolic angles on the de Sitter hyperboloid, whose metric is

$$ds^2 = d\psi^2 - \cosh^2\psi\,d\varphi^2\,.$$

The general solution can be found by means of separation variables. The result is

$$S = \sum_{m=-\infty}^{\infty} \mathrm{e}^{im\varphi}(\cosh\psi)^{-1/2}\left[a_m P^{1/2}_{m-\frac{1}{2}}(\tanh\psi) + b_m P^{-1/2}_{m-\frac{1}{2}}(\tanh\psi)\right]\,.$$

Here a_m and b_m are integration constants, P is the associated Legendre function. Putting this formula into the definition of the total charge we have

$$Q = \sqrt{\frac{2}{\pi}}\frac{b_0}{\alpha}\,.$$

2. Quantization and definition of the vacuum state

We define the vacuum state, as usual, dividing the solution of the equation of motion into positive and negative frequency parts. Consider the following integral over the light cone $kk = 0$, $k^0 > 0$ in the three-dimensional momentum space:

$$\frac{\xi}{\sqrt{2\pi}}\int \frac{d^2k}{k^0}\,\mathrm{e}^{-ikx}\mathrm{e}^{im\varphi_k} = \sqrt{\pi}\mathrm{e}^{im\varphi}(\cosh\psi)^{-1/2}$$
$$\times\left[P^{1/2}_{m-\frac{1}{2}}(\tanh\psi) + i|m|P^{-1/2}_{m-\frac{1}{2}}(\tanh\psi)\right] := 2\sqrt{\frac{|m|}{\alpha}}f_m(\psi,\varphi)\,,$$

where

$$x^0 = \xi\sinh\psi\,,$$
$$x^1 = \xi\cosh\psi\,\cos\varphi\,,\quad k^1 = k^0\cos\varphi_k\,,$$
$$x^2 = \xi\cosh\psi\,\sin\varphi\,,\quad k^2 = k^0\sin\varphi_k\,.$$

Since the integral on the left-hand side contains only positive frequencies in the three-dimensional Minkowski space-time, we assume that the right-hand side is a positive frequency solution in the de Sitter hyperboloid. In other words we define a positive frequency solution in the de Sitter space-time by means of a projection from the Minkowski space-time. The definition above makes no sense for $m = 0$. In this case we postulate for the vacuum state $|0\rangle$:

$$Q|0\rangle = 0\,,\quad \langle 0|Q = 0\,.$$

Using the positive frequency solutions f_m we quantize our model as follows:

$$S = S_0 - \alpha Q\arcsin\tanh\psi + \sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty} c_m f_m(\psi,\varphi) + \text{h.c.}\,,$$

$$[Q, S_0] = i\,,\quad [c_m, c_n^\dagger] = \delta_{mn}\,,\quad [S_0, c_m] = 0\,,$$
$$[Q, c_m] = 0\,,\quad [c_n, c_m] = 0\,,\quad [c_n^\dagger, c_m^\dagger] = 0\,,$$
$$Q|0\rangle = 0\,,\quad \langle 0|Q = 0\,,\quad c_m|0\rangle = 0\,,\quad \langle 0|c_m^\dagger = 0\,.$$

3. The set of Coulomb states

We define a pure Coulomb state as the vector $e^{-iS_0}|0\rangle$ [5]. The constant term S_0 in the partial waves expansion of S, being a rotational invariant, is not a Lorentz invariant quantity. Thus S_0 depends on a three-velocity vector u which indicates the inertial reference frame in which the partial waves expansion is carried out. The vector u is a unit future oriented time-like three-vector (in the three-dimensional Minkowski space-time). The state $|u\rangle := e^{-iS_0(u)}|0\rangle$ is spherical symmetric in the rest frame of u. Compose the superposition of the Coulomb states

$$|f\rangle = \int d^2u\, f(u)|u\rangle$$

by means of a smooth function f of compact support on the set of all three-velocity vectors which is identical with two-dimensional Lobachevsky space.

$$d^2u = \frac{du^1 \wedge du^2}{u^0}$$

is the invariant measure on the Lobachevsky space. We define the bilinear form on the set of such states by

$$\langle g|f\rangle = \int d^2v\, d^2u\, \overline{g(v)} f(u)\langle v|u\rangle.$$

The matrix element $\langle v|u\rangle$ can be computed as follows. This element is Lorentz invariant, so depends only on the hyperbolic angle λ between v and u. Assume, without loss of generality, that the rest frame of v moves in the x^2-direction in that of u. Furthermore

$$S_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi\, S|_{\psi=0}\,.$$

The following identities will be useful:

$$\tanh\psi_u|_{\psi_v=0} = \frac{\sinh\lambda\,\sin\varphi_v}{\sqrt{\cos^2\varphi_v + \cosh^2\lambda\,\sin^2\varphi_v}}$$

and

$$f_m(\psi_u,\varphi_u)|_{\psi_v=0} = \sqrt{\frac{\alpha}{2|m|}}\left(\frac{\cos\varphi_v + i\cosh\lambda\,\sin\varphi_v}{\sinh\lambda\,\sin\varphi_v - i\,\mathrm{sign}\,m}\right)^m .$$

In this way we obtain

$$S_0(v) = S_0(u) + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} i^m \sqrt{\frac{\alpha}{2|m|}} \left(\frac{\cosh\lambda - 1}{\sinh\lambda}\right)^{|m|} c_m(u) + \text{h.c.}.$$

Using this expression, the commutation relations and the Baker-Hausdorff identity one finds

$$\langle v|u\rangle = (\cosh\tfrac{1}{2}\lambda)^{-\alpha}.$$

Lobachevsky space is a symmetric space isomorphic to the factor group $SL(2,\mathcal{R})/SO(2)$. Hence one can construct the Fourier transform in it. The following two equations summarize the Fourier theory. They are derived in the Appendix.

$$\check{f}(k,\nu) = \int d^2u\, f(u)(ku)^{i\nu-\frac{1}{2}}$$

$$f(u) = \frac{1}{(2\pi)^2}\int_0^{\infty} d\nu\, \nu \tanh(\pi\nu) \int dk\, \check{f}(k,\nu)(ku)^{-i\nu-\frac{1}{2}}.$$

Here k is a future oriented null vector, dk is the Lorentz invariant measure on the set of null directions.

Let us investigate the form $\langle g|f\rangle$ in the dual space. Put in the Fourier transform $\check{g}, \check{f}$. We have

$$\int d^2u\, f(u)\langle v|u\rangle =$$

$$\frac{1}{(2\pi)^2}\int_0^{\infty} d\nu\, \nu\tanh(\pi\nu)\int dk\, \check{f}(k,\nu)\int du\,(ku)^{-i\nu-\frac{1}{2}}\langle v|u\rangle.$$

The order of integrations was changed. It is allowed for $\alpha > 1$, the last integral being convergent. Owing to the Lorentz invariance one can write the last integral in the form

$$\int du\,(ku)^{-i\nu-\frac{1}{2}}\langle v|u\rangle = (kv)^{-i\nu-\frac{1}{2}}K(\nu,\alpha).$$

The function $K(\nu,\alpha)$ can be calculated by means of standard methods in the rest frame of u. It can be expressed by the beta function:

$$K(\nu,\alpha) = 2^{\alpha}B\left(\frac{1}{2},\frac{\alpha-1}{2}\right)B\left(\frac{\alpha-1}{2}+i\nu,\frac{\alpha-1}{2}-i\nu\right).$$

Therefore

$$\langle g|f\rangle = \frac{2^\alpha B\left(\frac{1}{2},\frac{\alpha-1}{2}\right)}{(2\pi)^2} \times \int_0^\infty d\nu\, \nu \tanh(\pi\nu)\, B\left(\frac{\alpha-1}{2}+i\nu, \frac{\alpha-1}{2}-i\nu\right) \int dk\, \overline{\breve{g}(k,\nu)}\breve{f}(k,\nu)\,.$$

This formula is correct for $\alpha > 1$. To find the corresponding equation valid for $0 < \alpha < 1$ one has to notice what follows. The above equation can be rewritten in the form

$$\langle g|f\rangle = \frac{2^\alpha B\left(\frac{1}{2},\frac{\alpha-1}{2}\right)}{2\pi} \int d^2v\, d^2u\, \overline{g(v)}f(u) \times \int_0^\infty d\nu\, \nu \tanh(\pi\nu)\, B\left(\frac{\alpha-1}{2}+i\nu, \frac{\alpha-1}{2}-i\nu\right) P_{i\nu-\frac{1}{2}}(vu)\,.$$

The integrand of the last integral is a holomorphic function of the complex variable ν everywhere except for its poles. Positions of these poles in the complex ν-plane depend on the value of α. Certain poles cross the path of integral at the point $\alpha = 1$. Hence the integral over ν treated as a function of α is not an analytic function at the point $\alpha = 1$. On the other hand the original form

$$\langle g|f\rangle = \int d^2v\, d^2u\, \overline{g(v)}f(u)\langle v|u\rangle$$

is an analytic function for $\alpha > 0$. Thus the correct expression for the form $\langle g|f\rangle$ valid for $0 < \alpha < 1$ in the dual space is the analytic extension of the function of α given by the integral:

$$\int_0^\infty d\nu\, \nu \tanh(\pi\nu)\, B\left(\frac{\alpha-1}{2}+i\nu, \frac{\alpha-1}{2}-i\nu\right) P_{i\nu-\frac{1}{2}}(vu)\,.$$

One finds the analytic continuation subtracting residua of those poles which cross the path of integral at the point $\alpha = 1$. The result is

$$\langle g|f\rangle = \frac{2^\alpha B\left(\frac{1}{2},\frac{\alpha-1}{2}\right)}{(2\pi)^2} \times \int_0^\infty d\nu\, \nu \tanh(\pi\nu)\, B\left(\frac{\alpha-1}{2}+i\nu, \frac{\alpha-1}{2}-i\nu\right) \int dk\, \overline{\breve{g}(k,\nu)}\breve{f}(k,\nu) + \frac{2^{\frac{3}{2}\alpha-2}}{\left[B\left(\frac{1}{2},\frac{1-\alpha}{2}\right)\right]^2} \int \frac{dl\, dk}{(lk)^{\alpha/2}}\, \breve{g}\left(l, i\frac{1-\alpha}{2}\right) \breve{f}\left(k, i\frac{1-\alpha}{2}\right)$$

for $0 < \alpha < 1$.

The integrand of the first term (the only one for $\alpha > 1$)

$$\int dk\, \overline{\check{g}(k,\nu)}\check{f}(k,\nu)$$

is the scalar product for representations of the three-dimensional proper ortochronous Lorentz group from the main series corresponding to the value of the Casimir operator

$$-\tfrac{1}{2}M^{\mu\sigma}M_{\mu\sigma} = \tfrac{1}{4} + \nu^2 .$$

It is easy to show using *e.g.* the theory of group representations given by Gelfand, Graev and Vilenkin in Ref. [4].This term contains continuous spectrum of all scalar products from the main series, with the weight

$$\nu \tanh(\pi\nu)\, B\left(\frac{\alpha-1}{2} + i\nu, \frac{\alpha-1}{2} - i\nu\right) .$$

The second term

$$\int \frac{dl\, dk}{(lk)^{\alpha/2}}\, \check{g}\left(l, i\frac{1-\alpha}{2}\right) \check{f}\left(k, i\frac{1-\alpha}{2}\right),$$

necessary for $\alpha < 1$, is the scalar product for representations from the supplementary series. The eigenvalue of Casimir operator in this case is

$$-\tfrac{1}{2}M^{\mu\sigma}M_{\mu\sigma} = \tfrac{1}{4}\alpha(2-\alpha) .$$

4. The set of charged states at plus infinity

Let us consider the state $:\mathrm{e}^{-iS}:|0\rangle$. Normal ordering makes the exponential function of S a well defined quantity. One finds easily that this is an eigenstate of the total charge operator to the eigenvalue 1:

$$Q:\mathrm{e}^{-iS}:|0\rangle = :\mathrm{e}^{-iS}:|0\rangle .$$

A general charged state at plus infinity $\psi = \infty$ is

$$|f\rangle = \int dk\, f(\boldsymbol{k}):\mathrm{e}^{-iS(\boldsymbol{k})}:|0\rangle ,$$

where $\boldsymbol{k}$ is a unit vector on the circle $\psi = \infty$ and dk is the measure on the set of such vectors. Consider now the bilinear form on the set of these states

$$\langle g|f\rangle = \int dl\, dk\, \overline{g(\boldsymbol{l})} f(\boldsymbol{k}) \langle 0| :\mathrm{e}^{iS(\boldsymbol{l})}::\mathrm{e}^{-iS(\boldsymbol{k})}: |0\rangle .$$

Using the commutation relations and the Baker-Hausdorff identity one finds that

$$\langle 0| :e^{iS(l)}::e^{-iS(k)}: |0\rangle = (2\sin\tfrac{1}{2}\vartheta)^{-\alpha},$$

where ϑ is the angle between $\boldsymbol{k}$ and $\boldsymbol{l}$. The vectors $\boldsymbol{k}$ and $\boldsymbol{l}$ can be understood as the space-like components of future oriented null vectors k and l, respectively. Absorbing the factor $2^{-\alpha/2}$ into the functions f and g we have

$$\langle g|f\rangle = \int \frac{dl\, dk}{(kl)^{\alpha/2}}\, \overline{g(l)} f(k).$$

The above form is Lorentz invariant, dk being the Lorentz invariant measure on the set of null directions, if the integrand is homogeneous of degree -1. See [3], where the proof for the three-dimensional light cone is given. This bilinear form in stereographic coordinates was studied by Gelfand, Graev and Vilenkin [4], as the scalar product for the supplementary series of unitary representations of the group $SL(2,\mathcal{R})$. It is positive defined for $0 < \alpha < 1$ and negative defined for $1 < \alpha < 2$. One has to note that for $1 < \alpha < 2$ the integral is divergent and must be understood as the finite value distribution.

The analogous unitarity condition $0 < \alpha < \pi$ in the three-dimensional de Sitter space-time was found by Staruszkiewicz [1]. The scalar product was expressed by the integral

$$\langle g|f\rangle = \int \frac{dl\, dk}{(kl)^{\alpha/\pi}}\, \overline{g(l)} f(k),$$

convergent for $\alpha < \pi$. However, one can treat the above integral as the finite value distribution. Then this form is negative defined for $\pi < \alpha < 2\pi$ [4].

5. Conclusions

There is an analogy between real massless scalar field in three-dimensional and two-dimensional de Sitter space-time. In both cases there is a critical value of the fine structure constant, π in three dimensions and 1 in two dimensions, which separates two kinematically different regimes of the quantum Coulomb field. Apparently the same phenomenon will appear in any dimension: there is always a number $\alpha(n)$ which separates two kinematically distinct regimes of the quantum Coulomb field.

Part of this paper was done during my graduate studies. I am grateful to Professor Andrzej Staruszkiewicz for helpful suggestions.

Appendix

In this appendix we derive the Fourier transform and its inverse in the two-dimensional de Sitter space-time. The general harmonic analysis on a symmetric space was formulated at the beginning of the sixties. At present, one can find the Fourier transform theory in some textbooks (*e.g.* [6]) on the representation theory of Lie groups.

If a Lie group G acts on a connected symmetric manifold $\mathcal{H}$ transitively and a Lie subgroup K of the group G is the stability group at a certain point $o \in \mathcal{H}$, then the factor group $G/_K = \mathcal{H}$ (isomorphism of homogeneous spaces).

In our case $G = SL(2,\mathcal{R})$, $K = SO(2)$, $\mathcal{H}$ is the Lobachevsky space, and $o = (1,0,0) \in \mathcal{H}$. The action $g \in G$ on $x \in \mathcal{H}$

$$g \circ x := gxg^{\dagger}$$

is defined by means of identification

$$x = (x^0, x^1, x^2) = \begin{pmatrix} x^0 + x^2 & x^1 \\ x^1 & x^0 - x^2 \end{pmatrix}.$$

Furthermore, if the Lie algebra $\mathfrak{g}$ of the group G is semisimple and non-compact and the group K is connected, then the Iwasawa decomposition of $\mathfrak{g}$ into a direct sum (with respect to the Killing form) of subalgebras $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$ takes place:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}, \quad \mathfrak{n} = \bigoplus_{\beta \in \Sigma_+} \mathfrak{g}_\beta,$$

where $\mathfrak{k}$ is the Lie algebra of the group K; $\mathfrak{a}$ is the Cartan subalgebra of $\mathfrak{g} \ominus \mathfrak{k}$; $\mathfrak{g}_\beta$ is the root space of the algebra $\mathfrak{g}$ connected with the root $\beta \in \mathfrak{a}^*$ ($\mathfrak{a}^*$ is the dual space of $\mathfrak{a}$); Σ_+ is the set of positive roots for a fixed Weyl chamber $\mathfrak{a}_+$ in the algebra $\mathfrak{a}$. The group G also can be decomposed, *e.g.*, the mapping on the Cartesian product

$$K \times A \times N \ni (k, a, n) \to kan \in G$$

is a diffeomorphism of differentiable manifolds. Here A and N are Lie groups whose Lie algebras are $\mathfrak{a}$ and $\mathfrak{n}$, respectively. That diffeomorphism allows to define the function $\mathcal{A}$ on G:

$$\mathcal{A} : G \ni (k, a, n) \to a \in A.$$

In our case $\mathfrak{a}$ is formed by the totality of diagonal traceless matrices. There are two root spaces $\mathfrak{g}_{\beta_1}$, $\mathfrak{g}_{\beta_2}$ of upper and lower triangular matrices. Only one of the roots for the Weyl chamber

$$\mathfrak{a}_+ := \left\{ \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} : x > 0 \right\}$$

is positive:

$$\mathfrak{a}^* \ni \beta_1 : \mathfrak{a} \ni \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \to 2x \in \mathcal{R} .$$

The boundary of a symmetric space is defined as the factor group K/M, where

$$M := \{k \in K : \mathrm{Ad}\, k|_{\mathfrak{a}} = \mathrm{id}\} .$$

Our M group is very simple: $M = \{-1, 1\}$. Hence the boundary of the Lobachevsky space is isomorphic to the set of null directions in the three-dimensional Minkowski space-time.

Plane wave on a symmetric space $\mathcal{H}$ with frequency $\nu \in \mathfrak{a}^*$, and a normal $kM \in K/M$ is defined as

$$e_{\nu,kM}(g \circ o) := \exp\left(-\nu(\log \mathcal{A}(g^{-1}k))\right) .$$

By the Fourier transform of a function f on a symmetric space $\mathcal{H}$ we understand the function $\check{f}$ on the Cartesian product $\mathfrak{a}^* \times K/M$, given by the following integral:

$$\check{f}(\nu, kM) = \int_{\mathcal{H}} du\, f(u)\, e_{-i\nu+\varrho,kM}(u) ,$$

where $\varrho = {}^1\!/_2 \sum_{\beta \in \Sigma_+} \dim \mathfrak{g}_\beta\, \beta$ and du is the G-invariant measure on $\mathcal{H}$.

Using few popular tricks we get

$$\check{f}(\nu, k) = \int d^2u\, f(u)(ku)^{i\nu - \frac{1}{2}}.$$

Here $\nu \in \mathcal{R}$, because the form of $\mathfrak{a}$ implies that $\mathfrak{a}^*$ is isomorphic to $\mathcal{R}$.

We find the inversion formula of the Fourier transform constructing the Harish-Chandra c-function on the dual space $\mathfrak{a}^*_{\mathcal{C}}$ of the complexification of $\mathfrak{a}$

$$c(\nu) = \frac{I(i\nu)}{I(\varrho)} ,$$

where I is expressed by the beta function:

$$I(\nu) = \prod_{\beta \in \Sigma_+} B\left(\tfrac{1}{2} \dim \mathfrak{g}_\beta, \tfrac{1}{4} \dim \mathfrak{g}_{\frac{\beta}{2}} + \frac{(\nu|\beta)}{(\beta|\beta)}\right) .$$

We get the form $(\cdot|\cdot)$ restricting to $\mathfrak{a}$, extending onto $\mathfrak{a}_{\mathcal{C}}$, and moving onto $\mathfrak{a}^*_{\mathcal{C}}$ the Killing form.

In our case the Harish-Chandra c-function is

$$c = \frac{1}{\pi} B\left(\tfrac{1}{2}, i\nu\right) .$$

The Fourier transform satisfies the inversion formula given by

$$f(u) = \int_{\mathfrak{a}_+^*} d\nu \int_{K/M} d(kM)\, |c(\nu)|^{-2} \check{f}(\nu, kM) e_{i\nu+\varrho,kM}(u) .$$

The integration is over, so called, dual space of $\mathcal{H}$: $\mathfrak{a}_+^* \times K/M$, where

$$\mathfrak{a}_+^* := \left\{ \nu \in \mathfrak{a}^* : \nu|_{\mathfrak{a}_+} > 0 \right\} .$$

$d\nu$ denotes the Lebesgue measure on $\mathfrak{a}^*$, and the measure $d(kM)$ on the factor group K/M is defined by the integrals

$$\int_{K/M} d(kM) \int_M dm\, \phi(km) = \int_K dk\, \phi(k) ,$$

for a measurable function ϕ on K, where dm and dk are the Haar measures on M and K, respectively.

To return to the Lobachevsky space, the set $\mathfrak{a}_+^*$ is isomorphic to $\mathcal{R}_+$. The measure on the boundary of the Lobachevsky space is the Lorentz invariant measure on the set of null directions. In this way we obtain

$$f(u) = \frac{1}{(2\pi)^2} \int_0^\infty d\nu\, \nu \tanh(\pi\nu) \int dk\, \check{f}(\nu, k)(ku)^{-i\nu-\frac{1}{2}} .$$

REFERENCES

[1] A. Staruszkiewicz, *Ann. Phys.* (N.Y.) **190**, 354 (1989).

[2] W. Thirring, *A Course in Mathematical Physics*, Vol.2, Springer-Verlag New York, Wien Inc. 1979.

[3] D. Zwanziger, *Phys. Rev.* **D 14**, 2570 (1976).

[4] I.M. Gelfand, M.I. Graev, N.Ya. Vilenkin, *Generalized Functions*, Vol.5, Moscow 1962, in Russian.

[5] A. Staruszkiewicz, *Acta. Phys. Pol.* **B23**, 591 (1992).

[6] A. Wawrzyńczyk, *Group Representations and Special Functions*, D. Reidel Publishing Company, Dordrecht-Boston-Lancaster, PWN – Polish Scientific Publishers, Warszawa, 1984.