ON THE "CLASSICAL" LIMIT OF SOME q-COMMUTATION RELATIONS

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Two types of operators (I, II) are constructed in terms of bosonic quantum (q-)oscillator operators. These operators satisfy a relation analogous to the q-commutation relation for fermionic quantum (q-)oscillators. Both types of operators (I, II) can be identified with the fermionic harmonic oscillator in the limit $q \to 1$ by exploiting the presence of certain arbitrary functions of the deformation parameter q. For I they may be identified as parameters of U(1) transformations while for II they correspond to parameters of GL(2,R) transformations. Appropriate fermionic number operators in the limit $q \to 1$ are also constructed.

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1. Introduction

Quantum (q-) oscillators [1, 10–15] provide a rich arena for testing many of the basic theoretical tools of quantum physics. They are inextricably related to quantum groups which have given new insight in quantum inverse scattering problems [2], quantum deformations of Lie algebras [3], quantum field theory and statistical mechanics [4, 5].

On the other hand, quantum oscillators promise interesting physical applications [6–9]. The motive in this paper is to investigate the possibilities of q-bosonization as an analogue of the usual bosonization. Certain ideas related to q-bosonization have been discussed in [14]. Here the perspective is different *viz.* transformations resembling canonical q-transformations [12]. We utilise an algebraic method to construct two types of operators (I and II) in terms of bosonic q-oscillators. These new operators satisfy a relation similar to the q-commutation relation for fermionic q-oscillators. For type I operators, nilpotency can be obtained by choosing two arbitrary functions of q which occur in the solutions to be equal. But a suitable q-fermionic number operator cannot be constructed. For type II operators, nilpotency cannot be obtained for any realistic choice of arbitrary functions which are present. Neither can one construct a q-fermionic number operator.

However, in the "classical" limit (*i.e.* when the deformation parameter $q \to 1$) both types of operators (I and II) can be made to satisfy the usual fermionic harmonic oscillator anticommutation relations including the nilpotency conditions by suitably choosing the behaviour of the arbitrary functions as $q \to 1$. For operators of type I the arbitrary phase factor may be interpreted as a parameter of U(1) transformations of these operators while for those of type II these functions correspond to parameters of $GL(2,R)$ transformations. Appropriate number operators in the limit $q \to 1$ can also be constructed.

In Section 2 the first (I) type of the abovementioned operators are constructed. Section 3 describes operators of the second (II) type. In Section 4 the origin of the various arbitrary functions and phase factors occurring in the solutions are discussed. We summarize our conclusions in Section 5.

2. Operators of type I

The equations characterizing the q-deformed bosonic oscillator system are

$$aa^\dagger - qa^\dagger a = q^{-N}, \quad N^\dagger = N, \tag{1}$$

$$Na = a(N-1), \quad Na^\dagger = a^\dagger(N+1). \tag{2}$$

$a, a^\dagger$ and N are the annihilation, creation and number operators, respectively, $q = \mathrm{e}^s$ is the deformation parameter and we take it to be real. Standard bosonic oscillators are described by

$$\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1, \quad \widehat{N} = \hat{a}^\dagger\hat{a} = \widehat{N}^\dagger, \tag{3}$$

$$\widehat{N}\hat{a} = \hat{a}(\widehat{N}-1), \quad \widehat{N}\hat{a}^\dagger = \hat{a}^\dagger(\widehat{N}+1). \tag{4}$$

Fermionic q-oscillators are described by the equations

$$bb^\dagger + qb^\dagger b = q^M, \quad b^2 = (b^\dagger)^2 = 0, \tag{5}$$

$$M^\dagger = M = M^2, \quad Mb = b(M-1), \quad Mb^\dagger = b^\dagger(M+1), \tag{6}$$

where $b, b^\dagger$ and M correspond to annihilation, creation and number operators, respectively. The fermionic harmonic oscillator is defined through the relations

$$\hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b} = 1, \quad \hat{b}^2 = (\hat{b}^\dagger)^2 = 0, \tag{7}$$

$$\widehat{M} = \hat{b}^\dagger\hat{b} = \widehat{M}^\dagger = \widehat{M}^2, \quad \widehat{M}\hat{b} = \hat{b}(\widehat{M}-1), \quad \widehat{M}\hat{b}^\dagger = \hat{b}^\dagger(\widehat{M}+1). \tag{8}$$

We shall also utilise the relations:

$$[N,\ \widehat{N}] = [M,\ \widehat{M}] = [N,\ M] = [N,\ \widehat{M}] = [\widehat{N},\ \widehat{M}] = 0\,. \tag{9}$$

It is known that the operator N is a function of q and $\widehat{N}$ [1]. Therefore they commute. The case of M and $\widehat{M}$ is similar [15]. From usual quantum mechanics it is known that $\widehat{N}$ and $\widehat{M}$ commute. Therefore, it is plausible to assume that $[N,\ M] = [N,\ \widehat{M}] = 0$. The expression (9) is therefore justified. Let us transform to new operators $\tilde{b}$ and $\tilde{b}^\dagger$:

$$\tilde{b} = \hat{a}u(\widehat{N})\,, \quad \tilde{b}^\dagger = u^*(\widehat{N})\hat{a}^\dagger\,, \tag{10}$$

where u is a function of ($\widehat{N}$ and q) to be subsequently determined, and $*$ denotes complex conjugation. We further demand that

$$\tilde{b}\tilde{b}^\dagger + q\tilde{b}^\dagger\tilde{b} = q^{\widetilde{M}}, \tag{11a}$$

$$\widetilde{M}^\dagger = \widetilde{M}\,, \tag{11b}$$

$$[N,\ \widehat{N}] = [\widetilde{M},\ \widehat{M}] = [N,\ \widetilde{M}] = [N,\ \widehat{M}] = [\widehat{N},\ \widehat{M}] = 0\,, \tag{11c}$$

the operator $\widetilde{M}$ being assumed to be some function of $\widehat{N}$ and q *viz.* $\phi(q, \widehat{N})$. Our assumption is based on the relations (11c) and (10) and the discussion following equation (9).

Substituting the transformations (10) in Eq. (11a) and using (3) and (4) we have

$$U(\widehat{N}+1) + qU(\widehat{N}) = q^{\widetilde{M}} = q^{\phi(q,\widehat{N})}, \tag{12}$$

where

$$U(\widehat{N}) = \widehat{N}u^*(\widehat{N})u(\widehat{N})\,. \tag{13}$$

The assumption $\widetilde{M} = \phi(q, \widehat{N})$ combined with Eq. (4) gives

$$q^{\widetilde{M}}\hat{a} \;= \hat{a}q^{\widetilde{M}-1}, \tag{14a}$$

$$q^{\widetilde{M}}\hat{a}^\dagger = \hat{a}^\dagger q^{\widetilde{M}+1}. \tag{14b}$$

Putting Eq. (12) in either of Eqs (14) leads to

$$U(\widehat{N}) = q^2U(\widehat{N}-2)\,. \tag{15}$$

The solution of Eq. (15) is found to be

$$U(q, \widehat{N}) = q^{\widehat{N}}X(q) + q^{-\widehat{N}}Y(q)\,, \tag{16}$$

where the arbitrary functions $X(q)$ and $Y(q)$ are given by

$$X(q) = \tfrac{1}{2} q^{\phi(q,0)-1}, \tag{17a}$$

$$Y(q) = U(q,0) - X(q). \tag{17b}$$

Finite difference equations are analogous to the discrete versions of ordinary differential equations. Hence to solve them one requires the specification of initial conditions. The two arbitrary functions $X(q)$ and $Y(q)$ encode the two initial conditions $u(q,0)$ and $\phi(q,0)$. (11a) as embodied in (12) is insensitive to the choice of these initial conditions.

It should be noted here that if $U(\widehat{N})$ is a solution of (15) then $U(-\widehat{N})$ is *not* a solution. Using the definitions (13) we thus arrive at

$$\begin{aligned} \tilde{b} &= \hat{a} \left(\frac{q^{\widehat{N}} X(q) + (-q)^{\widehat{N}} Y(q)}{\widehat{N}} \right)^{1/2} \exp\left(i\nu(q,\widehat{N}) \right), \\ \tilde{b}^\dagger &= \left(\frac{q^{\widehat{N}} X(q) + (-q)^{\widehat{N}} Y(q)}{\widehat{N}} \right)^{1/2} \exp\left(-i\nu(q,\widehat{N}) \right) \hat{a}^\dagger, \end{aligned} \tag{18}$$

where $\nu(q,\widehat{N})$ is an arbitrary phase factor.

The relations (18) satisfy (11a) with the identification

$$\widetilde{M} = \widehat{N} + 1 + \frac{1}{s} \ln(2X). \tag{19}$$

The squares of the operators $\tilde{b}$ and $\tilde{b}^\dagger$ are

$$(\tilde{b})^2 = \left(\tilde{b}^\dagger\right)^2 = 0 \quad \text{for} \quad X(q) = Y(q). \tag{20}$$

Hence for $X(q) = Y(q)$, the operators $\tilde{b}$, $\tilde{b}^\dagger$ are nilpotent. However, the operator $\widetilde{M}$ *cannot* be identified with any q-fermionic number operator as it is evident from (19) that $\widetilde{M}^2 \neq \widetilde{M}$ for any $X(q)$ because $X(q)$ is always restricted to be a function of q only.

In the "classical" limit $q \to 1$

$$\begin{aligned} \tilde{b} &\to b_1 = \hat{a} \left(\frac{X(1) + (-1)^{\widehat{N}} Y(1)}{\widehat{N}} \right)^{1/2} \exp\left(i\nu(1,\widehat{N}) \right), \\ \tilde{b}^\dagger &\to b_1^\dagger = \left(\frac{X(1) + (-1)^{\widehat{N}} Y(1)}{\widehat{N}} \right)^{1/2} \exp\left(-i\nu(1,\widehat{N}) \right) \hat{a}^\dagger. \end{aligned} \tag{21}$$

Then one has

$$b_1 b_1^\dagger + b_1^\dagger b_1 = 1 \quad \text{for} \quad X(1) = Y(1) = \tfrac{1}{2}\,, \tag{22a}$$

$$(b_1)^2 = 0 \quad \text{for} \quad X(1) = Y(1)\,, \tag{22b}$$

$$\left(b_1^\dagger\right)^2 = 0 \quad \text{for} \quad X(1) = Y(1)\,. \tag{22c}$$

The number operator in this limit $q \to 1$ *should not be obtained* from $\lim_{q\to 1} \widetilde{M}$ because we have already shown that $\widetilde{M}$ is never a q-fermionic number operator. Rather, the correct number operator is

$$M_1 = b_1^\dagger b_1$$

i.e.

$$M_1 = b_1^\dagger b_1 = \tfrac{1}{2}\left(1 + (-1)^{\widehat{N}}\right) \quad \text{for} \quad X(1) = Y(1) = \tfrac{1}{2} \tag{23}$$

and it can be easily verified that $M_1^2 = M_1$.

The number operator *by definition* should also satisfy the following

$$[M_1, b_1] = -b_1\,, \quad [M_1,\, b_1^\dagger] = b_1^\dagger\,. \tag{24}$$

This is verified as follows:

$$\begin{aligned}
&[M_1, b_1] \\
&= \tfrac{1}{2}\left(\left(1 + (-1)^{\widehat{N}}\right)\hat{a} - \hat{a}\left(1 + (-1)^{\widehat{N}}\right)\right)\left(\frac{M_1}{\widehat{N}}\right)^{1/2} \exp\left(i\nu(1,\widehat{N})\right) \\
&= \tfrac{1}{2}\left(\hat{a}\left(1 - (-1)^{\widehat{N}}\right) - \hat{a}\left(1 + (-1)^{\widehat{N}}\right)\right)\left(\frac{M_1}{\widehat{N}}\right)^{1/2} \exp\left(i\nu(1,\widehat{N})\right) \\
&= -b_1\,.
\end{aligned}$$

Here we have used

$$\left(1 - (-1)^{\widehat{N}}\right)\left(1 + (-1)^{\widehat{N}}\right) = 0$$

and

$$\left(\tfrac{1}{2}\left(1 \pm (-1)^{\widehat{N}}\right)\right)^2 = \tfrac{1}{2}\left(1 \pm (-1)^{\widehat{N}}\right)\,.$$

3. Operators of type II

We now consider the following new transformations for the operators $\tilde{b}$ and $\tilde{b}^\dagger$ satisfying Eq. (11a):

$$\tilde{b} = \hat{a} f(\hat{N}) + g(\hat{N})\hat{a}^\dagger , \quad (25a)$$

$$\tilde{b}^\dagger = f^*(\hat{N})\hat{a}^\dagger + \hat{a} g^*(\hat{N}) . \quad (25b)$$

f and g are functions to be determined, $*$ denotes complex conjugation and Eqs (11b) and (11c) are still valid. We now show that consistent solutions exist to the present problem under the assumption that $\widetilde{M}$ *is some periodic function of* $\hat{N}$ *i.e.* $\widetilde{M} = \phi(q, \hat{N})$, *where* $\phi(q, \hat{N}) = \phi(q, \hat{N}+1)$. Hence, in so far as commuting operators through $\widetilde{M}$ are concerned, $\widetilde{M}$ behaves as a constant ("operator") term.

With these new expressions (25) for $\tilde{b}$, $\tilde{b}^\dagger$ in (11a) and proceeding as described in Section 2 one arrives at

$$F(\hat{N}+1) + qF(\hat{N}) + G(\hat{N}) + qG(\hat{N}+1) = q^{\widetilde{M}} , \quad (26)$$

$$f(\hat{N})g^*(\hat{N}+1) = -qg^*(\hat{N})f(\hat{N}+1) , \quad (27)$$

$$f^*(\hat{N})g(\hat{N}+1) = -qg(\hat{N})f^*(\hat{N}+1) , \quad (28)$$

where

$$F(\hat{N}) = \hat{N} f^*(\hat{N}) f(\hat{N}) , \quad G(\hat{N}) = \hat{N} g^*(\hat{N}) g(\hat{N}) . \quad (29)$$

As $\widetilde{M}$ is hermitian $q^{\widetilde{M}}$ is also hermitian since q is real. The requirement that all non-hermitian quantities vanish gives the relations (27) and (28). Moreover, in view of (11c) and the assumption that $\widetilde{M}$ is some periodic function of $\hat{N}$, $q^{\widetilde{M}}$ is like some "constant" (operator) term.

Multiplying (27) and (28) gives

$$G(\hat{N}+1)F(\hat{N}) = q^2 G(\hat{N})F(\hat{N}+1) \quad (30)$$

whose solution is

$$G(\hat{N}) = q^{2\hat{N}} W(q, \hat{N}) F(\hat{N}) . \quad (31)$$

Here the arbitrary function of q $W(q, \hat{N}) = W(q, \hat{N}+1)$ and belongs to what we call the class **P** of the periodic functions [11, 12]. Considering the feature:

$$[W, \hat{a}] = [W, \hat{a}^\dagger] = [W, \hat{N}] = [W, \hat{b}] = [W, \hat{b}^\dagger] = [W, \widetilde{M}] = 0 ,$$

we can take W without loss of generality to be a function of q only [11, 12].

Substituting (31) in (26) we get

$$F(\widehat{N}+1)\left\{1+q^{2(\widehat{N}+2)}\widetilde{W}\right\} = q^{\widetilde{M}} - qF(\widehat{N})\left\{1+q^{2\widehat{N}}\widetilde{W}\right\}, \quad (32)$$

where $\widetilde{W} = q^{-1}W$. To solve the functional Eq. (32) we first determine the solution $F_0(\widehat{N})$ to the corresponding homogeneous equation

$$F_0(\widehat{N}+1)\left\{1+q^{2(\widehat{N}+2)}\widetilde{W}\right\} = (-q)F_0(\widehat{N})\left\{1+q^{2\widehat{N}}\widetilde{W}\right\}. \quad (33)$$

Following the method elaborately described in Ref. [12] $F_0(\widehat{N})$ is found to be

$$F_0(\widehat{N}) = Q(\widehat{N})Q(\widehat{N}+1)P(\widehat{N},q), \quad (34)$$

$$\bar{Q}(\widehat{N}) = \frac{(-q)^{\widehat{N}/2}}{1+q^{2\widehat{N}}\widetilde{W}}, \quad (35)$$

where $P(\widehat{N},q)$ is an arbitrary **P** function. For a suitable choice of initial conditions the general solution of (32) is independent of $P(\widehat{N},q)$.

The general solution of (32) may be given as

$$F(\widehat{N}) = F_0(\widehat{N})Z(\widehat{N}), \quad (36)$$

where now $Z(\widehat{N})$ has to be determined. Solving for $Z(\widehat{N},q)$ the final solutions are (for $Z(0,q)=0$)

$$F(\widehat{N}) = \frac{q^{\widetilde{M}}\left\{1+Wq^{2\widehat{N}}+(-1)^{\widehat{N}+1}q^{\widehat{N}}(1+W)\right\}}{(1+q)\left(1+q^{2\widehat{N}-1}W\right)\left(1+q^{2\widehat{N}+1}W\right)}, \quad (37a)$$

$$G(\widehat{N}) = q^{2\widehat{N}}WF(\widehat{N}), \quad (37b)$$

where $\widetilde{M}$ is some periodic function of $\widehat{N}$. The above solutions are independent of P as promised. However, the dependence on W is non-trivial. We will say more about W in Section 4. It is easily verified that the solutions (37) satisfy the fundamental relation (32).

Using the definition (29) we have

$$f(\widehat{N}) = \left|f(\widehat{N})\right| \exp(i\alpha),$$

$$g(\widehat{N}) = q^{\widehat{N}}W^{1/2}\left|f(\widehat{N})\right| \exp(i\beta), \quad (38)$$

where

$$\mathbf{|}f(\widehat{N})\mathbf{|} = \left\{ \frac{F(\widehat{N})}{\widehat{N}} \right\}^{1/2}$$

with $F(\widehat{N})$ given in equation (37a). Here $\alpha(q, \widehat{N})$ and $\beta(q, \widehat{N})$ are some arbitrary phase factors subject to the condition that

$$\exp\left(i\{\alpha(\widehat{N}) - \beta(\widehat{N}+1)\}\right) = \exp\left(i\{\alpha(\widehat{N}+1) - \beta(\widehat{N}) \pm (2m+1)\pi\}\right), \tag{39}$$

where $m = 0, 1, 2, \ldots$. This condition guarantees the validity of Eqs (27) and (28).

We can therefore write the transformations (25) as

$$\begin{aligned} \tilde{b} &= \mathbf{|}f(\widehat{N}+1)\mathbf{|} \exp\left(i\alpha(\widehat{N}+1)\right) \hat{a} + q^{\widehat{N}} W^{1/2} \mathbf{|}f(\widehat{N})\mathbf{|} \exp\left(i\beta(\widehat{N})\right) \hat{a}^\dagger, \\ \tilde{b}^\dagger &= \mathbf{|}f(\widehat{N})\mathbf{|} \exp\left(-i\alpha(\widehat{N})\right) \hat{a}^\dagger \\ &\quad + q^{\widehat{N}+1} W^{1/2} \mathbf{|}f(\widehat{N}+1)\mathbf{|} \exp\left(-i\beta(\widehat{N}+1)\right) \hat{a}, \end{aligned} \tag{40}$$

where $\mathbf{|}f\mathbf{|}$ is given in Eq. (38) and α and β satisfy (39). It is straightforward to verify that the expressions (40) satisfy (11a). However, the operators $\tilde{b}$ and $\tilde{b}^\dagger$ *are not nilpotent.* Nilpotency cannot be achieved for any consistent choice of the arbitrary functions W, α and β. It may be verified by direct calculation that if $W(q) \neq 0$, then nilpotency can be obtained only if

$$W(q) \to \infty, \quad \alpha \to \infty, \quad \beta \to \infty.$$

In the limit $q \to 1$ (40) becomes

$$\begin{aligned} \tilde{b} \to b_2 &= \left(\frac{1 + (-1)^{\widehat{N}}}{2(\widehat{N}+1)(1 + W(1))} \right)^{1/2} \exp\left(i\alpha(1, \widehat{N}+1)\right) \hat{a} \\ &\quad + W^{1/2}(1) \left(\frac{1 - (-1)^{\widehat{N}}}{2\widehat{N}(1 + W(1))} \right)^{1/2} \exp\left(i\beta(1, \widehat{N})\right) \hat{a}^\dagger, \end{aligned} \tag{41a}$$

$$\begin{aligned} \tilde{b}^\dagger \to b_2^\dagger &= \left(\frac{1 - (-1)^{\widehat{N}}}{2\widehat{N}(1 + W(1))} \right)^{1/2} \exp\left(-i\alpha(1, \widehat{N})\right) \hat{a}^\dagger \\ &\quad + W^{1/2}(1) \left(\frac{1 + (-1)^{\widehat{N}}}{2(\widehat{N}+1)(1 + W(1))} \right)^{1/2} \exp\left(-i\beta(1, \widehat{N}+1)\right) \hat{a}. \end{aligned} \tag{41b}$$

We then find that

$$b_2 b_2^\dagger + b_2^\dagger b_2 = 1\,. \tag{42}$$

It can be readily shown that if we choose $\lim_{q\to 1} W(q) = 0$ then the operators are also nilpotent, *i.e.*

$$(b_2)^2 = \left(b_2^\dagger\right)^2 = 0\,. \tag{43}$$

The choice of the limit $\lim_{q\to 1} W(q) = 0$ does not alter the fundamental anticommutation relation (42). For $W(1) = 0$, $G(1,\widehat{N})$ is also zero and the relations (27) and (28) are also absent. The equation (39) is then superfluous and $\alpha(1,\widehat{N})$ is the only arbitrary phase factor. So here also we have the fermionic oscillator anticommutation relation as well as nilpotency of operators. Proceeding as in Section 2 one can show that the Eqs (42) and (43) do indeed describe a fermionic harmonic oscillator with the number operator M_2 defined as

$$\begin{aligned} M_2 &= b_2^\dagger b_2 = \tfrac{1}{2}\left(1 - (-1)^{\widehat{N}}\right)\,, \\ M_2^2 &= M_2\,. \end{aligned} \tag{44}$$

For $W(1) = 0$, (41) becomes

$$\begin{aligned} b_2 &= \left(\frac{1 + (-1)^{\widehat{N}}}{2\left(\widehat{N}+1\right)}\right)^{1/2} \exp\left(i\alpha(1,\widehat{N}+1)\right)\hat{a}\,, \\ b_2^\dagger &= \left(\frac{1 - (-1)^{\widehat{N}}}{2\widehat{N}}\right)^{1/2} \exp\left(-i\alpha(1,\widehat{N})\right)\hat{a}^\dagger\,. \end{aligned} \tag{45}$$

Choosing $M_2 = b_2^\dagger b_2$ it can again be shown that M_2 satisfies all the requirements of a fermionic number operator.

The reasons why the results following from (10) cannot strictly be obtained as a special case of (25) will become clear in the following Section.

4. On the arbitrary functions and phases

Here we comment on the interpretation of the arbitrary functions and phases occurring in the solutions. Consider the limit $q \to 1$, assuming that all the phases are independent of $\widehat{N}$ in this limit.

For operators of type I we may write

$$b_1 = e^{i\nu} b_1' ,$$
$$b_1^\dagger = e^{-i\nu} b_1'^{\dagger} , \tag{46}$$

where

$$b_1' = \hat{a} \left(\frac{1}{2\hat{N}} \left(1 + (-1)^{\hat{N}} \right) \right)^{1/2} . \tag{47}$$

Viewing $(b_1', b_1'^{\dagger})$ as elements of some quantum space of vectors, the transformations (46) imply a U(1) transformation of $(b_1', b_1'^{\dagger})$, the U(1) parameter being ν, preserving the relations (22). For $q \neq 1$, a similar approach would mean that $\nu(q, \hat{N})$ is the parameter of some type of q-deformed U(1) transformations. However, for $q \neq 1$, a consistent number operator *cannot* be obtained.

For the type II operators the relations (41) may be written as

$$\begin{pmatrix} b_2 \\ b_2^\dagger \end{pmatrix} = \begin{pmatrix} (1+W)^{-1/2} e^{i\alpha} & \left(\frac{W}{1+W}\right)^{1/2} e^{i\beta} \\ \left(\frac{W}{1+W}\right)^{1/2} e^{-i\beta} & (1+W)^{-1/2} e^{-i\alpha} \end{pmatrix} \begin{pmatrix} b_2' \\ b_2'^{\dagger} \end{pmatrix} , \tag{48}$$

where

$$b_2' = \hat{a} \left(\frac{1}{2\hat{N}} \left(1 - (-1)^{\hat{N}} \right) \right)^{1/2} . \tag{49}$$

The determinant of the transformations (48) is $(1 - W)/(1 + W)$. Hence for the above transformation to be invertible it is necessary that $W(1) \neq \pm 1$. Now $W(q) = G(0, q)/F(0, q)$. Hence for invertibility specifying the ratio is not enough. There is also the constraint that $G(0, 1) \neq \pm F(0, 1)$. Therefore, the four quantities $F(0, 1)$, $G(0, 1)$, $\alpha(1)$ and $\beta(1)$ can be viewed as the four parameters of $GL(2, R)$ transformations of $(b_2', b_2'^{\dagger})$ which preserve the relation (42). For $q \neq 1$, $F(0, q)$, $G(0, q)$ $\alpha(q, \hat{N})$ and $\beta(q, \hat{N})$ should then be parameters of some q-deformed $GL(2, R)$ transformations. It should be noted that for $W \neq 0$ the operators are not nilpotent.

For $W(1) = 0$ these transformations reduce to U(1) transformations similar to (46) with α the relevant parameter and a different definition for b_1' *viz.*

$$b_1' = \hat{a} \left(\frac{1}{2\hat{N}} \left(1 - (-1)^{\hat{N}} \right) \right)^{1/2} . \tag{50}$$

The case for $q = 1$ can be similarly generalized.

5. Conclusions

(i) Two types of operators (I, II) have been constructed in terms of bosonic q-oscillator operators. These new operators satisfy a relation reminiscent of the fermionic q-commutator relation.

(ii) For operators of type I in the limit $q \to 1$, by appropriately choosing the behaviour of the arbitrary functions of the deformation parameter q the fermionic harmonic oscillator is obtained. An arbitrary phase factor ν occurring in the solutions can be identified as the parameter of U(1) transformations of the operators preserving the basic fermionic anticommutator relation. For $q \neq 1$, $\nu(q, \widehat{N})$ should then correspond to the parameter of some q-deformed U(1) transformation that preserves a fermionic q-commutator relation.

(iii) For the type II operators in the limit $q \to 1$ one can again obtain the fermionic harmonic oscillator by suitably choosing the arbitrary functions present in the solutions. Four arbitrary parameters of the solution can be identified with the four real parameters of $GL(2, R)$ transformations of some two-dimensional vector (defined in terms of bosonic harmonic oscillator operators) — the transformations again preserving the basic fermionic anticommutator relation. The $q \neq 1$ case then corresponds to some q-deformed $GL(2, R)$ transformations which preserve a fermionic q-commutator relation.

(iv) Appropriate number operators in the limit $q \to 1$ for both types of operators have been obtained.

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