Vol. 25(1994) *ACTA PHYSICA POLONICA B* No 6

INTRODUCTION TO MATRIX MODELS AND STATISTICAL MECHANICS ON RANDOM LATTICES*

L. Molinari

Dip. di Fisica and INFN
Via Celoria 16, 20133 Milan, Italy

(Received January 11, 1994)

Matrix models are a powerful technique to solve some models of statistical mechanics on random planar lattices. A simple introduction is given, to illustrate the basic analytical methods and a few applications.

PACS numbers: 05.50. +q, 05.70. -a

1. Introduction

The $1/N$ expansion, where N characterizes the dimension of a symmetry group of the system, is an effective technique to achieve non perturbative results widely used in physics. In quantum mechanics, the radial equation for a O(3) symmetric problem with a complicated potential can be solved approximately by a $1/N$ expansion after transferring it in N dimensions and setting finally $N = 3$. Fields in the vector representation of O(N) or U(N) occur in statistical mechanics or field theory, and the solution becomes simpler in the large N limit, the starting point for a $1/N$ expansion.

A great challenge in physics is the understanding of the structure of hadrons from QCD, a theory based on the local symmetry gauge group SU(3). Strong interactions between quarks are described by the exchange of colour-charged particles, the gluons, which mathematically correspond to self-interacting 3×3 matrix fields. Though high energy physics, because of asymptotic freedom, is well described by this simple picture, quark-gluon bound states require a non perturbative analysis, where elementary exchange processes are summed to all orders.

A brilliant idea by 't Hooft [1] was to introduce in the theory a new expansion parameter $1/N$, by raising the physical Yang Mills symmetry from

* Presented at the VI Symposium on Statistical Physics, Zakopane, Poland, September 20–29, 1993.

SU(3) to SU(N). Qualitative arguments suggested that yet the leading term is significant for the physics of mesons. Although the project still remains very ambitious, the paper by 't Hooft contained a beautiful result, that was the source for a wide research in matrix models, with applications in different and unexpected directions. He showed that, for matrix models, the $1/N$ expansion of the partition function or Green functions corresponds to a topological classification of the Feynman graphs arising from their ordinary perturbative expansion. A Feynman graph is constructed with propagators crossing at interaction vertices: to draw it on a surface may require handles that avoid intersections not corresponding to vertices. The leading order in the $1/N$ expansion is the "planar limit", and corresponds to the sum of Feynman graphs which can be drawn on the surface of the sphere. The next term is given by the sum of graphs which can be drawn on the surface of a torus, and so on. This property is true in any dimension D of space-time, and for any polynomial potential.

In a fundamental work, Brézin, Itzykson, Parisi and Zuber [2] were able to compute analytically, by a saddle point integral, the planar limit of the ϕ^3 and ϕ^4 models in D=0. Their solution was shortly afterwards obtained by Bessis [3], with a very elegant and powerful technique based on orthogonal polynomials.

The solution of the planar limit in D=0 attracted much interest since the planar Feynman graphs that are so generated and exactly summed, can be viewed as random lattices for 2D statistical models. Vertices are the lattice sites, connected in all possible ways by propagators which, by a suitable definition of the potential, can have different weights. The thermodynamic limit corresponds to the large order limit in the perturbation expansion of the planar term.

Kazakov and Boulatov [4] have shown that a two-matrix model, which can be solved exactly in the planar limit in D=0, corresponds to an Ising model on 2D planar random lattices with constant magnetic field. The critical exponents turned to be different from those found by Onsager and Yang for the regular lattice.

An important difference to note between a statistical model on a regular lattice and the same model on the random lattices given by the planar limit of a matrix model is that, in the latter case, the lattices themselves are configurational variables. For example, the partition function of the Ising model by Kazakov not only involves a summation over spin configurations on k sites, but also a summation over all planar graphs with k vertices allowed by the potential of the matrix model. Such a tremendous task is accomplished quite easily through the very effective analytical tools of matrix models.

One-matrix models with cubic or quartic potential generate, through a duality relation, all random triangulations or quadrangulations on the sphere, on the torus *etc.* [5]. A summation on random lattices can then be viewed as the discrete formulation, or even the definition, of a functional integral on surfaces, as occurring in 2D quantum gravity or string theory, provided a continuum limit exists. This is indeed the case, when the parameters of the matrix model are suitably chosen. For this reason, the above statistical models on random lattices are also described as models of matter coupled to gravity, for which a fundamental work by Knizhnik, Polyakov and Zamolodchikov [6] predicts correctly the critical behaviour.

2. The topological expansion

Before showing the basic property that the $1/N$ expansion coincides with a topological classification of Feynman graphs of matrix models, it is useful to spend a few words about their definition in D=0.

A field theory in D=0 is extremely simple, all is sitting on a single point: a real scalar field is a real variable, the partition function and Green functions are ordinary integrals. Nevertheless, Feynman graphs can still be built from the perturbative expansion, although with building elements that are constants. A theory in D=0 is then a way to count them.

For example, the euclidean partition function for the ϕ^4 model in D=0 is:

$$Z(g) = C \int_{-\infty}^{+\infty} d\phi \mathrm{e}^{-\frac{1}{2}m^2\phi^2 - g\phi^4} = \sum_{V=0}^{\infty} \left(-\frac{4g}{m^4}\right)^V \frac{1}{V!}\Gamma\left(2V-\frac{1}{2}\right), \quad (2.1)$$

where C is a constant such that $Z(0) = 1$. The perturbative series can be given a graphical counterpart in terms of graphs with quartic vertices, with weight g, connected by lines with weight $1/m^2$ always ending on vertices (vacuum graphs). If V and P are the numbers of vertices and lines in a graph $\mathcal{G}$, the relation $4V = 2P$ holds, and the graph contributes a factor $x^V C(\mathcal{G})$, where $x = g/m^4$. The number $C(\mathcal{G})$ counts all possible ways to connect lines to produce the graph, modulo some symmetries. To a given order V, many different graphs contribute, and the coefficient of x^V is, up to a sign, their total number. Some of these graphs are the union of distinct graphs of lower order: to remove all disconnected graphs, one introduces the free energy $E(x) = -\log Z$, whose perturbative expansion counts connected vacuum graphs

$$E(x) = 3x - 48x^2 + 1584x^3 - 78336x^4 + \ldots . \quad (2.2)$$

The number of graphs grows factorially with the number of vertices, a reason for the perturbation series to be divergent. To reorder Feynman graphs into a topological expansion, the scalar variable ϕ must be replaced with N^2 variables in a Hermitian $N \times N$ matrix, and the partition function is rewritten with rescaled parameters as follows

$$\mathrm{Z}_N(g) = C_N \int \prod_i dH_{ii} \prod_{i<j} d\,\mathrm{Re}\,H_{ij}\, d\,\mathrm{Im}\,H_{ij} e^{-\frac{1}{2}m^2\mathrm{Tr}(H^2)-\frac{g}{N}\mathrm{Tr}(H^4)} \qquad (2.3)$$

For short we shall denote by $[dH]$ the integration over matrix variables. Note that for $N = 1$ we obtain (2.1). One can again expand the partition function in g and evaluate the free energy; the first two terms are

$$\mathrm{E}_N(x) = (2N^2 + 1)x - (18N^2 + 30)x^2 + \dots \qquad (2.4)$$

Each coefficient of x^V corresponds to the sum of Feynman graphs with V quartic interaction, but different weights in N. To calculate the weight in N of a Feynman graph, it is useful to introduce a double line notation, corresponding to the flow of the two matrix indices. A propagator is couple of oriented lines, like a road with two opposite traffic flows. A quartic vertex is the crossing of four roads with the rule that the driver must always turn right. A planar graph is a road net with no bridges. In a vacuum graph each line is a loop and implies a summation on the flowing matrix index, thus a factor N. To obtain meaningful results for $N \to \infty$, it is necessary to scale appropriately the weights of vertices.

We recognize, for example, that the only graph of first order in x has multiplicity 3 (2.2); when the three realizations are drawn in double line notation, two are planar, the other (where opposite legs of the vertex are connected) is non planar and has the topology of a torus (2.4).

When the perturbative series is reordered as a series in $1/N$, one obtains

$$E_N(x) = N^2 \left[E_0(x) + \frac{1}{N^2} E_1(x) + \frac{1}{N^4} E_2(x) + \dots \right], \qquad (2.5)$$

where $E_0(x) = 2x - 18x^2 + \dots$ is the planar term, summing all planar Feynman graphs.

To show that a coefficient $E_i(x)$ of the $1/N^2$ expansion corresponds to the sum of Feynman graphs with the same topology, we consider the general potential in D=0:

$$\mathrm{V}(H) = \frac{m^2}{2}\,\mathrm{Tr}\,(H^2) + \sum_{k>2} \frac{g_k}{N^{(k-1)/2}} \mathrm{Tr}(H^k)\,. \qquad (2.6)$$

Let us fix our attention on a vacuum Feynman graph $\mathcal{G}$ with V_k vertices of coordination $k = 3, 4 \ldots$. Since there are no external lines, the number of double line propagators is $P = \frac{1}{2}\sum_k kV_k$, and the total number of vertices is $V = \sum_k V_k$. Each closed single line can be viewed as the boundary of the face of a polyhedron, with two faces joining along a double line propagator, and k faces merging into a vertex of coordination k. The total surface is closed and can have a complicated topology described by Euler's number $\chi = 2 - 2h$, where h is the number of handles ($\chi = 2$ for the sphere, 0 for the torus *etc.*). If F is the total number of faces, the following famous relation holds: $\chi = F - V + P$.

The Feynman graph contributes to the partition function through a product of terms: *(i)* a combinatorial factor $C(\mathcal{G})$, due to the fact that the same graph can be built in various ways, *(ii)* a space time factor, which is a constant in D=0, *(iii)* the following factor related to the structure of the graph (we include the propagators, coming from *(ii)*:

$$\prod_k \left(\frac{g_k}{N^{(k-1)/2}}\right)^{V_k} \left(\frac{1}{m^2}\right)^P N^F = N^\chi \prod_k \left(\frac{g_k}{m^k}\right)^{V_k} \tag{2.7}$$

by the above relations. This proves the assertion that equal weights in N correspond to the same topological structure (Euler number) of the manifold where the graph is drawn, such that all intersecting lines correspond to vertices. The planar vacuum graphs are generated by the planar free energy:

$$E_0(x) = -\lim_{N\to\infty} \frac{1}{N^2} \log Z_N(x) \tag{2.8}$$

whose evaluation is described in the next section.

3. Analytical methods

The first step in both the analytical methods that will be described, saddle point and orthogonal polynomials, is to express the integral over matrix elements as an integral over eigenvalues. This reduction is possible, since the potential is constructed with traces. Writing $H = U\Lambda U^\dagger$, with Λ real diagonal and U unitary, the measure transforms according to

$$[dH] = |\Delta(\lambda_1, \ldots \lambda_N)|^\beta \prod_{i=1}^N d\lambda_i \, [dU] \quad , \quad \Delta(\lambda_1, \ldots \lambda_N) = \prod_{i<j} (\lambda_i - \lambda_j) \, , \tag{3.1}$$

where $\beta = 2$, and $[dU]$ is the Haar measure of the group SU(N). One could also consider real symmetric ($\beta = 1$), quaternionic self-dual ($\beta = 4$), complex matrices, or even rectangular matrices [7]. Once the eigenvalues are rescaled to extract a factor $\sqrt{N}$, one obtains

$$Z_N = C \int \prod_{i=1}^{N} d\lambda_i \Delta(\lambda_1, \ldots \lambda_N)^2 \exp\left(-N \sum_{i=1}^{N} V(\lambda_i)\right), \qquad (3.2)$$

where $V(\phi)$ is the potential for N=1. The quadratic case is very important, and defines the Wigner and Dyson Gaussian ensembles of random matrices, that were introduced to describe the statistical properties of spectra of complex and chaotic systems [8].

3.1. The saddle point integral

The method is very effective to evaluate the large N limit of the free energy and Green functions. Another advantage is that it gives the boundaries, in the space of the parameters of the potential, of the various phases that characterize the large N limit of matrix models.

In the large N limit the integrals of the partition function can be computed in the Gaussian approximation around the saddle point configuration, which minimizes the effective potential

$$U(\lambda_1, \ldots \lambda_N) = \frac{1}{N} \sum_{k=1}^{N} V(\lambda_k) - \frac{2}{N^2} \sum_{i<j} \log|\lambda_i - \lambda_j|. \qquad (3.3)$$

The saddle point equations are:

$$\frac{1}{2} V'(\lambda_i) = \frac{1}{N} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \qquad i = 1 \ldots N. \qquad (3.4)$$

In the quadratic case, these equations are exactly solved by the zeros of a Hermite polynomial of order N. In the general case, and for large N, one introduces a density of eigenvalues $\rho(\lambda)$ for the saddle point configuration, solving

$$\int_L d\lambda \rho(\lambda) = 1, \quad \frac{1}{2} V'(\lambda) = P \int_L d\mu \frac{\rho(\mu)}{\lambda - \mu} \qquad \lambda \in L. \qquad (3.5)$$

L is the support of the distribution, which is also unknown. The first equation is the normalization of the eigenvalue density. The second is a singular integral equation for the density; the symbol P stands for Cauchy integration, where the segment $\lambda \pm \epsilon$ is avoided. Once the density is found, the planar energy and Green functions are:

$$E_0 = \int_L d\lambda V(\lambda)\rho(\lambda) - \int_L d\lambda \int_L d\mu \rho(\lambda)\rho(\mu)\log|\lambda-\mu| \tag{3.6a}$$

$$G^{pl}_{2n} = \lim_{N\to\infty} \frac{1}{N^{n+1}}\langle \mathrm{Tr} H^{2n}\rangle = \int_L d\lambda \lambda^{2n}\rho(\lambda)\,. \tag{3.6b}$$

To solve the integral equation there are various methods; the simplest is based on the analytic properties of the resolvent, and works well for a polynomial potential. Another possibility rests on a formula by Poincaré and Bertrand, concerning the exchange of two Cauchy integrals [9]. In the first method one introduces the function of complex $z \notin L$

$$F(z) = \lim_{N\to\infty} \frac{1}{N}\langle \mathrm{Tr}[(z-H)^{-1}]\rangle = \int_L d\mu \frac{\rho(\mu)}{z-\mu}\,. \tag{3.7}$$

The Cauchy integral equation for $\rho(\lambda)$ and Plemely's formulae:

$$\frac{1}{\lambda-\mu\pm i\epsilon} = P\frac{1}{\lambda-\mu} \mp i\pi\delta(\lambda-\mu) \tag{3.8}$$

imply the following boundary equations for $F(z)$, $z = \lambda \pm i\epsilon$, $\lambda \in L$:

$$F(\lambda+i\epsilon) + F(\lambda-i\epsilon) = V'(\lambda) \tag{3.9a}$$

$$F(\lambda+i\epsilon) - F(\lambda-i\epsilon) = -\,2\pi i\rho(\lambda)\,. \tag{3.9b}$$

The first equation is solved by $F(z) = (1/2)V'(z) - F_0(z)$, where $F_0(z)$ is an analytic function that changes sign across the cut L, typically a square root. To proceed, one must make an hypothesis on the support of the density $\rho(\lambda)$. Since in the quadratic case L is a segment in the real line, the "perturbative" solution corresponding to the planar graphs of the perturbative expansion, is found by taking $L = (a,b)$. In this case one obtains: $F(z) = (1/2)V'(z) - P(z)\sqrt{(z-b)(z-a)}$, where $P(z)$ is an unknown polynomial. The polynomial and the endpoints a and b are completely determined by the requirement that for large z, due to the normalization condition of the density, $F(z) = 1/z + o(1/z)$. Once $F(z)$ is found, the density is obtained through Eq. (3.9b):

$$\rho(\lambda) = \frac{1}{\pi}\sqrt{(b-\lambda)(\lambda-a)}P(\lambda)\,, \quad \rho(\lambda) = 0 \quad \lambda \notin L \tag{3.10}$$

which generalizes Wigner's semicircle law valid for the Gaussian ensembles. However, for certain values of the parameters of the potential, the above single cut solution may develop a zero in the polynomial part, for λ inside the support. This marks a transition to a new phase with a splitted support, $L_1 \cup L_2$, to ensure $\rho(\lambda) \geq 0$ on it. The multicut solutions have first been investigated by Shimamune [10]. A particular occurrence is the vanishing of the polynomial part at one edge of the support. The "edge singularity" produces a different edge behaviour of the density and is related to the radius of convergence of the perturbative expansion of the free energy.

3.2. Orthogonal polynomials

The partition function for Hermitian matrix models can be formally computed by introducing a suitable family of orthogonal polynomials, as suggested by Bessis [3]. For a polynomial potential this method is very powerful, allowing also to evaluate the $1/N$ corrections to the planar free energy.

The method exploits the invariance properties of determinants. Since $\Delta(\lambda_1 \ldots \lambda_N)$ in (3.1) is the determinant of the Vandermonde matrix $[\lambda_i^{k-1}]$, $i, k = 1 \ldots N$, one can replace its matrix elements with monic polynomials $P_{k-1}(\lambda_i) = \lambda_i^{k-1} + \ldots$ without altering the value of Δ. The N polynomials are chosen orthogonal with respect to the positive measure $d\mu(\lambda) = \exp[-NV(\lambda)]d\lambda$:

$$\int_{-\infty}^{+\infty} d\mu(\lambda) P_r(\lambda) P_s(\lambda) = h_r \delta_{rs} \,. \tag{3.12}$$

These constraints completely determine the polynomials and the positive numbers h_r, in terms of the moments of the measure. For example, $P_0(\lambda) = 1$ and $h_0 = \int d\mu(\lambda)$, $P_1(\lambda) = \lambda - \mu_1$ and $h_1 = \mu_2 h_0 - h_0 \mu_1^2$ etc., where $\mu_k = (1/h_0) \int d\mu(\lambda) \lambda^k$.

Because of the orthogonality requirement, expanding the squared determinant, one can get rid of all integrals in the partition function (3.2)

$$Z_N = CN! h_0 h_1 \ldots h_{N-1} \,. \tag{3.13}$$

For large N, to compute all coefficients h_r is certainly hopeless, and a recursive scheme must be searched. To this end, note that the orthogonality condition involves a truncated recurrence rule

$$\lambda P_r(\lambda) = P_{r+1}(\lambda) + A_r P_r(\lambda) + R_r P_{r-1}(\lambda) \,, \tag{3.14}$$

where $R_0 = 0$, and $A_0 = \mu_1$. Multiplying (3.14) by $P_{n-1}(\lambda)$ and integrating in $d\mu(\lambda)$ one gets $R_k = h_k/h_{k-1}$, positive for $k > 0$. These coefficients will play the relevant role: for example the free energy is

$$\mathrm{E}_N = -\frac{1}{N^2}\log(CN!) - \frac{1}{N}\log h_0 - \frac{1}{N}\sum_{k=1}^{N-1}\left(1-\frac{k}{N}\right)\log R_k \tag{3.15}$$

and it is suited for the continuum (planar) limit $x = k/N$, $R(x) = R_k$, $N \to \infty$.

To compute the coefficients R_k and A_k, one starts from the identities

$$kh_{k-1} = \int_{-\infty}^{\infty} d\mu(\lambda)P_{k-1}(\lambda)\frac{d}{d\lambda}P_k(\lambda)\,, \quad 0 = \int_{-\infty}^{\infty} d\lambda\frac{d}{d\lambda}[\mathrm{e}^{-NV(\lambda)}P_k^2(\lambda)]\,.$$

An integration by parts, gives the following equations, which for polynomial potentials become two finite recursive relations, with the aid of (3.14):

$$\frac{k}{N}h_{k-1} = \int_{-\infty}^{+\infty} d\mu(\lambda)V'(\lambda)P_{k-1}(\lambda)P_k(\lambda)\,, \quad 0 = \int_{-\infty}^{+\infty} d\mu(\lambda)V'(\lambda)P_k^2(\lambda)\,. \tag{3.16}$$

To solve the recursive equations one needs a finite number of initial conditions $R_0, R_1 \ldots, A_0, A_1 \ldots$, which must be calculated by hand. In the large N limit the two sets may collapse on two single values, 0 and μ_1, or may take different limit values. In the first case only two interpolating functions $R(k/N) = R_k$ and $A(k/N) = A_k$ are needed, and solve two coupled algebraic equations resulting from the continuum limit of (3.16), with boundary conditions $R(0) = 0$ and $A(0) = \mu_1$. This case corresponds to the single cut saddle point solution [11]. The other case, investigated by myself [12], requires more functions and more equations and corresponds to multicut solutions [13].

3.3. The quartic potential

To illustrate the two methods, let us solve the planar limit of the matrix model with quartic potential:

$$V(\lambda) = \tfrac{1}{2}\alpha\lambda^2 + g\lambda^4\,. \tag{3.17}$$

Since the potential is symmetric, we expect the density to be symmetric, with support $L = [-2a, 2a]$. Following Brèzin, Itzykson, Parisi and Zuber

[2] we write and expand $F(z) = \frac{1}{2}\alpha z + 2gz^3 - (2gz^2 + B)\sqrt{z^2 - 4a^2} = z(\frac{1}{2}\alpha - B + 4ga^2) + \frac{1}{z}(2a^2B + 4ga^4) + \ldots$ To meet the normalization condition $F(z) \to 1/z$, the first coefficient must be set to zero, and the second is set to one. The final result is:

$$\rho_1(\lambda) = \tfrac{1}{2\pi}\sqrt{4a^2 - \lambda^2}\,(4g\lambda^2 + \alpha + 8ga^2)\,, \quad 12ga^4 + \alpha a^2 - 1 = 0\,. \quad (3.18)$$

A positive solution a^2 exists for $(g/\alpha^2) \geq -1/48$. At this particular value, indicated as the "edge singularity", $P(\pm 2a) = 0$, so that the density $\rho(\lambda)$ vanishes at the edge of the support with an exponent different than 1/2. Setting $x = g/\alpha^2$, the planar free energy, subtracted of its $g = 0$ part, is

$$\mathrm{E}_0(x) = \tfrac{1}{24}(\alpha a^2 - 1)(9 - \alpha a^2) - \tfrac{1}{2}\log(\alpha a^2) = -\sum_{k=1}^{\infty}(-12x)^k \tfrac{(2k-1)!}{k!(k+2)!}\,. \quad (3.19)$$

By Stirling's formula we get the asymptotic behaviour of the terms in the perturbative series: $(1/2\sqrt{\pi})(-48x)^k k^{-7/2}$ where we read the finite radius $|x| < 1/48$. The same exponent 7/2 is found for cubic graphs, supporting the idea that the edge singularity is the point where to compute the continuum limit of random surfaces [5]. Brèzin *et al.* have shown in D=0,1 that planar graphs, although outnumbered by all non-planar ones, provide the main contribution to the free energy even for N=1. This fact supports the 1/N expansion as a good nonperturbative approximation. We are however concerned in the 1/N expansion as the tool for generating planar graphs. For $\alpha < 0$ the potential is double-well, and if $g/\alpha^2 = 1/16$ the density (3.18) vanishes at $\lambda = 0$. Beyond the critical value 1/16 one has to consider a double cut solution on $L = [-B, -A] \cup [A, B]$

$$\rho_2(\lambda) = \frac{1}{\pi}2g|\lambda|\sqrt{(B^2 - \lambda^2)(\lambda^2 - A^2)}\,,\ B^2 = -\frac{\alpha}{4g} + \frac{1}{\sqrt{g}}\,,\ A^2 = -\frac{\alpha}{4g} - \frac{1}{\sqrt{g}}\,. \quad (3.20)$$

Nonsymmetric solutions with one or two cuts are also allowed, but they are non-unique.

Let us illustrate the solution with orthogonal polynomials [3]. Since the potential is even, the polynomials have definite parity and $A_r = 0$. Using repeatedly (3.14) to remove λ factors coming from the potential, the first equation in (3.16) gives

$$\tfrac{k}{N} = R_k[\alpha + 4g(R_{k+1} + R_k + R_{k-1})] \quad (3.21)$$

with the initial conditions $R_0 = 0$ and $R_1 = \mu_2$. For $\alpha > 0$ and large N, a saddle point integration gives $\mu_2 = (2\alpha N)^{-1}$, so that it is reasonable to interpolate all R_k with a single function $R(x)$, with $x = k/N$ between 0 and 1. The continuum limit equation is

$$x = \alpha R(x) + 12gR^2(x)\,, \quad R(0) = 0 \quad (3.22)$$

and the planar free energy is

$$E_0 = -\int_0^1 dx(1-x)\log R(x)\,. \tag{3.23}$$

In the case of double well potential ($\alpha \le 0$), $R_1 = -2g/\alpha$ is finite, and the interpolation requires two distinct functions [12], $R_0(2k/N) = R_{2k}$ and $R_1(2k+1/N) = R_{2k+1}$, with $R_0(0) = 0$ and $R_1(0) = R_1$. The solution corresponds to the symmetric two-cut density.

3.4. Beyond the one-matrix problem

Models with two or more coupled matrices are in general very difficult, since they involve integrations over the angular SU(N) variables. An exception is provided by the following models with p Hermitian matrices [14]

$$Z_{N,p}(c) = \int \prod_{i=1}^{p} [dH_i] e^{-\sum_{i=1}^{p} \mathrm{Tr}\, V_i(H_i) - 2c \sum_{i=1}^{p-1} \mathrm{Tr}(H_i H_{i+1})} \tag{3.24}$$

which can be reduced to integrals on eigenvalues by means of a remarkable formula by Harish–Chandra [11][15]

$$\int [dU] e^{\beta \mathrm{Tr}[AUBU^*]} = \beta^{-\frac{N(N-1)}{2}} \left(\prod_{k=1}^{N-1} k! \right) \frac{\det[\exp \beta a_i b_j]}{\Delta(a)\Delta(b)}\,, \tag{3.25}$$

where A and B are Hermitian matrices with eigenvalues $\{a_i\}$ and $\{b_i\}$.

4. Ising models on random lattices

In the standard Ising model, a certain 2D lattice $\mathcal{G}$ is given, with a finite number of sites V, connected in a definite way by bonds $\langle ij\rangle$. A spin variable $\sigma_i = \pm 1$ is located on each site, giving rise to 2^V different spin configurations of the lattice. Each bond $\langle ij\rangle$ carries a spin-spin interaction energy proportional to $\sigma_i\sigma_j$. Including a uniform magnetic field H, the partition function is

$$Z(\mathcal{G}) = \sum_{\{\sigma\}} \exp\left(\frac{\beta}{2} \sum_{\langle ij\rangle} \sigma_i\sigma_j + H \sum_{i}^{V} \sigma_i \right)\,. \tag{4.1}$$

The computation of the partition function is, in general, an extremely difficult task. The above planar lattice, however, can be thought as the Feynman graph of the V-th order perturbative expansion of some matrix model, for example with quartic potential if the lattice has coordination four. It is much simpler to compute exactly a partition function where a sum over all planar lattices $\mathcal{G}_V$ with V quartic vertices is performed, each weighted by its combinatorial factor $C(\mathcal{G}_V)$. As Kazakov has shown in the case of zero magnetic field [16], and then Boulatov and Kazakov [4] for $H \neq 0$, matrix models can provide the exact evaluation of

$$\mathrm{Z}_V(\beta, H) = \sum_{\mathcal{G}_V} C(\mathcal{G}_V)\mathrm{Z}(\mathcal{G}_V) . \tag{4.2}$$

For coordination four, this partition function can be calculated exactly from the solvable quartic two matrix model

$$\mathrm{Z}_N = \int [dM_1][dM_2] \exp -\mathrm{Tr}\left(M_1^2 + M_2^2 - 2cM_1M_2 + \tfrac{g}{N}e^H M_1^4 + \tfrac{g}{N}e^{-H}M_2^4\right) , \tag{4.3}$$

where M_1 and M_2 are Hermitian $N \times N$ matrices, H is a real parameter, $0 < c < 1$ and $g \geq 0$. The planar free energy of the model $\mathrm{E}_0(c, g, H)$ is the sum of all connected quartic graphs. At the perturbative order V in the g expansion, we meet graphs with $V = V_1 + V_2$ vertices, where V_1 are "spin up" vertices generated by $\mathrm{Tr}M_1^4$ giving a factor ge^H, and V_2 are "spin down" vertices generated by $\mathrm{Tr}M_2^4$ with a factor ge^{-H}. These vertices are connected by p and $\tilde{p}$ double line propagators with weights

$$\langle\uparrow\uparrow\rangle = \langle\downarrow\downarrow\rangle = \langle\frac{1}{N}\mathrm{Tr}(M_1^2)\rangle = \langle\frac{1}{N}\mathrm{Tr}(M_2^2)\rangle = \frac{1}{2}\frac{1}{1-c^2} , \tag{4.4a}$$

$$\langle\uparrow\downarrow\rangle = \langle\downarrow\uparrow\rangle = \langle\frac{1}{N}\mathrm{Tr}(M_1M_2)\rangle = \frac{1}{2}\frac{c}{1-c^2} . \tag{4.4b}$$

Up to the combinatoric and the planar N^2 factors, the weight of the graph with $V = V_1 + V_2$ vertices and $p + \tilde{p} = 2V$ propagators, is

$$\begin{aligned} & g^V e^{H(V_1 - V_2)} \left(\frac{1}{2(1-c^2)}\right)^p \left(\frac{c}{2(1-c^2)}\right)^{\tilde{p}} \\ &= \left[\frac{gc}{4(1-c^2)^2}\right]^V e^{H(V_1-V_2)} \left(\frac{1}{\sqrt{c}}\right)^p (\sqrt{c})^{\tilde{p}} . \end{aligned} \tag{4.5}$$

The perturbative expansion of the planar free energy is

$$\mathrm{E}_0(g, c, H) = \sum_V \left[\frac{gc}{(1-c^2)^2}\right]^V \mathrm{F}_V(c, H) , \tag{4.6}$$

where $F_V(c, H)$ coincides with $Z_V(\beta, H)$ with the identification $c = \exp(-\beta)$, and H the magnetic field in both cases.

By means of the Harish–Chandra formula, the partition function of the two matrix model can be written in terms of eigenvalues:

$$Z_N = \int \prod_{i=1}^{N} d\lambda_i d\mu_i \Delta(\lambda)\Delta(\mu) e^{-N\sum_{i=1}^{N} W(\lambda_i,\mu_i)}, \tag{4.7}$$

where $W(\lambda, \mu) = \lambda^2 + \mu^2 - 2c\lambda\mu + ge^H\lambda^4 + ge^{-H}\mu^4$. Following Mehta [15] and Boulatov–Kazakov [4], one defines two sets of monic orthogonal polynomials,

$$\int d\lambda d\mu e^{-NW(\lambda,\mu)} P_i(\lambda) Q_j(\mu) = h_i \delta_{ij} \tag{4.8}$$

that allow all integrals in the partition function to be done. Assuming that the relevant ratio $R_k = (6g/c)h_k/h_{k-1}$ has a continuous limit $R(x)$, with $x = k/N$, one obtains the planar energy

$$E_0(g, c, H) = \tfrac{1}{2}\log(6g/c) - \int_0^1 dx(1-x)\log R(x) \tag{4.9}$$

and a fifth order equation for $R(x)$, resulting from several equations for the recurrence coefficients of orthogonal polynomials

$$gx = \frac{c^2}{9}R^3 - \frac{c^2}{3}R + \frac{1}{3}\frac{R}{(1-R)^2} + \frac{1}{3}\frac{R^2}{(1-R^2)^2}B, \quad \mathrm{B} = 2\mathrm{ch}H - 2. \tag{4.10}$$

To study the critical behaviour one is interested in the thermodynamic limit, given by the large V behaviour of $F_V(c, H)$ which can be found by studying the radius of convergence of the perturbative series (4.6) of $E_0(g, c, H)$:

$$F_V(c, H) \approx \left(\frac{cg(c)}{(1-c^2)^2}\right)^{-V}, \tag{4.11}$$

where $g(c)$ is obtained from the conditions $x'(R) = 0$, $x(R) = 1$. Quoting only results, for a weak field H two different critical relations $g = g(c)$ are found, corresponding to a phase with spontaneous magnetization for $0 < c < 1/4$, and a high temperature phase $1/4 < c < 1$. The critical exponents are different from those of the Onsager solution (in parenthesis), and coincide with those of the 3D spherical model: $\beta = 1/2$ $(1/8)$, $\gamma = 2$ $(7/4)$ and $\delta = 5$ (15). Widom's scaling relation $\gamma = \beta(\delta - 1)$ is satisfied.

With cubic planar graphs, the same critical exponents were obtained. This supports the hypothesis of universality, or independence in the limit of random surfaces, from the discretization procedure. Matrix models, therefore, appear as a good route to study models on random surfaces.

The Ising model on the dual lattices was investigated by Johnston [17], and the critical points were found to obey the usual duality relation. A generalization to a q-state Potts model on random planar lattices, corresponding to q-matrix model, was studied by Kazakov [18].

5. The $O(n)$ model on cubic graphs

The Boltzmann weight for the Ising model with $H = 0$ can be rewritten as $\mathrm{ch}\,\beta^V \prod_i (1 + \mathrm{th}\,\beta\sigma_i\sigma_j)$, the starting point for a high temperature expansion. A generalization is provided by the $O(n)$ model, where the site variable is a real n-component vector $\vec{S}_i$, of fixed length n

$$Z_{\mathcal{G}} = C \int \prod_i d^n S_i \prod_{\langle ij \rangle} \left(1 + K \vec{S}_i \cdot \vec{S}_j\right) . \tag{5.1}$$

C is a normalization constant for $K = 0$. The Ising model corresponds to the case $n = 1$. The critical behaviour of the $O(n)$ model on the regular honeycomb lattice was explored by Nienhuis [19]. The same partition function (5.1) also works for a generic cubic graph $\mathcal{G}$. In any case, expanding the product only integrations with even powers of a site variable contribute, so that the product is restricted to closed nonintersecting paths on the lattice. Equivalently, we may think to replace the graph with a collection of replicas where some links are "coloured" to form closed nonintersecting loops. Since there are n colours, each loop yields a factor n, and each visited site gives a factor K. The result is a counting problem of self-avoiding closed random walks on $\mathcal{G}$:

$$Z_{\mathcal{G}} = \sum_L n^{|L|} K^{V_1} , \tag{5.2}$$

where $|L|$ is the total number of loops on the replica and V_1 is their total length. As Kostov [20] has shown, if the planar graphs $\mathcal{G}$ are allowed to vary in the class of cubic graphs, with any number V of vertices, and the partition function extended to

$$Z_{O(n)} = \sum_{\mathcal{G}} C(\mathcal{G}) e^{-\beta V} Z_{\mathcal{G}} \tag{5.3}$$

one obtains a statistical model that can be described exactly by the planar free energy of a matrix model with cubic couplings

$$Z_N(z,g) = \int [dM] \prod_{i=1}^{n} [d\Phi_i]$$

$$\exp - \mathrm{Tr}\left(\frac{1}{2}M^2 - \frac{g}{\sqrt{N}}M^3 + \frac{1}{2}\sum_{i=1}^{n}\Phi_i^2 - \frac{z}{\sqrt{N}}\sum_{i=1}^{n} M\Phi_i^2\right), \qquad (5.4)$$

where M and Φ_i, $i = 1 \ldots n$, are Hermitian $N \times N$ matrices. In the large N limit it generates planar graphs with cubic vertices of weight z, where a line of the field M interacts with two "coloured" lines of type Φ, and cubic vertices of three M lines, with weight g. Each graph can then be viewed as a configuration of the $O(n)$ model: if V_1 and V_2 are the two types of vertices, with $V_1 + V_2 = V$, and $|L|$ is the number of loops with lines Φ, the graph contributes a factor $z^{V_1} g^{V-V_1} n^{|L|}$. Comparison with formula (5.2) gives the identification $g = e^{-\beta}$, $g^{-1}z = K$.

The Gaussian integrals on the n matrices Φ_i can be done, and one remains with an integral involving only the eigenvalues of M, with a nonlocal interaction

$$Z_N = \int \prod_{j=1}^{N} d\lambda_j \Delta^2(\lambda) \exp{-N}\left[\sum_{i=1}^{N}\left(\frac{1}{2}\lambda_i^2 - g\lambda_i^3\right) + \frac{n}{2}\sum_{ij}\log\left(\frac{1}{z} - \lambda_i - \lambda_j\right)\right]. \qquad (5.5)$$

The saddle point equation for the eigenvalue distribution could not be solved by the standard technique, and was translated by Gaudin and Kostov into a Wiener–Hopf problem [21]. The model has an interesting critical behaviour, that is described in detail in [22]. We are currently investigating the model (5.4) with n complex matrices, replacing Φ^2 with $\Phi^\dagger\Phi$. Loops are now charged, and in the large N limit are empty of lines and have a definite orientation: this leads to the interpretation of a statistical models of random surfaces with holes.

6. A colouring problem

The graphical interpretation of the large N limit of matrix models, makes them an interesting tool for the theory of graphs itself. The most famous problem in this context, which actually was the source of the theory of graphs, is the Four-colour conjecture, by Guthrie in 1852. It states that any planar map can be coloured with four colours in such a way that countries with a common border always have different colours. For higher genus maps the number is greater, and found by Heawood at the turn of the century. Apart from a long computer proof (Haken and Appel, 1976), no analytical proof of the conjecture for planar maps exists. It has been shown that a proof for maps with only cubic vertices is sufficient, and that the four-colouring of regions is in correspondence with the three-colouring of bonds of the same map. We therefore introduced the following three-matrix model [23]

$$Z_N(g_1, g_2) = \int [dA][dB][dC] e^{-\frac{1}{2}\mathrm{Tr}(A^2+B^2+C^2) - \frac{g_1}{\sqrt{N}}\mathrm{Tr}(ABC) - \frac{g_2}{\sqrt{N}}\mathrm{Tr}(ACB)} . \qquad (6.1)$$

When $g_1 = g_2$ the model describes cubic graphs with bonds of colour A, B and C joining in each vertex. To show that all graphs of this theory also appear in the graph expansion of the one-matrix cubic model [2], which generates all cubic graphs, means that any cubic graph is three-colourable, thus proving the conjecture. We were not able to do so much, but investigated the critical behaviour of the model, which has similarities with an $O(n)$ model. Integrating over one matrix one obtains

$$Z_N(g_1, g_2) = \int [dA][dB] e^{-\frac{1}{2}\mathrm{Tr}(A^2+B^2) + \frac{1}{N}(g_1+g_2)\mathrm{Tr}(ABAB) + \frac{1}{N} g_1 g_2 \mathrm{Tr}(A^2B^2)} \qquad (6.2)$$

with quartic vertices where lines of colour A and B cross and touch. Putting $g_1 + g_2 = 0$, the model describes a gas of self-avoiding loops with similar properties of the dense phase of the $O(n)$ model on random graphs.

The author is grateful to the organizers of the Symposium in Statistical Mechanics in Zakopane and to Dr. Karol Życzkowski for their kind hospitality.

REFERENCES

Note: there are hundreds of papers in matrix models, often very difficult for people non introduced in string, conformal, or quantum gravity theories; I here restrict to a few titles of accessible papers.

[1] G. 't Hooft, *Nucl. Phys.* **B72**, 461 (1974).
[2] E. Brézin, C. Itzykson, G. Parisi, J.B. Zuber, *Commun. Math. Phys.* **59**, 35 (1978).
[3] D. Bessis, *Commun. Math. Phys.* **69**, 147 (1979).
[4] D.V. Boulatov, V.A. Kazakov, *Phys. Lett.* **B186**, 379 (1987).
[5] F. David, *Nucl.Phys.* **B257** [FS14], 45 (1985).
[6] V.G. Knizhnik, A.M. Polyakov, A.B. Zamolodchikov, *Mod. Phys. Lett.* **A3**, 819 (1988).
[7] G.M. Cicuta, L. Molinari, E. Montaldi, *J. Math. Phys.* **28**, 1716 (1987).

[8] M.L. Mehta *Random Matrices*, 2nd ed. Academic Press.
[9] N.I. Muskhelishvili, *Singular Integral Equations*, Dover (reprint of 1953 ed.).
[10] Y. Shimamune, *Phys. Lett.* **B108**, 407 (1982).
[11] C. Itzykson, J.B. Zuber, *J. Math. Phys.* **21**, 411 (1980).
[12] L. Molinari, *J. Phys.* **A21**, 1 (1988).
[13] R.C. Brower, N. Deo, S. Jain, C.I. Tan, *Nucl. Phys.* **B405**, 166 (1993).
[14] S. Chadha, G. Mahoux, M.L. Mehta, *J. Phys.A* **14**, 579 (1981).
[15] M.L. Mehta, *Commun. Math. Phys.* **79**, 327 (1981).
[16] V.A. Kazakov, *Phys. Lett.* **A119**, 140 (1986).
[17] D.A. Johnston, *Phys. Lett.* **B314**, 69 (1993).
[18] V.A. Kazakov, *Nucl. Phys.* **B** (Proc. Suppl.) **4**, 94 (1988).
[19] B. Nienhuis, *Phys. Rev. Lett.* **49**, 1062 (1982).
[20] I.K. Kostov, *Mod. Phys. Lett.* **A4**, 217 (1989).
[21] M. Gaudin, I. Kostov, *Phys. Lett.* **B220**, 200 (1989).
[22] B. Duplantier, I.K. Kostov, *Nucl. Phys.* **B340**, 491 (1990).
[23] G.M. Cicuta, L. Molinari, E. Montaldi, *Phys. Lett.* **B306**, 245 (1993).

Recent Review papers on matrix models:

— L.Alvarez-Gaumé, *Random Surfaces, Statistical Mechanics and String Theory*, *Helvetica Physica Acta* **64**, 359–526 (1991);
— *Two Dimensional Quantum Gravity and Random Surfaces*, 8th Jerusalem Winter School for Theor. Phys., World Scientific, Singapore 1992;
— Y.Lozano, J.L.Manes, *Introduction to Nonperturbative 2D Quantum Gravity*, *Fortschr. Phys.* **41**, 45–85 (1993).