GRAVITATING SOLITONS
AND HAIRY BLACK HOLES

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A brief review of recent research on soliton and black hole solutions of Einstein's equations with nonlinear field sources is presented and some open questions are pointed out.

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1. Introduction

During the past few years there has been a lot of interest in soliton (by which I mean non-singular finite energy stationary solution) and black hole solutions of Einstein's equations with nonlinear field sources. In this brief survey I shall outline the most interesting, in my view, results of these studies[1]. Since my intention is to emphasize the model-independent aspects of the problem I shall restrict the details of specific models to a minimum (with one important exception of the Einstein–Yang–Mills theory which deserves separate analysis).

It is rather surprising that although solitons in various nonlinear field theories in flat space were intensively investigated in the past, the analogous problem in general relativity received little attention until recently. Presumably this lack of interest in Einstein's equations with nonlinear field sources was due to two widely accepted beliefs (apart from the psychological factor due to the formidable structure of equations). First, it has been thought that the gravitational interaction, being very weak, cannot change qualitatively the spectrum of soliton solutions of a theory in which gravity was neglected. Second, according to the so called "no-hair" conjecture,

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[1] I tried to make this survey a complementary up-date of an excellent review on this subject written two years ago by Gibbons [1].

a stationary black hole is uniquely determined by global charges given by surface integrals at spatial infinity, such as mass, angular momentum, and electric (magnetic) charge. It was believed that this property of black holes, following from the uniqueness [2] and the non-existence [3] results proven rigorously for various linear field sources, persists for general matter sources.

The situation has changed radically when it was discovered that the Einstein–Yang–Mills (EYM) equations admit static non-abelian soliton [4] and black hole [5] solutions. This came as a surprise because *i)* neither the vacuum Einstein equations [6], nor pure Yang–Mills equations [7] have soliton solutions, and *ii)* "colored" black holes have non-abelian hair which is not associated with any conserved charge. These unexpected results have put into question the two beliefs mentioned above and have launched intensive investigations of Einstein's equations with nonlinear field sources.

There are several, more or less independent, motivations to study this subject. Let us mention some of them. First and most important, this research may have serious conceptual implications. Our understanding of general relativity is to large extent based on the knowledge of few exact solutions (*e.g.* Kerr-Newman, Friedmann). However, as the results of [4] and [5] showed, the intuitions based on solutions with linear field sources fail in more general situations. In this sense the analysis of Einstein's equations with nonlinear field sources may shed new light on generic properties of solutions of Einstein's equations.

Particularly interesting are *non-perturbative* effects of gravity, as, for instance, the existence of gravitational equilibria of non-abelian gauge fields [4] with finite energy. They result from the cancellation of gauge and gravitational singularities which implements the old idea due to Einstein that gravity may regularize ultraviolet divergencies in field theory. Such nonperturbative phenomena may have deep consequences at the quantum level.

Second, although we do not have yet a unified quantum theory of all interactions, we obviously can couple gravity to the standard model at the classical level. Then, it is natural to expect that the effects of gravity become important when the relevant energy scale is close to the Planck scale, *i.e.* in the very early Universe. This fact is notoriously ignored in cosmological literature. For example, in discussions how inflation solves the monopole problem the gravitational properties of monopoles are neglected, which might be a serious omission.

Third, the long time behaviour of perturbed solitons is closely related to the main unsolved problem in general relativity: cosmic censorship hypothesis.

Finally, there is a purely mathematical motivation to study these problems. In the spherically symmetric case (which is physically most interesting) Einstein's equations with nonlinear field sources reduce to dynamical

systems having very rich structure. It seems that modern methods of bifurcation and critical point theories may be succesfully applied to these systems.

The rest of this brief review is organized into two sections dealing with solitons and black holes, respectively. In Section 2 the general properties of gravitating solitons are described. For weak coupling the perturbative effects of gravity are examined, while for strong coupling the critical dependence on the coupling constant is analyzed. Next, the non-perturbative phenomena are discussed, in particular the Bartnik-Mckinnon solutions of the EYM equations. Section 3 is devoted to black holes with nonlinear hair and the current status of "no-hair" conjecture. This review is concerned with fundamental aspects of gravitating solitons and hairy black holes and I shall not discuss here an interesting problem of possible astrophysical and cosmological relevance of these solutions.

I shall use units in which $c = 1$; all other dimensional parameters are explicitly written.

2. Globally regular solutions

Let us consider a nonlinear field theory in flat space and suppose that the field equations have a soliton solution. For the purpose of this review I define the term soliton to mean a stationary solution which is everywhere non-singular and has finite energy[2]. The following question arises: What happens when the gravitational interaction is included into the model? In particular, how does gravity affect the spectrum of soliton solutions? To study this problem we have to consider Einstein's equations with a solitonic field as the source.

Before doing this, it is worth recalling that a theory which satisfies the dominant energy condition and is scale invariant, cannot have soliton solutions. The reason is that under the scaling transformation $x \rightarrow \lambda x$, the energy scales as $E \rightarrow \lambda E$, so a stationary solution, being the extremum of energy, must have $E = 0$. Since the energy is positive this implies that the solution is trivial. Thus, our flat space theory, having a soliton, must have a scale of length, call it L_0, which determines the characteristic size of the soliton (of course there is also a corresponding scale of energy). The gravitational coupling introduces into the model the additional dimensional parameter, Newton's constant G, which allows to define the second scale of length L_g (and energy). Thus typically the Einstein-solitonic-matter system has two scales of length and the ratio $\alpha = L_g/L_0$ is a dimensionless

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[2] I do not require stability, so this meaning is broader than that usually used in literature.

parameter characterizing the model. Although my considerations are supposed to be model-independent, it is helpful to keep in mind a specific model for illustration. For this purpose I shall use the Einstein-Skyrme model [8, 9]. In this model the two scales of length are given by $L_0 = 1/ev$ and $L_g = G^{1/2}/e$, where v and e are dimensional parameters, and $\alpha = G^{1/2}v$ (*cf.* [8]).

2.1. Weak coupling

For small α one may apply the standard perturbative argument to prove the existence of gravitating solitons. The field equations may be expressed schematically in the form

$$F(f, \alpha) = 0, \tag{1}$$

where F is the differential operator and f denotes the collection of all unknown metric and matter field variables. Suppose that the parameters which determine L_0 are fixed, hence the limit $\alpha \to 0$ corresponds to switching-off gravity. By assumption, for $\alpha = 0$ (without gravity) there exists a soliton solution f_0 satisfying

$$F(f_0, 0) = 0. \tag{2}$$

A natural idea to show the existence of solutions of Eq. (1) is to use the implicit function theorem . In the present context the appropriate version of the implicit function theorem is the following. Let X, Y be Banach spaces, $F : X \times R^1 \to Y$ be a smooth mapping and let a point $p = (f_0, 0) \in X \times R^1$ be a solution of Eq. (2). Then, if the derivative mapping $D_f F(p)$ is bijective, the theorem guarantees that in the neighbourhood of p there exists a solution $f(\alpha)$ of Eq. (1), such that $f(0) = f_0$. The derivative $D_f F$ is just the linearized operator $\delta F/\delta f$ one obtains when linearizing F. In the most interesting physical situation the soliton f_0 is linearly stable which means that the eigenvalue problem

$$\frac{\delta F}{\delta f}(p)\xi = \omega^2 \xi \tag{3}$$

has only positive eigenvalues, so the linear operator $\delta F/\delta f(p)$ is invertible, and, after choosing appropriate Banach spaces, bijective. If the spectrum of Eq. (3) is not positive (as it happens for unstable solitons), we have to make sure in addition that there are no zero modes[3]. We conclude therefore that, in general, the soliton solution persists for sufficiently small α. Since the

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[3] In gauge invariant theories there are pure gauge zero modes which one has to remove before applying the implicit function argument. This can be achieved either by fixing the gauge or by working with the space of gauge orbits.

gravitational binding energy is negative, the energy of a gravitating soliton decreases with α. In "everyday" situations α is extremely small, so the gravitational effects are negligible.

It might seem from the above argument that in the region of weak coupling nothing spectacular can happen. However, sometimes even for weak coupling interesting things occur. To see this, notice that I assumed above that $\alpha = 0$ corresponds to $L_g = 0$ ($G = 0$). However, since $\alpha = L_g/L_0$, the limit $\alpha \to 0$ may be obtained in another way, namely by keeping L_g fixed and taking $L_0 \to \infty$. This results in Einstein's equations with some truncated matter sources. For example in the Einstein–Skyrme theory in this limit ($v \to 0$ with G fixed) the sigma-model term disappears from the Lagrangian. An interesting situation arises when such a limiting theory, albeit usually unphysical, has a soliton solution. Then, in analogy to Eq. (1) we have

$$\bar{F}(\bar{f}, \alpha) = 0 \,, \tag{1a}$$

and for $\alpha = 0$ there is a solution $\bar{f}_0$ satisfying

$$\bar{F}(\bar{f}_0, 0) = 0 \,. \tag{2a}$$

The bar over F and f means that although Eqs (1) and (1a) are equivalent when $\alpha > 0$, the operators F and $\bar{F}$ are different; Eq. (1a) is obtained from Eq. (1) by a scaling transformation (depending on α). Eqs (2) and (2a) describe different limiting theories; the procedures of rescaling and taking the limit $\alpha \to 0$ do not commute. If the operator $\delta\bar{F}/\delta\bar{f}(p)$ is bijective, the implicit function theorem guarantees the existence of a solution $\bar{f}(\alpha)$ for sufficiently small α. Then for small α there are two distinct solutions, f and $\bar{f}$, which are perturbations of f_0 and $\bar{f}_0$, respectively. It should be emphasized that although the solution $\bar{f}$ is obtained by the perturbative argument, its occurence, from the point of view of the original flat space model, is non-perturbative. This interesting phenomenon was observed and described in more detail in the Einstein–Skyrme model [8]. It seems that this effect is characteristic for a certain class of models possessing two scales of length.

2.2. Strong coupling

I have argued above that, as long as the dimensionless coupling constant α is sufficiently small, a flat-space soliton survives the gravitational coupling. Will such a gravitating solution persist for large values of α? The gravitational interaction becomes important when $\alpha \sim 1$ since then the size of the soliton is of the order of its Schwarzschild radius. For example the typical size of the skyrmion is $R \sim 1/ev$ while its mass is $M \sim v/e$, so

$GM/R \sim \alpha^2$. Hence, one might expect that for $\alpha \sim 1$ the gravitating soli-
ton becomes unstable with respect to the gravitational collapse and there
is a critical value $\alpha_0 \sim 1$ beyond which no solitons exist. In fact, this ex-
pectation was confirmed numerically in several models [8–12]. For example
in the Einstein–Skyrme model $\alpha_0 \approx 0.2$. It is plausible that this behaviour
is generic for stable strongly gravitating solitons[4].

To understand better the existence of a critical value α_0 let us recall
the standard technique of proving existence of solitons in flat space. Typi-
cally one starts from a topological argument showing that the configuration
space has a nontrivial topological structure and splits into homotopy classes
labeled by a winding number (topological charge). Then one shows that
within a given homotopy class the energy functional is bounded from below
by a positive constant proportional to the topological charge and therefore
there exists a positive lower bound for energy

$$E_0 = inf \, E[f] \,. \tag{4}$$

The idea is to prove that this bound is attained by a regular function f_0
which is thereby a stationary solution. Technically this is a three step
procedure:

1. Construct a minimizing sequence $\{f_n\}$, where f_n belong to the space
   M of regular finite energy functions, such that $\lim_{n\to\infty} E[f_n] = E_0$.
   The existence of such a sequence is guaranted by Eq. (4).
2. Prove that the sequence $\{f_n\}$ has a limit $f_0 \in M$.
3. Finally, show that $E[f_0] = \lim_{n\to\infty} E[f_n]$. This step is nontrivial be-
   cause usually $E[f]$ is not continuous in M.

The proofs of existence of topological solitons along these lines were
given for several models involving non-abelian gauge fields [14].

Now, let us return to gravitating solitons and follow the above proce-
dure. First notice that as long as we deal with asymptotically flat solutions
with R^3 topology on $t = $ const slices, the topological arguments are still
valid for gravitating solitons (as we shall see below in the black hole context
the situation may be different). Similarly, the mass is bounded from below
by a positive constant. However, when one tries to repeat the steps 1–3 in
the case of nonzero α, it turns out that for large α a minimizing sequence
$\{f_n\}$ has no limit in M. In other words, the minimizing solution f_0 is singu-
lar [11]. This shows that one should be careful with topological arguments.
The topological argument is only a hint for existence of a solution and has
to be supplemented by the real proof, which, as the above example teaches
us, might not be a "mere technicality".

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[4] Unstable gravitating solitons may exist for all values of α [13].

The behaviour of gravitating solitons in the strong coupling region is not yet well understood. In particular, a general argument for the existence of a critical value α_0 is lacking. One might try to approach the problem along the lines of [15], where some universal upper bounds for the binding energy of static configurations were derived. In specific spherically symmetric models another approach seems to be easier. Impose regular initial data at $r = 0$ and show that, when α is large, a local solutions cannot be extended to a global solution. Actually, this is exactly how such systems are studied numerically.

A closely related issue is the question of stability of gravitating solitons. Let us recall that the the static solution f is said to be linearly stable if all linear perturbations $\delta f(t)$ around it remain bounded (in a suitable norm) in time. The usual procedure of investigating linear stability is to assume that $\delta f = e^{i\omega t}\xi$, where ξ is time-independent and linearize time-dependent version of Eq. (1) to get

$$\frac{\delta F}{\delta f}\xi = \omega^2 W\xi\,, \tag{5}$$

where W is some positive weight function. The solution f is unstable if there exists at least one eigenmode ξ (satisfying appropriate boundary conditions) with negative eigenvalue ω^2, since then δf will grow exponentially in time. Otherwise the solution is linearly stable. If the flat space soliton is stable then its gravitating counterpart will also be stable for small α, because the eigenvalues ω^2 change continuously with α. The mixed numerical and analytical analysis shows that the eigenvalues ω^2 decrease with α and for $\alpha \to \alpha_0$ the lowest eigenvalue ω^2 tends to zero. This is not a coincidence, but follows from the general relationship between zero modes of the linearized operator and bifurcation points[5]. This connection is one of the basic results of the catastrophe theory which allows to study stability problems in the essentially non-dynamical way [16]. For gravitating solitons this fact was first observed in the Einstein–Skyrme model [8], where at α_0 the branch of fundamental gravitating skyrmions $f(\alpha)$ merges with another branch of unstable solutions $\bar{f}(\alpha)$ (discussed in the text below Eq. (2a)).

2.3. Gravitational desingularization

Above I have discussed Einstein's equations with solitonic matter sources. Now, consider a different situation and suppose that a field source has no soliton solution in flat space. Can the gravitational coupling help in this respect, *i.e.* might there exist globally regular solutions in the coupled

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[5] In the presence of symmetries there might be zero eigenvalues which have nothing to do with bifurcations (Goldstone modes).

Einstein-non-solitonic-matter theory? First, notice that if the field the-
ory is scale invariant then, according to the remarks above, the necessary
condition for existence of a soliton in Einstein-matter system is that grav-
ity breaks scale invariance. For this reason the Einstein-massless scalar or
the Einstein–Maxwell equations cannot have a soliton, while the Einstein–
Yang–Mills (EYM) equations might have, because the EYM theory has a
scale of length given by $\sqrt{G}/e$, where e is the gauge coupling constant.

Even when a field theory has no globally regular solutions, it usually
will have solutions which behave correctly at large distances (but are singu-
lar at the origin). A typical example is the monopole solution (electric or
magnetic) in electrodynamics. It should be stressed that the leading asymp-
totic behaviour at spatial infinity of such solutions will not be altered by
the gravitational coupling because gravity decouples from sources at infin-
ity. However, gravity will modify the short distance behaviour of solutions,
and in particular may regulate their singularities. This idea is traced back
to Einstein who believed that ultraviolet divergencies in field theory will
be eliminated in some unified nonlinear theory. Let us illustrate such a
desingularization phenomenon with two examples.

Consider a spherical shell of radius r with uniform charge and mass
density. Let e be its total charge and m_0 its bare mass. Ignoring gravity,
the energy is given by

$$m(r) = m_0 + \frac{e^2}{r} \tag{6}$$

which, of course, diverges as r tends to zero. Taking into account the
gravitational binding energy, this formula is replaced by

$$m(r) = m_0 + \frac{e^2}{r} - \frac{Gm^2}{r} \, . \tag{7}$$

Solving this equation for $m(r)$ and taking the limit $r \to 0$, yields

$$m(r = 0) = \frac{e}{\sqrt{G}} \, . \tag{8}$$

This heuristic argument can be made rigorous by solving the constraint
equations in the Einstein–Maxwell theory [17]. The result (8) may be viewed
as non-perturbative cancellation of two infinite self-energies: positive elec-
trostatic and negative gravitational.

To show another example of similar type let us consider a very simple
theory in which gravity is modelled by a massless scalar field called dila-
ton (scalar gravity). Dilaton ϕ couples in a universal way to matter (with
Lagrangian L_m) through the term $e^{-2a\phi}L_m$, where a is the dilaton cou-
pling constant. If we take the electromagnetic field as matter, the static

Maxwell-dilaton equations

$$d(e^{-2a\phi} \star F) = 0, \qquad \nabla^2\phi + \frac{a}{2}e^{-2a\phi}F^2 = 0 \qquad (9)$$

have a simple spherical solution with magnetic charge $1/e$ [18]

$$eF = d\vartheta \wedge \sin\vartheta d\varphi, \qquad a\phi = \ln(1 + \frac{a}{er}). \qquad (10)$$

Without a dilaton the energy density of the monopole diverges at $r = 0$ as $T_{00} \sim 1/r^4$, whereas in the present case $T_{00} \sim e^{-2a\phi}F^2 \sim 1/r^2$, hence the total energy is finite (equal to $1/ea$). Although the total energy of the dilatonic monopole is finite, the solution (10) is still singular at $r = 0$. Below I shall discuss the EYM theory where gravitational coupling leads to non-perturbative globally regular solutions.

2.4. Bartnik–Mckinnon solutions

Probably a single most important result in the study of gravitating solitons was the discovery of solitons in the EYM system by Bartnik and Mckinnon (BM) in 1988 [4]. Since then many properties of BM solutions were understood but still some questions remain to be answered. Below I discuss briefly the main features of BM solitons and point out open problems.

The Einstein–YM coupled system is described by the action

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{G}R - \mathcal{F}^2 \right], \qquad (11)$$

where R is a scalar curvature and $\mathcal{F} = d\mathcal{A} + e\mathcal{A} \wedge \mathcal{A}$ is the Yang–Mills curvature of a YM connection $\mathcal{A}$ which takes values in the Lie algebra of the gauge group $\mathcal{G}$. BM considered $\mathcal{G} = \mathrm{SU}(2)$. There are two dimensional parameters in the theory: Newton's constant G (of dimension *length/mass*) and the gauge coupling constant e (of dimension $mass^{-1/2}length^{-1/2}$).

Let us consider static spherically symmetric configurations. It is convenient to parametrize the metric in the following way

$$ds^2 = -A^2N dt^2 + N^{-1}dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \qquad (12)$$

where A and N are functions of r.

Assuming that the electric part of the YM field vanishes (actually this is not a restriction because one can show, [19], that there are no globally regular static solutions with nonzero electric field) the purely magnetic static

spherically symmetric SU(2)–YM connection can be written, in the Abelian gauge, as [20]

$$e\mathcal{A} = w\tau_1 d\vartheta + (\cot \vartheta \tau_3 + w\tau_2) \sin \vartheta d\varphi, \qquad (13)$$

where τ_i $(i = 1, 2, 3)$ are Pauli matrices and w is a function of r. The corresponding YM curvature is given by

$$e\mathcal{F} = w'\tau_1 dr \wedge d\vartheta + w'\tau_2 dr \wedge \sin \vartheta d\varphi - (1 - w^2)\tau_3 d\vartheta \wedge \sin \vartheta d\varphi, \qquad (14)$$

where prime denotes derivative with respect to r.

Inserting these ansätze into the action (11) yields the reduced Lagrangian[6] (where $S = 16\pi \int L dt$)

$$L = - \int\limits_0^\infty \left[\frac{1}{2G} r A'(1 - N) + AU \right] dr, \qquad (15)$$

where

$$U = \frac{1}{e^2} \left[Nw'^2 + \frac{(1 - w^2)^2}{2r^2} \right]. \qquad (16)$$

Variation of L with respect of $w, A,$ and N yields the field equations[7]

$$(ANw')' + \frac{1}{r^2} Aw(1 - w^2) = 0, \qquad (17)$$

$$[r(1 - N)]' = 2GU, \qquad (18)$$

$$A' = \frac{2G}{e^2 r} Aw'^2. \qquad (19)$$

Note that, using Eq. (19), A may be eliminated from Eq. (17).

These equations have three explicit abelian solutions. The first two are the vacua $w = \pm 1$, $A = N = 1$ with zero mass (or Schwarzschild $A = 1, N = 1 - 2GM/r$). The third solution

$$w = 0, \qquad A = 1, \qquad N = 1 - \frac{2GM}{r} + \frac{G}{e^2 r^2} \qquad (20)$$

describes the Reissner–Nordström black hole with mass M and magnetic charge $1/e$.

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[6] Strictly speaking, to obtain (15) one has to add a surface term to the action (11).

[7] The principle of symmetric criticality [21] guarantees that the variation of S within the spherically symmetric ansatz gives the correct equations of motion.

In order to construct non-abelian solutions which are globally regular one has to impose the boundary conditions which ensure regularity at $r = 0$ and asymptotic flatness. The asymptotic solutions of Eqs (17), (18) satisfying these requirements are

$$\pm w = 1 - br^2 + O(r^4), \qquad N = 1 - 4b^2 r^2 + O(r^4) \tag{21}$$

near $r = 0$, and

$$\pm w = 1 - \frac{c}{r} + O(\frac{1}{r^2}), \qquad N = 1 - \frac{d}{r} + O(\frac{1}{r^2}) \tag{22}$$

near $r = \infty$. Here b, c, d are arbitrary constants.

Notice that, for the asymptotic behaviour (22), the radial magnetic curvature, $\mathcal{B}_r = \tau_3 (1 - w^2)/r^2$, falls-off as $1/r^3$, and therefore all globally regular solutions have zero YM magnetic charge.

Bartnik and Mckinnon gave strong numerical evidence that there exists a countable sequence of initial values b_n $(n \in N)$ determining globally regular solutions w_n, A_n, N_n. The index n labels the number of nodes of the function w_n. The sequence of masses M_n of these solutions increases with n and tends to $M_\infty = 1$ for $n \to \infty$ (the unit of mass is given by $1/G^{1/2}e$). If one defines the "fine structure" constant $\gamma^2 = \hbar e^2$, then the mass of BM solitons is of the order m_{Pl}/γ while their effective size is of the order l_{Pl}/γ. Notice that the classical treatment of such objects makes sense physically only if γ is small since then the Compton radius is much less than the effective radius of BM solitons.

The BM discovery raised a number of questions. Let us enumerate the most important ones:

- **Existence:** The first rigorous proof of existence of BM solitons was given by Smoller *et al.* [22]. Another proof was recently presented by Breitenlohner *et al.* [23]. Both proofs use the methods of dynamical system theory to analyze the a priori behaviour of orbits starting with regular initial data at $r = 0$ and find that there exists a countable family of connecting orbits (separatrices) which correspond to the BM solutions. Both proofs leave open the question of uniqueness of BM solutions (amongst globally regular solutions of Eqs (17)–(19)).

- **Stability:** The linear stability analysis of BM solutions was carried out by Straumann and Zhou [24]. They derived an eigenmode equation, such as Eq. (5), which governs the time evolution of small perturbations around the BM solitons and demonstrated that it has at least one negative eigenvalue (unstable mode). Later [25] they also analyzed the long time behaviour of perturbed BM solitons and argued that, depending on an initial perturbation, they either disperse to infinity or collapse to form the Schwarzchild black hole.

- **Raison d'être**: In the study of nonlinear equations perhaps even more important than proving existence of a solution is to understand the reason for its existence. The existence proofs mentioned above do not really explain what are the essential features of the SU(2)-YM gauge group which are responsible for the existence of BM solutions and their basic properties, such as discreteness and instability. Soon after the discovery of BM solutions Mazur suggested (private communication, 1990) that they should be regarded as gravitational sphalerons. Let us recall that sphaleron is a saddle point solution whose existence follows from the Lusternik–Snirelman mini-max (mountain pass) construction applied to the space of non-contractible loops (or paths joining topologically non-equivalent vacua) of field configurations. Sphalerons were first discovered by Taubes and Manton in spontanously broken gauge theories (YM–Higgs) [26], but they exist in many other models which possess non-contractible loops in configuration space [27]. The first published attempt [28] to interpret the BM solutions as sphalerons was unsuccesful, because the authors considered loops with too weak decay condition for the YM connection which led to a disaster of having the infimum of energy over all loops equal to zero (*cf.* footnote 9).
  Recently Sudarsky and Wald (SW) [29] proposed, in the language of Hamiltonian formulation of general relativity, a very interesting modification of the original Taubes–Manton argument. In the case of $n = 1$ BM solution, the SW argument is, in essence, equivalent to the mini-max procedure for paths joining two topologically inequivalent vacua. However, in contrast to the mini-max construction, the SW mechanism can be naturally extended to account also for the existence of $n > 1$ BM solutions. The SW argument got support by the discovery of solutions similar to BM solutions in another theory involving SU(2)-YM field [18]. It is very unlikely that the SW argument it its present form can be converted into a genuine proof. However, in the spherically symmetric case the situation simplifies considerably and seems to be tractable rigorously. I shall come back to this point when discussing the YM field on Schwarzschild background.

- **Spectrum of masses**: The discrete spectrum of masses of BM solutions has an accumulation point (upper bound) at $M = 1/(eG^{1/2})$. This follows from the fact that for $n \to \infty$ the BM solutions tend (non-uniformly) to the extremal Reissner-Nordström solution with magnetic charge equal to $1/e$. Can one construct an inequality $M \leq 1/(eG^{1/2})$ which is saturated by the limiting solution?

- **Spherical symmetry**: In the EYM system the Birkhoff theorem is not valid since a spherically symmetric configuration need not be static. Is the converse result true, *i.e.* does staticity imply spherical symmetry?

Showing this would be the first step towards the proof of uniqueness of BM solutions (of static SU(2)-YM equations). One could try to attack this problem using the technique of Bunting and Masood-ul-Alam [30] of finding a conformal transformation of spatial 3-metric into a zero-mass metric with non-negative scalar curvature. As far as I am aware (W. Simon, private communication) this approach encounters serious technical difficulties. A closely related question: are there stationary axisymmetric analogues of BM solutions?

• **Desingularization:** The BM solutions are non-perturbative in the gravitational constant G — *nota bene* in the EYM theory there is no dimensionless parameter which might serve as a perturbative parameter (changing the coupling constants e and G only changes the scale). Although the flat-space static YM equations have no globally regular solutions [7], they possess spherically symmetric solutions which at large distances behave similarly to BM solutions. These are solutions of Eq. (17) with $A \equiv N \equiv 1$ [31]. Their asymptotic behaviour at infinity is given by Eq. (22), whereas near $r = 0$ they behave as

$$w \sim \sqrt{r} \sin(\frac{\sqrt{3}}{2} \ln r + \text{const}). \tag{23}$$

so the YM curvature $\mathcal{F}$ is singular at $r = 0$ and the total energy is infinite. I believe that the BM solutions may be viewed as the regularized (by gravity) version of these singular flat-space solutions. In a sense the proof of Breitenlohner *et al.* implements this idea, because they start from the phase portrait of Eq. (17) in flat space and analyze how it is modified by gravity. It would be of interest to pursue this idea further. One possible direction is to search for exact form of BM solutions, since having it we probably could see how the scale of length $G^{1/2}/e$ provides a cut-off for singular behaviour (23). Recent results in the EYM-dilaton theory (Bizoń and Chmaj, work in progress) suggest that the hope of finding an analytical expression for BM solutions is not unrealistic.

3. Black holes

Let us now turn to another interesting class of asymptotically flat solutions, namely black holes. We restrict attention to the region outside the horizon. Then, in the stationary case, Einstein's equations reduce to the elliptic boundary value problem between horizon and infinity.

There are more black hole solutions than globally regular ones. This is due to two important differences between these two cases. First, stationary globally regular solutions are extrema of mass, while stationary black holes extremize the mass provided that the area of the horizon is fixed (and the

angular momentum is fixed, if there is nonzero rotation)[8] [29]. This follows from the first law of black hole dynamics

$$G \delta M = \frac{1}{8\pi} \kappa \delta A + \Omega \delta J \,, \qquad (24)$$

where M, κ, A, J, and Ω are the black hole mass, surface gravity, area, angular momentum, and angular velocity of the horizon. In other words the presence of horizon introduces the boundary condition which breaks scale invariance. This allows for nontrivial black hole solutions in scale invariant theories, as for instance the Schwarzschild solution.

Second, solutions which are singular in flat space may give rise to perfectly regular hair on black holes if the singularities are hidden behind the horizon. The Reissner–Nordström black hole is a typical example.

In analogy to the discussion of globally regular solutions we can distinguish several situations depending on scaling properties. The first possibility is that the model is scale invariant, and the scale is given solely by the area of the horizon. In this case we have a continuous family of black holes, such as Schwarzschild or Reissner–Nordström solutions.

The second possibility is that the corresponding flat space model has a soliton and a scale L_0. The coupling of gravity gives a second scale L_g and the presence of horizon provides a third scale r_H, where r_H is the radius of the horizon (for simplicity I consider spherically symmetric case). Thus, in analogy to Eq. (1), we can write the field equations in the form

$$F(f, \alpha, \beta) = 0 \,, \qquad (25)$$

where $\alpha = L_g/L_0$ and $\beta = r_H/L_0$. Note that the status of parameters α and β is different — α is a combination of coupling constants while β is determined by the boundary condition (in the nonspherical case it is even not clear how to define β). Nevertheless, as long as $r_H > 0$, by rewriting the equations in the variable $x = r/r_H$, β is eliminated from the boundary condition and appears as a second (apart from α) dimensionless parameter in the equations. It is very convenient to think about Eq. (25) as describing black holes when $\beta > 0$, and globally regular solutions when $\beta = 0$. That is, we require f to belong to a space X of asymptotically flat configurations which for $\beta > 0$ satisfy the black-hole boundary conditions at r_h, while for $\beta = 0$ (when black hole becomes singular since the horizon shrinks to zero) are regular at the origin. The advantage of working with X is that the flat-space solution f_0, satisfying $F(f_0, 0, 0) = 0$, belongs to X and may be used

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[8] In both cases there might be additional contributions to the variation of mass coming from other extensive parameters (such as *e.g.* charge). Mass is extremized at stationary solutions if these parameters are fixed [29].

as the basis of the perturbative argument. Such an argument was proposed by Kastor and Traschen [32] who argued, using the implicit function theorem, that for sufficiently small α and β in the neighbourhood of f_0 there exist black hole solutions of Eq. (25). Although their heuristic argument is probably basically correct, it certainly should be pursued by more precise analysis which would take care of some technical subtleties, such as the effects of degenerate ellipticity at r_{H}. Anyhow, this reasoning is strongly supported by numerical results in several models having soliton solutions, where black hole counterparts with small horizon (sometimes referred to as black holes inside solitons) were found [8–12, 33]. When $\beta \to 0$ these black holes tend to the corresponding globally regular solutions.

It turns out that such black holes exist only in a bounded region of the (α, β)-plane. As was shown before, for $\beta = 0$ there is a maximal value α_0 beyond which no globally regular solutions exist. Numerical and analytical analyses show that when β grows, the critical value $\alpha_0(\beta)$ monotonically decreases and goes to zero for some β_0 [8–12, 33]. Thus there is an upper bound for the radius of horizon. The heuristic explanation is that when the horizon gets larger, more and more of solitonic hair is swallowed and finally, when the radius of the horizon is comparable to the size of the soliton ($\beta \sim 1$), the whole soliton disappears inside the black hole (no "cloud" outside the horizon is left). The technical reason of non-existence of black holes with solitonic hair for large β is the following. The standard no-hair proof (see papers by Bekenstein [3]) consists in constructing an identity which has the form of a divergence equaling a non-negative definite quantity. After integrating this identity between horizon and infinity the divergence term vanishes, hence the integral of non-negative quantity is zero which implies that the field (hair) is trivial. For a nonlinear source one can also construct a similar divergence identity but usually its right side has both positive and negative contributions, hence the proof fails. However in some cases one can show that when β is large enough the positive terms take over so the right side of the divergence identity is non-negative and the no-hair proof goes through.

Note that the limiting case $\alpha = 0$ corresponds to the decoupling of gravity. Thus to find a maximal value β_0 it is sufficient to study solitonic fields on fixed Schwarzschild background. For skyrmionic hair this was done in [34].

The third possibility is that the flat space model has no solitons, and gravity brings a scale L_{g}. Then the field equations have the form $F(f, \gamma) = 0$, where $\gamma = r_{\mathrm{H}}/L_{\mathrm{g}}$. This situation arises in the EYM theory so let us now consider black holes in this model. As above, it is convenient to treat the limit $\gamma = 0$ as corresponding to a regular solution. As I have already described, in this case there is a countable family of BM solutions; let us

denote them by f_n^0 (superscript 0 refers to $\gamma = 0$). For sufficiently small γ we can repeat Kastor and Traschen's heuristic argument to predict the existence of a countable family of black holes $f_n(\gamma)$, which are perturbations of the corresponding BM solutions. These solutions, so called colored black holes (CBHs), were found numerically independently by several authors [5] soon after the BM discovery. Recently the existence of CBHs was established rigourously [22, 23]. Both numerical and analytical results show that there is no upper bound for γ, *i.e.* for every γ there is a countable family of CBHs, $f_n(\gamma)$. It should be stressed that CBHs with different γ are essentially different; in particular they are *not* related by any simple scaling (contrary to some statements scattered in literature). When $\gamma \to \infty$ the YM and gravitational fields decouple and we end up with the YM field on fixed Schwarzschild background. This model is extremely simple, nevertheless it seems to contain all the essential features which are responsible for the existence of BM-type solutions. For this reason it is an ideal ground for converting heuristic arguments (such as the SW argument) into rigorous results. To my knowledge this model was not described in literature so I take this opportunity to give some details.

3.1. SU(2)-YM on Schwarzschild background

Let us rewrite Eqs (17)–(19) using a dimensionless coordinate $x = r/r_{\mathrm{H}}$ and take the limit $\gamma = er_{\mathrm{H}}/G^{1/2} \to \infty$. In this limit the right side of Eqs.(18,19) vanishes, hence they are solved by the Schwarzschild metric

$$N = 1 - \frac{1}{x}, \qquad A = 1, \tag{26}$$

and therefore Eq. (17) reduces to

$$\left((1 - \frac{1}{x})w' \right)' + \frac{1}{x^2}w(1 - w^2) = 0. \tag{27}$$

This equation follows from the variation of the energy functional

$$E[w] = \int\limits_1^\infty \left[(1 - \frac{1}{x})w'^2 + \frac{1}{2x^2}(1 - w^2)^2 \right] dx. \tag{28}$$

It was shown first numerically and recently analytically (Bizoń, unpublished) that Eq. (27) has a countable family of solutions $\{w_n\}$ ($n \in Z$) which are regular for $x \geq 1$. The first member of this family is the vacuum $w_0 = 1$, for which the energy functional attains the global minimum

$E[w_0] = 0$. For a discrete set of initial values $w_n(1) \in (0,1)$ there exist essentially non-abelian solutions w_n which oscillate $(n-1)$-times within a strip $(-1,1)$ and tend to $(-1)^n$ for $x \to \infty$. The discrete spectrum of energies $E_0 = 0, E_1 \approx 0.4795, E_2 \approx 0.4994, E_3 \approx 0.4999, \ldots$ has an accumulation point at $E_\infty = {}^1\!/_2$. This follows from the fact that for $n \to \infty$ the solutions w_n tend pointwise (on any finite interval $[1, x_0)$) to the abelian solution $w_\infty = 0$ with energy $E[w_\infty] = 1/2$. Using Eq. (27) one can easily show that on-shell the energy is given by

$$E[w] = \int\limits_1^\infty \frac{1}{2x^2}(1 - w^4)dx \,, \tag{29}$$

which proves that the solution $w_\infty = 0$ has maximal energy.

For completeness let us note that the solution w_1 is known analytically [35]

$$w_1 = \frac{c - x}{x + 3(c - 1)} \,, \tag{30}$$

where $c = (3 + \sqrt{3})/2$.

Notice that our system has Z_2 symmetry, hence along with $\{w_n\}$ there is a corresponding family of reflected solutions $\{-w_n\}$. The limiting solution w_∞ is a fixed point of reflection.

The Hessian of the energy functional is given by

$$\delta^2 F(w)(\xi, \xi) = \int\limits_1^\infty \left[(1 - \frac{1}{x})\xi'^2 + \frac{1}{x^2}(3w^2 - 1)\xi^2 \right] dx \,. \tag{31}$$

A very interesting feature (found numerically) is that the Hessian evaluated at a solution w_n has exactly n negative eigenvalues. This property is a strong hint for a mechanism which gives rise to the family $\{w_n\}$. Before discussing it let me make the following clarifying remark.

Remark. The key feature of the SU(2)-YM gauge group is the existence of large gauge transformations, *i.e.* topologically inequivalent cross sections of the YM bundle. They arise as follows. Choose the gauge $A_0 = 0$ and consider the spatial YM potential A_i on the $t = $ const slice Σ of R^3 topology. The pure gauge configurations have the form $A_i = \partial_i U U^{-1}$, where U is a map from Σ to the group manifold SU(2). Demanding that $U \to 1$ (sufficiently fast[9]), this becomes a homotopically nontrivial map $U : S^3 \to$

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[9] It is essential that $r\partial_i U \to 0$ (and therefore $rA_i \to 0$) as $r \to \infty$. Otherwise the YM bundle is topologically trivial, as one can easily demonstrate by constructing curves of connections (with $A_i = O(1/r)$ decay) joining large gauge transformated vacua. I thank R. Bartnik for pointing this out to me.

S^3, classified by integer winding numbers. Thus, there is a countable set of topologically inequivalent vacua which cannot be continuously deformed into one another. This fact is the basis for the Taubes–Manton mini-max construction of sphalerons and for the Sudarsky–Wald construction of BM solutions. However, in the black hole setting the situation is different. The point is that now the space Σ has topology $S^2 \times R^1$ and the mapping $U : \Sigma \to S^3$ is topologically trivial (it is not possible to compactify Σ by demanding U to be identity on the horizon).

Although the topology of the SU(2)-YM field on the Schwarzschild background is trivial, the presence of two distinct vacua and Z_2 symmetry seems to be sufficient for making the mini-max construction. Let Γ be a space of functions w for which the energy functional (28) is finite. Consider all paths in Γ connecting w_0 and $-w_0$. Since w_0 and $-w_0$ are global minima of energy, on each path there is a point of maximal energy. The infimum over these maximal energies gives a saddle point (with exactly one unstable mode), which we identify as w_1. It is tempting to repeat this procedure using w_1 and its reflection $-w_1$. However, this fails because there are paths joining w_1 and $-w_1$ which go below the energy level $E[w_1]$. One could try to remedy this obstacle by defining a space Γ_1 consisting of all maxima on paths joining w_0 and $-w_0$, and making mini-max on Γ_1. Unfortunately, it is very doubtful that Γ_1 will be sufficiently well-behaved to allow such construction[10], in particular almost surely Γ_1 will not be a surface in Γ. Recently R. Wald suggested (private communication) that to obtain higher n solutions one should apply mini-max method not for paths but for higher dimensional objects, such as n-spheres which are invariant under reflections.

Another very interesting approach was pursued by Popp (unpublished). The idea is to show that the energy functional is a Morse function on $\Gamma \setminus w_\infty$, i.e. that all critical points of $E[w]$ (except w_∞) have finite index (in the language of Morse theory, index = number of negative eigenvalues of the Hessian). To prove this one approximates $E[w]$ by some truncated functional $F_\lambda[w]$, defined by certain integral over the interval $[1, \lambda]$. For λ close to one, $F_\lambda[w]$ has only three critical points: $\pm w_0 = 1$ and $w_\infty = 0$ and all these points have zero index. When λ increases new critical points appear. At each bifurcation point λ_n the solution w_∞ "sheds" a pair of new solutions $(w_n, -w_n)$, and at the same time it acquires one additional negative eigenvalue (i.e. the index of w_∞ increases by one). Near the bifurcation point λ_n the solution w_n is close to w_∞, and therefore it has the same index (equal to n). As λ increases the index of w_n remains constant (this fact is not yet proven rigorously but there is strong evidence that it is true). Showing that $F_\lambda[w] \to E[w]$ for $\lambda \to \infty$ concludes Popp's argument.

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[10] I thank R. Wald for pointing this out to me.

3.2. No-hair conjecture

The "no-hair" conjecture belongs to the folklore of general relativity since the late sixties (for a list of references, see [36]). The conjecture concerns the possible forms of stationary black holes. The idea is that the only classical degrees of freedom of black holes are those corresponding to non-radiative multipole moments, because "everything that can be radiated away will be radiated away" (*cf.* [37]). For a massless bosonic field with spin s, radiative multipoles have moments $l \geq s$, hence for pure gravity ($s = 2$) the conjecture allows for a monopole (mass) and a dipole (angular momentum) degrees of freedom. If there is an electromagnetic field ($s = 1$), the electromagnetic monopoles (electric and magnetic charge) are also allowed. The conjecture exludes the massless scalar field ($s = 0$) and all massive fields, because for these fields all multipoles are radiative.

At the time when the "no-hair" conjecture was formulated this picture was perfectly consistent with the fact that the only known stationary black hole was the Kerr–Newman solution and nobody doubted that this was a unique stationary black hole solution of the Einstein–Maxwell equations (although this fact was established rigorously later on [2]). Morever, the conjecture was supported by several no-go results which showed that such fields as massless and massive scalar, massive vector or spinor cannot reside on stationary black holes [3].

The "no-hair" conjecture was formulated rather vaguely and therefore admits many interpretations. Nowadays most people agree that it should be meant as a statement concerning the uniqueness of stationary black holes. Let us adopt this viewpoint and try to formulate the "no-hair" conjecture more precisely[11]. For this purpose let me define a term global charge to mean a conserved charge associated with a massless gauge field, which is given by a surface integral at spatial infinity. Mass, angular momentum, and electric and magnetic charges are examples of global charges. Then the strongest variant of "no-hair" conjecture is

Version 1. A stationary black hole is uniquely described by global charges.

For a long time no counterexample to this version was known. However, recent discoveries of various black holes with nonlinear matter fields made clear that one cannot speak about uniqueness without specifying the sources. To see this consider an example of Einstein–Maxwell-dilaton theory. In this model Gibbons found very interesting black hole solutions [39]. In the so called string-inspired case (where the dilaton coupling constant a is equal

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[11] I am concerned here with what the "no-hair" conjecture really means, and not with mathematical assumptions underlying the uniqueness theorems (*cf.* [38]).

to $\sqrt{G}$), the electrically charged solution is given by

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r(r - \frac{Q^2}{M})d\Omega^2 \,,$$

$$F = e^{2\sqrt{G}\phi} \frac{Q}{r^2}\, dt \wedge dr \,, \qquad e^{2\sqrt{G}\phi} = 1 - \frac{Q^2}{Mr} \,, \qquad (32)$$

where ϕ is a dilaton. It was shown recently that this (and the analogous magnetically charged solution) is a unique static black hole solution in this model [40]. However, the dilaton field is not associated with any conserved charge, and therefore at infinity this solution is indistinguishable from the Reissner–Nordström solution with the same mass and charge. Thus an observer at infinity would not be able to determine a solution by measuring global charges, unless she knows the matter content of the world. This leads us to the following modification:

Version 2. *Within a given model* a stationary black hole is uniquely determined by global charges.

Although in many cases, such as the electrovacuum, this version is actually a rigorous theorem, there are theories in which it is false. For example, as I discussed above, in the EYM theory there is a countable family of colored black holes. They carry no charge so their non-abelian field leaves no imprint at infinity. For a given mass (which is the only global parameter) there are infinitely many CBHs with different areas of the horizon which clearly violates *Version 2.* However, this is not a physically serious counterexample, because CBHs are unstable [41] — under small perturbation they loose non-abelian hair and decay into Schwarzschild (for such unstable hair the word wig seems to be more appropriate). We should therefore add to *Version 2* the following qualification:

Version 3. Within a given model a *stable* stationary black hole is uniquely determined by global charges.

In the EYM theory this unique stable solution is, of course, Schwarzschild. Unfortunately, there are models which violate even this version [8, 33]. For example, in the Einstein–Skyrme model, for a given mass there are three black holes, two stable and one unstable. One of the stable black holes is Schwarzschild while another one has skyrmionic hair. Since the skyrmionic hair is not connected to any conserved charge (and morever is topologically trivial), the existence of these two stable black holes provides a counterexample to *Version 3.*

There seems to be no further possibility of relaxing *Version 3* to accomodate such cases as the Einstein-Skyrme model. We conclude therefore that there is no universally valid formulation of the "no-hair" conjecture.

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