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ON SOME CONSERVATION LAWS IN THE MAXWELL ELECTRODYNAMICS IN VACUUM

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The group theoretical foundation of some conservation laws in the Maxwell electrodynamics in vacuum is considered. In particular the "mysterious" conservation laws of Lipkin [4] are found to be a consequence of the Noether theorem.

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1. Introduction

It is well known that the Maxwell electrodynamics in vacuum, being a special example of a linear vacuum field theory, admits an infinite number of conserved quantities. Some of them, as for example: the energy, the momentum or the angular momentum, have a well defined group theoretical origin founded on the Noether theorem. One can also find the non-local conserved quantity generated by the duality rotation group according to the Noether theorem [1–3]. However, we know explicitly many conserved quantities where the group theoretical foundation is not clear or, in fact, unknown [4–9]. The group theoretical analysis of some of those conserved quantities is given in [7] and [9] but this analysis concerns the symmetry of the source-free Maxwell equations and not the variational symmetry. In other words the group theoretical analysis given in [7] or [9] is *non-Noether.*

The aim of the present work is to find some conserved quantities in the vacuum Maxwell electrodynamics which arise from the Noether theorem. We employ the Fourier representation (the *$\vec{p}$-representation*) of the electrodynamics and we assume that the group generators in this representation are linear with respect to the electric field $\vec{\mathcal{E}}$ and the vector potential $\vec{\mathcal{A}}$ (compare with [2, 6, 9]). Then we consider symmetries of the source-free Maxwell electrodynamics (Section 2). In the *$\vec{x}$-representation* these symmetries appear to be non-local or general Lie-Bäcklund (hidden) symmetries.

They lead, as it is shown in Section 3, to infinite number of conserved quantities according to the Noether theorem. Thus one gets the total energy, the total momentum as well as the *zilch* of Lipkin [4–8] or the non-local conserved quantity generated by the duality rotation group [1–3]. In Section 4 the conserved quantity $Z_{00} = {}^1\!/_2 Z^{00}_{(\mathrm{Lipkin})}$ is considered in some details. Then concluding remarks (Section 5) close our paper.

2. Some symmetries of the source-free Maxwell equations

In this section we intend to study some symmetries of the source-free Maxwell equations. These symmetries lead to conservation laws which include, as their special cases, the conservations law generated by the duality rotation [1–3] as well as the "mysterious" conservation laws of Lipkin [4–8].

For our purpose it is convenient to deal with the Maxwell electrodynamics within the first order formalism [1, 2]. In this formalism the action reads

$$S[\vec{E}, A_\mu] = -\int d^4x\{\vec{E}\cdot\partial_t\vec{A} + \tfrac{1}{2}[\vec{E}^2 + (\nabla\times\vec{A})^2] + A_0\nabla\cdot\vec{E}\}\,, \qquad (2.1)$$

where A_μ defines the electromagnetic potential by $A_\mu = (-\Phi, \vec{A})$ and $\vec{E}$ stands for the electric field vector. (In this paper the space-time metric tensor $(g_{\mu\nu}) = \mathrm{diag}(-1,1,1,1)$; Greek indices run through 0,1,2,3 and Latin ones through 1,2,3; the speed of light $c = 1$). The least action principle $\delta S[\vec{E}, A_\mu] = 0$ yields the source-free Maxwell equations

$$\partial_t\vec{A} - \nabla A_0 + \vec{E} = 0\,, \quad \partial_t\vec{E} - \nabla\times(\nabla\times\vec{A}) = 0\,, \qquad (2.2)$$

$$\nabla\cdot\vec{E} = 0\,. \qquad (2.3)$$

Using the gauge

$$A_0 = 0\,, \quad \nabla\cdot\vec{A} = 0\,, \qquad (2.4)$$

one brings (2.2), (2.3) to the form

$$\partial_t\vec{A} + \vec{E} = 0\,, \quad \partial_t\vec{E} + \nabla^2\vec{A} = 0\,, \qquad (2.5)$$

$$\nabla\cdot\vec{A} = 0\,. \qquad (2.6)$$

(Observe that if $\partial_t\vec{A} + \vec{E} = 0$ and $\nabla\cdot\vec{A} = 0$ then also $\nabla\cdot\vec{E} = 0$).

Let $\vec{\mathcal{E}}$ and $\mathcal{A}_\mu$ denote the Fourier transforms of $\vec{E}$ and A_μ respectively, *i.e.*

$$\vec{\mathcal{E}}(t,\vec{p}) = (2\pi)^{-3/2}\int d^3x\,\vec{E}(t,\vec{x})\mathrm{e}^{-i\vec{p}\cdot\vec{x}}\,, \qquad (2.7)$$

$$\mathcal{A}_\mu(t,\vec{p}) = (2\pi)^{-3/2}\int d^3x\,A_\mu(t,\vec{x})\mathrm{e}^{-i\vec{p}\cdot\vec{x}}\,. \qquad (2.8)$$

Then, in terms of $\vec{\mathcal{E}}$ and $\mathcal{A}_\mu$, equations (2.2) and (2.3) take the form

$$\partial_t \vec{\mathcal{A}} - i\mathcal{A}_0 \vec{p} + \vec{\mathcal{E}} = 0\,, \quad \partial_t \vec{\mathcal{E}} + \vec{p} \times (\vec{p} \times \vec{\mathcal{A}}) = 0\,, \tag{2.9}$$

$$\vec{p} \cdot \vec{\mathcal{E}} = 0\,, \tag{2.10}$$

where $\vec{\mathcal{A}} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$. By the Perseval–Plancherel formula one gets

$$S[\vec{E}, A_\mu] = \mathcal{S}[\vec{\mathcal{E}}, \vec{\mathcal{E}}^*, \mathcal{A}_\mu, \mathcal{A}_\mu^*] \doteq \int dt d^3p \{ -\tfrac{1}{2} [\vec{\mathcal{E}} \cdot \partial_t \vec{\mathcal{A}}^* + \vec{\mathcal{E}}^* \cdot \partial_t \vec{\mathcal{A}} + \vec{\mathcal{E}} \cdot \vec{\mathcal{E}}^* + (\vec{p} \times \vec{\mathcal{A}}) \cdot (\vec{p} \times \vec{\mathcal{A}}^*) - i\mathcal{A}_0 \vec{p} \cdot \vec{\mathcal{E}}^* + i\mathcal{A}_0^* \vec{p} \cdot \vec{\mathcal{E}}] \}\,, \tag{2.11}$$

where the star ' * ' stands for the complex conjugation. Then the least action principle $\delta\mathcal{S} = 0$ leads to equations (2.9) and (2.10) and to their complex conjugation. In terms of $\mathcal{A}_\mu$ the gauge (2.4) reads

$$\mathcal{A}_0 = 0\,, \quad \vec{p} \cdot \vec{\mathcal{A}} = 0\,. \tag{2.12}$$

Thus with (2.12) assumed, the Maxwell equations (2.9), (2.10) can be written as follows

$$\partial_t \vec{\mathcal{A}} + \vec{\mathcal{E}} = 0\,, \quad \partial_t \vec{\mathcal{E}} - p^2 \vec{\mathcal{A}} = 0\,, \tag{2.13}$$

$$\vec{p} \cdot \vec{\mathcal{A}} = 0\,. \tag{2.14}$$

(Note that if $\partial_t \vec{\mathcal{A}} + \vec{\mathcal{E}} = 0$ and $\vec{p} \cdot \vec{\mathcal{A}} = 0$ then also $\vec{p} \cdot \vec{\mathcal{E}} = 0$). It is an easy matter to show that (2.13) and their complex conjugation can be derived from the least action principle

$$\delta\mathcal{S}'[\vec{\mathcal{E}}, \vec{\mathcal{E}}^*, \vec{\mathcal{A}}, \vec{\mathcal{A}}^*] = 0\,,$$

$$\mathcal{S}' = \int dt d^3p \mathcal{L}\,, \ \mathcal{L} \doteq -\tfrac{1}{2} (\vec{\mathcal{E}} \cdot \partial_t \vec{\mathcal{A}}^* + \vec{\mathcal{E}}^* \cdot \partial_t \vec{\mathcal{A}} + \vec{\mathcal{E}} \cdot \vec{\mathcal{E}}^* + p^2 \vec{\mathcal{A}} \cdot \vec{\mathcal{A}}^*)\,. \tag{2.15}$$

(Observe that if (2.12) holds then $\mathcal{S} = \mathcal{S}'$). We are going to find some symmetries of the system (2.13) and their complex conjugate. To this end we use the jet space formalism [10–13]. Denote the set (2.13) and their complex conjugate by

$$\mathcal{F} = 0\,. \tag{2.16}$$

Let the general infinitesimal operator X of a symmetry group G of (2.16) be of the form

$$X = Y_k \frac{\partial}{\partial \mathcal{E}_k} + Z_k \frac{\partial}{\partial \mathcal{A}_k} + \text{c.c.}\,,$$

$$Y_k = a_{kl}\mathcal{E}_l + b_{kl}\mathcal{A}_l + e_k\,, \quad Z_k = c_{kl}\mathcal{E}_l + d_{kl}\mathcal{A}_l + f_k\,, \tag{2.17}$$

where a_{kl}, b_{kl}, c_{kl}, d_{kl}, e_k and f_k are some functions of $\vec{p}$ such that

$$a^*_{kl}(\vec{p}) = a_{kl}(-\vec{p}), \quad b^*_{kl}(\vec{p}) = b_{kl}(-\vec{p}), \ldots, etc. \tag{2.18}$$

and c.c. denotes the complex conjugation. Then the invariance condition for (2.16) reads

$$X^1(\mathcal{F})_{|\mathcal{F}=0} = 0, \tag{2.19}$$

where X^1 is the first prolongation of X Refs [10–13]. In local coordinates $(t, p_k, \mathcal{E}_k, \mathcal{A}_k, \mathcal{E}_{k,\nu_1\ldots\nu_s}, \mathcal{A}_{k,\nu_1\ldots\nu_s}, \ s \geq 1)$ in the relevant jet spaces one has

$$\begin{aligned} X^1 =& Y_k \frac{\partial}{\partial \mathcal{E}_k} + Z_k \frac{\partial}{\partial \mathcal{A}_k} + Y_{k,\mu} \frac{\partial}{\partial \mathcal{E}_{k,\mu}} + Z_{k,\mu} \frac{\partial}{\partial \mathcal{A}_{k,\mu}} + \text{c.c.} \\ & Y_{k,\mu} = D_\mu Y_k, \quad Z_{k,\mu} = D_\mu Z_k, \end{aligned} \tag{2.20}$$

with D_μ standing for the total derivative

$$\begin{aligned} D_\mu =& \partial_\mu + \mathcal{E}_{k,\mu} \frac{\partial}{\partial \mathcal{E}_k} + \mathcal{A}_{k,\mu} \frac{\partial}{\partial \mathcal{A}_k} + \text{c.c.} \\ &+ \sum_{s \geq 1} \left(\mathcal{E}_{k,\nu_1\ldots\nu_s\mu} \frac{\partial}{\partial \mathcal{E}_{k,\nu_1\ldots\nu_s}} + \mathcal{A}_{k,\nu_1\ldots\nu_s\mu} \frac{\partial}{\partial \mathcal{A}_{k,\nu_1\ldots\nu_s}} + \text{c.c.} \right), \\ \partial_\mu \doteq& (\partial_t, \partial_{\vec{p}}). \end{aligned} \tag{2.21}$$

Having all that one finds easily that the invariance condition (2.19) takes the form

$$\begin{aligned} (D_t \vec{Z} + \vec{Y})_{|\mathcal{F}=0} = 0 \quad \wedge \quad \text{c.c.} \\ (D_t \vec{Y} - p^2 \vec{Z})_{|\mathcal{F}=0} = 0 \quad \wedge \quad \text{c.c.}. \end{aligned} \tag{2.22}$$

where $D_t \doteq D_0$. Then, by (2.13) and (2.17) the conditions (2.22) yield

$$\begin{aligned} (a_{kl} - d_{kl})\mathcal{E}_l + (b_{kl} + p^2 c_{kl})\mathcal{A}_l + e_k = 0 \quad \wedge \quad \text{c.c.}, \\ -(b_{kl} + p^2 c_{kl})\mathcal{E}_l + p^2(a_{kl} - d_{kl})\mathcal{A}_l - p^2 f_k = 0 \quad \wedge \quad \text{c.c.}. \end{aligned} \tag{2.23}$$

Hence (remember that we don't assume (2.14))

$$e_k = 0, \quad f_k = 0, \quad d_{kl} = a_{kl}, \quad b_{kl} = -p^2 c_{kl}. \tag{2.24}$$

Finally the infinitesimal operator (2.17) for given $\hat{a} = (a_{kl})$ and $\hat{c} = (c_{kl})$ will be denoted by $X_{(\hat{a}\hat{c})}$

$$\begin{aligned} X_{(\hat{a}\hat{c})} &= Y_{(\hat{a}\hat{c})k} \frac{\partial}{\partial \mathcal{E}_k} + Z_{(\hat{a}\hat{c})k} \frac{\partial}{\partial \mathcal{A}_k} + \text{c.c.} \\ &= (a_{kl}\mathcal{E}_l - p^2 c_{kl}\mathcal{A}_l) \frac{\partial}{\partial \mathcal{E}_k} + (c_{kl}\mathcal{E}_l + a_{kl}\mathcal{A}_l) \frac{\partial}{\partial \mathcal{A}_k} + \text{c.c.}. \end{aligned} \tag{2.25}$$

Simple calculations show that the commutator $[X_{(\hat{a}\hat{c})}, X_{(\hat{a}'\hat{c}')}]$ of two infinitesimal operators $X_{(\hat{a}\hat{c})}$ and $X_{(\hat{a}'\hat{c}')}$ reads

$$[X_{(\hat{a}\hat{c})}, X_{(\hat{a}'\hat{c}')}] \doteq X_{(\hat{a}\hat{c})}X_{(\hat{a}'\hat{c}')} - X_{(\hat{a}'\hat{c}')}X_{(\hat{a}\hat{c})} = X_{(\hat{a}''\hat{c}'')}, \tag{2.26}$$

for

$$\hat{a}'' = [\hat{a}', \hat{a}] - p^2[\hat{c}', \hat{c}], \quad \hat{c}'' = [\hat{a}', \hat{c}] + [\hat{c}', \hat{a}].$$

Thus the set of all operators of the form (2.25) constitutes an infinite-dimensional Lie algebra g of some infinite-dimensional point transformation group G leaving the first set (2.13) of the Maxwell equations and their complex conjugation invariant. One quickly shows that the subalgebra $g_0 \subset g$ defined by

$$X_{(\hat{a}\hat{c})} \in g_0 \quad \dot{\Leftrightarrow} \quad a_{kl}p_k = \omega(\vec{p})p_l \wedge c_{kl}p_k = \sigma(\vec{p})p_l, \tag{2.27}$$

where $\omega = \omega(\vec{p})$ and $\sigma = \sigma(\vec{p})$ are functions of $\vec{p}$ such that $\omega^*(\vec{p}) = \omega(-\vec{p})$ and $\sigma^*(\vec{p}) = \sigma(-\vec{p})$, determines the subgroup $G_0 \subset G$ leaving the Maxwell equations (2.13), (2.14) and their complex conjugation invariant.

3. Conservation laws induced by G

In this section we find the conservation laws induced by the symmetry group G according to the Noether theorem [10, 11]. In our case (first order Lagrangian and vertical vector field) the Noether identity (see also [12]) takes the form

$$X^1(\mathcal{L}) = D_\mu N^\mu(\mathcal{L}) + \left(Y_k \frac{\delta}{\delta \mathcal{E}_k}(\mathcal{L}) + Z_k \frac{\delta}{\delta \mathcal{A}_k}(\mathcal{L}) + \text{c.c.}\right), \tag{3.1}$$

where N^μ is the *Legendre operator*

$$N^\mu = Y_k \frac{\partial}{\partial \mathcal{E}_{k,\mu}} + Z_k \frac{\partial}{\partial \mathcal{A}_{k,\mu}} + \text{c.c.} \tag{3.2}$$

and $\delta/\delta\mathcal{E}_k$, $\delta/\delta\mathcal{A}_k$ are the *Euler–Lagrange operators*

$$\frac{\delta}{\delta \mathcal{E}_k} = \frac{\partial}{\partial \mathcal{E}_k} - D_\nu \frac{\partial}{\partial \mathcal{E}_{k,\nu}}, \quad \frac{\delta}{\delta \mathcal{A}_k} = \frac{\partial}{\partial \mathcal{A}_k} - D_\nu \frac{\partial}{\partial \mathcal{A}_{k,\nu}}. \tag{3.3}$$

First we consider $X^1(\mathcal{L})$, where $\mathcal{L}$ and X are given by (2.15) and (2.25), respectively. One gets

$$\begin{aligned} X^1_{(\hat{a}\hat{c})}(\mathcal{L}) &= \Big(Y_{(\hat{a}\hat{c})k} \frac{\partial}{\partial \mathcal{E}_k} + Z_{(\hat{a}\hat{c})k} \frac{\partial}{\partial \mathcal{A}_k} + Y_{(\hat{a}\hat{c})k,\mu} \frac{\partial}{\partial \mathcal{E}_{k,\mu}} \\ &+ Z_{(\hat{a}\hat{c})k,\mu} \frac{\partial}{\partial \mathcal{A}_{k,\mu}} + \text{c.c.}\Big) \left(-\tfrac{1}{2}(\vec{\mathcal{E}} \cdot \partial_t \vec{\mathcal{A}}^* + \vec{\mathcal{E}}^* \cdot \partial_t \vec{\mathcal{A}} + \vec{\mathcal{E}} \cdot \vec{\mathcal{E}}^* + p^2 \vec{\mathcal{A}} \cdot \vec{\mathcal{A}}^*)\right) \\ &= -\tfrac{1}{2}\left\{a_{kl}(\mathcal{A}^*_{k,t}\mathcal{E}_l + \mathcal{E}^*_k\mathcal{E}_l + p^2\mathcal{A}^*_k\mathcal{A}_l + \mathcal{E}^*_k\mathcal{A}_{l,t}) + c_{kl}(-p^2\mathcal{A}^*_{k,t}\mathcal{A}_l \right. \\ &\left. -p^2\mathcal{E}^*_k\mathcal{A}_l + p^2\mathcal{A}^*_k\mathcal{E}_l + \mathcal{E}^*_k\mathcal{E}_{l,t}) + \text{c.c.}\right\}. \end{aligned} \tag{3.4}$$

Consequently, using (2.13) and their complex conjugate, we have the following formulas

$$X^1_{(\hat{a}\hat{c})}(\mathcal{L})_{|\mathcal{F}=0} = D_t\big(- \tfrac{1}{2}(a_{kl}\mathcal{A}^*_k\mathcal{E}_l + a^*_{kl}\mathcal{A}_k\mathcal{E}^*_l + c_{kl}\mathcal{E}^*_k\mathcal{E}_l + c^*_{kl}\mathcal{E}_k\mathcal{E}^*_l)\big)_{|\mathcal{F}=0}, \tag{3.5}$$

or

$$X^1_{(\hat{a}\hat{c})}(\mathcal{L})_{|\mathcal{F}=0} = D_t\big(- \tfrac{1}{2}(a_{kl}\mathcal{E}^*_k\mathcal{A}_l + a^*_{kl}\mathcal{E}_k\mathcal{A}^*_l - c_{kl}p^2\mathcal{A}^*_k\mathcal{A}_l - c^*_{kl}p^2\mathcal{A}_k\mathcal{A}^*_l)\big)_{|\mathcal{F}=0}. \tag{3.6}$$

Now one can also find the relation

$$D_\mu N^\mu(\mathcal{L}) = D_t\big(- \tfrac{1}{2}(a_{kl}\mathcal{E}^*_k\mathcal{A}_l + a^*_{kl}\mathcal{E}_k\mathcal{A}^*_l + c_{kl}\mathcal{E}^*_k\mathcal{E}_l + c^*_{kl}\mathcal{E}_k\mathcal{E}^*_l)\big). \tag{3.7}$$

Of course, the system of equations $\mathcal{F} = 0$ is equivalent to the following one

$$\frac{\delta\mathcal{L}}{\delta\mathcal{E}_k} = 0, \quad \frac{\delta\mathcal{L}}{\delta\mathcal{A}_k} = 0, \quad \frac{\delta\mathcal{L}}{\delta\mathcal{E}^*_k} = 0, \quad \frac{\delta\mathcal{L}}{\delta\mathcal{A}^*_k} = 0; \quad k = 1,2,3. \tag{3.8}$$

Therefore, using the Noether identity (3.1) and combining (3.5) with (3.7) and then (3.6) with (3.7) one arrives at the conservation laws

$$D_t\left(\tfrac{1}{2}(a_{kl} - a^*_{lk})(\mathcal{E}^*_k\mathcal{A}_l - \mathcal{A}^*_k\mathcal{E}_l)\right)_{|\mathcal{F}=0} = 0, \tag{3.9}$$

$$D_t\left(\tfrac{1}{2}(c_{kl} + c^*_{lk})(\mathcal{E}^*_k\mathcal{E}_l + p^2\mathcal{A}^*_k\mathcal{A}_l)\right)_{|\mathcal{F}=0} = 0. \tag{3.10}$$

From (2.7), (2.8) and (2.18) one quickly finds

$$\int d^3p[a_{kl}(\vec{p}) - a^*_{lk}(\vec{p})][\mathcal{E}^*_k(t,\vec{p})\mathcal{A}_l(t,\vec{p}) - \mathcal{A}^*_k(t,\vec{p})\mathcal{E}_l(t,\vec{p})] = \int d^3x\{E_k(t,\vec{x}) \times[a_{kl}(-i\nabla) - a_{lk}(i\nabla)]A_l(t,\vec{x}) - A_k(t,\vec{x})[a_{kl}(-i\nabla) - a_{lk}(i\nabla)]E_l(t,\vec{x})\} \doteq 2I \tag{3.11}$$

and

$$\int d^3p[c_{kl}(\vec{p}) + c^*_{lk}(\vec{p})][\mathcal{E}^*_k(t,\vec{p})\mathcal{E}_l(t,\vec{p}) + p^2\mathcal{A}^*_k(t,\vec{p})\mathcal{A}_l(t,\vec{p})] = \int d^3x\{E_k(t,\vec{x}) \times[c_{kl}(-i\nabla) + c_{lk}(i\nabla)]E_l(t,\vec{x}) - A_k(t,\vec{x})[c_{kl}(-i\nabla) + c_{lk}(i\nabla)]\nabla^2 A_l(t,\vec{x})\} \doteq 2J. \tag{3.12}$$

It is evident that $I^* = I$ and $J^* = J$. Thus we have found real quantities I and J which are conserved, *i.e.*

$$\partial_t I = 0 \quad \wedge \quad \partial_t J = 0, \tag{3.13}$$

if only equations (2.5) are satisfied. Now we are interested in the conservation laws in the Maxwell electrodynamics in vacuum, so we assume that both (2.5) and (2.6) are satisfied. From the well-known formula $\vec{B} = \nabla \times \vec{A}$ one finds the relation

$$\vec{\mathcal{B}} = i\vec{p} \times \vec{\mathcal{A}}, \tag{3.14}$$

where $\vec{\mathcal{B}} = \vec{\mathcal{B}}(t, \vec{p})$ denotes the Fourier transform of the magnetic field $\vec{B} = \vec{B}(t, \vec{x})$. Then taking into account (2.14) one gets

$$\vec{\mathcal{A}} = ip^{-2}\vec{p} \times \vec{\mathcal{B}}, \tag{3.15}$$

($\vec{A} = -\nabla^{-2}(\nabla \times \vec{B})$ in terms of $\vec{A}$ and $\vec{B}$). Thus our conserved quantities I and J can be written down in a gauge invariant form

$$\begin{aligned} I &= \tfrac{i}{2} \int d^3p [a_{kl}(\vec{p}) - a^*_{lk}(\vec{p})] p^{-2} p_i (\epsilon_{lij} \mathcal{E}^*_k \mathcal{B}_j + \epsilon_{kij} \mathcal{B}^*_j \mathcal{E}_l) \\ &= \tfrac{1}{2} \int d^3x \{ -E_k [a_{kl}(-i\nabla) - a_{lk}(i\nabla)] \epsilon_{ljm} \nabla^{-2} \partial_j B_m \\ &\quad + \epsilon_{kjm} (\nabla^{-2} \partial_j B_m) [a_{kl}(-i\nabla) - a_{lk}(i\nabla)] E_l \} \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} J &= \tfrac{1}{2} \int d^3p [c_{kl}(\vec{p}) + c^*_{lk}(\vec{p})] (\mathcal{E}^*_k \mathcal{E}_l + \epsilon_{kij} \epsilon_{lrs} p^{-2} p_i p_r \mathcal{B}^*_j \mathcal{B}_s) \\ &= \tfrac{1}{2} \int d^3x \{ E_k [c_{kl}(-i\nabla) + c_{lk}(i\nabla)] E_l - \epsilon_{kjm} (\nabla^{-2} \partial_j B_m) \\ &\quad \times [c_{kl}(-i\nabla) + c_{lk}(i\nabla)] \epsilon_{lrs} \partial_r B_s \}. \end{aligned} \tag{3.17}$$

Examples:

(3.a) $a_{kl}(\vec{p}) = 0$, $c_{kl}(\vec{p}) = \frac{i}{2} h(\vec{p}) p^{-2} p_i \epsilon_{ikl}$, where $h = h(\vec{p})$ is a real function such that $h(\vec{p}) = h(-\vec{p})$. Then $I = 0$ and using the condition $\vec{p} \cdot \vec{\mathcal{B}} = 0$ or $\nabla \cdot \vec{B} = 0$ one finds J to be

$$J = \tfrac{1}{2} \int d^3x \{ \vec{E} \cdot [h(-i\nabla) \nabla^{-2} \nabla \times \vec{E}] + \vec{B} \cdot [h(-i\nabla) \nabla^{-2} \nabla \times \vec{B}] \}, \tag{3.18}$$

If $h(\vec{p}) = 1$ then

$$J = \tfrac{1}{2} \int d^3x \{ \vec{E} \cdot (\nabla^{-2} \nabla \times \vec{E}) + \vec{B} \cdot (\nabla^{-2} \nabla \times \vec{B}) \},$$

appears to be the conserved quantity induced by the duality rotation group [1–3]. For $h(\vec{p}) = -p^2$ one gets

$$J = \tfrac{1}{2} \int d^3x \{ \vec{E} \cdot (\nabla \times \vec{E}) + \vec{B} \cdot (\nabla \times \vec{B}) \} \doteq Z_{00}. \tag{3.19}$$

(3.b) $a_{kl}(\vec{p}) = 0\,,\ c_{kl}(\vec{p}) = -\frac{i}{2}p_i\epsilon_{jkl}\,.$ Here, of course, $I = 0$. Then by some simple manipulations, using also the condition $\vec{p}\cdot\vec{\mathcal{B}} = 0$ one gets the following formula for J

$$J = -\tfrac{1}{2}\int d^3x\{(\vec{E}\times\partial_i\vec{E})_j + (\vec{B}\times\partial_i\vec{B})_j\} \doteq Z_{ji}\,. \tag{3.20}$$

(3.c) $a_{kl}(\vec{p}) = -\frac{1}{2}p^2\epsilon_{ikl}\,,\ c_{kl}(\vec{p}) = 0\,.$ Now $J = 0$ and I by (2.13), (2.14), (3.15) reads

$$I = -\tfrac{1}{2}\int d^3x\{(\vec{E}\times\partial_t\vec{E})_i + (\vec{B}\times\partial_t\vec{B})_i\} \doteq Z_{0i}\,. \tag{3.21}$$

(3.d) $a_{kl}(\vec{p}) = 0\,,\ c_{kl}(\vec{p}) = \frac{1}{2}\delta_{kl}\,.$ Here $I = 0$. Then, using also the condition $\vec{p}\cdot\vec{\mathcal{B}} = 0$, one finds

$$J = \tfrac{1}{2}\int d^3x(\vec{E}^2 + \vec{B}^2)\,, \tag{3.22}$$

i.e. J is the total energy of the electromagnetic field.

(3.e) $a_{kl}(\vec{p}) = \frac{i}{2}p_i\delta_{kl}\,,\ c_{kl}(\vec{p}) = 0$. Now $J = 0$ and I, when the conditions $\vec{p}\cdot\vec{\mathcal{B}} = 0$ and $\vec{p}\cdot\vec{\mathcal{E}} = 0$ are used, reads

$$I = \int d^3x(\vec{E}\times\vec{B})_i = P_i\,, \tag{3.23}$$

where $\vec{P} = (P_1, P_2, P_3)$ denotes the momentum of the electromagnetic field.

The conserved quantities (3.19), (3.20) and (3.21) can be written in a compact form. Indeed, if

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{3.24}$$

is the electromagnetic field tensor and

$$\breve{F}_{\mu\nu} = -\tfrac{1}{2}\epsilon_{\mu\nu\varrho\sigma}F^{\varrho\sigma}\,;\quad (\epsilon_{0123} = 1)\,, \tag{3.25}$$

is the dual tensor of $F_{\mu\nu}$, then one quickly shows that the conserved quantities (3.19), (3.20) and (3.21) can be presented in a concise form

$$Z_{\mu\nu} = \tfrac{1}{2}\int d^3x(\breve{F}^{0\varrho}\partial_\mu F_{\varrho\nu} - F^{0\varrho}\partial_\mu\breve{F}_{\varrho\nu})\,. \tag{3.26}$$

(Note that Z_{00} given by (3.26) is equal to Z_{00} given by (3.19) by virtue of (2.13), (2.14) and (3.15). One can easily show that if the electromagnetic field is assumed to vanish at the spatial infinity then $Z_{i0} = 0$).

Define the following tensor

$$Z_{\mu\nu}{}^{\sigma} \doteq \tfrac{1}{2}(\check{F}^{\sigma\varrho}\partial_\mu F_{\varrho\nu} - F^{\sigma\varrho}\partial_\mu \check{F}_{\varrho\nu}) . \tag{3.27}$$

From (3.26) and (3.27) one gets

$$Z_{\mu\nu} = \int d^3x Z_{\mu\nu}{}^{0} . \tag{3.28}$$

Then, using the Maxwell equations

$$\partial_\nu F^{\mu\nu} = 0 , \quad \partial_\nu \check{F}^{\mu\nu} = 0 , \tag{3.29}$$

we find the following conservation law

$$\partial_\sigma Z_{\mu\nu}{}^{\sigma} = 0 . \tag{3.30}$$

Formula (3.30) leads to our conserved quantities $Z_{\mu\nu}$ defined by (3.26). It is of some interest to compare our results with those of Lipkin presented in his paper [4]. (see also [5–8]). Lipkin [4] has found a tensor, which we denote here by $Z^{\mu\nu\sigma}_{(\mathrm{Lipkin})}$, such that it satisfies the conservation law

$$\partial_\sigma Z^{\mu\nu\sigma}_{(\mathrm{Lipkin})} = 0 . \tag{3.31}$$

Comparing the formulas (23) and (24) of [4] with our formula (3.27) one gets

$$Z^{\mu\nu\sigma}_{(\mathrm{Lipkin})} = 2Z^{\mu\nu\sigma} + \tfrac{1}{2}(g^{\mu\alpha}\epsilon^{\nu\sigma\beta\gamma} + g^{\nu\alpha}\epsilon^{\mu\sigma\beta\gamma} + g^{\sigma\alpha}\epsilon^{\mu\nu\beta\gamma})\partial_\gamma T_{\alpha\beta} , \tag{3.32}$$

where $T_{\alpha\beta}$ stands for the energy-momentum tensor of the electromagnetic field

$$T_{\alpha\beta} = F_{\alpha\gamma}F_{\beta}{}^{\gamma} - \tfrac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta} . \tag{3.33}$$

(Remember that in Lipkin's paper [4] the signature of the metric is $(+\ -\ -\ -)$. In the present paper we take the signature to be $(-\ +\ +\ +)$). Therefore

$$\partial_\sigma Z^{\mu\nu\sigma}_{(\mathrm{Lipkin})} = 2\partial_\sigma Z^{\mu\nu\sigma} + \epsilon^{\mu\nu\beta\gamma}\partial_\gamma\partial_\sigma T^{\sigma}{}_{\beta} \tag{3.34}$$

and, consequently

$$(\partial_\sigma Z^{\mu\nu\sigma} = 0 \quad \wedge \quad \partial_\sigma T^{\sigma}{}_{\beta} = 0) \Rightarrow \partial_\sigma Z^{\mu\nu\sigma}_{(\mathrm{Lipkin})} = 0 . \tag{3.35}$$

Moreover, assuming that the electromagnetic field vanishes at the spatial infinity, one gets from (3.32)

$$\int d^3x Z^{\mu\nu 0}_{(\text{Lipkin})} = 2\int d^3x Z^{\mu\nu 0} - \tfrac{1}{2}\epsilon^{\mu\nu\beta 0}\partial_t \int d^3x T_{0\beta}\,, \tag{3.36}$$

i.e. , as $\partial_t \int d^3x T_{0\beta} = 0$

$$\int d^3x Z^{\mu\nu 0}_{(\text{Lipkin})} = 2\int d^3x Z^{\mu\nu 0}\,. \tag{3.37}$$

The conserved quantity $\int d^3x Z^{\mu\nu 0}_{(\text{Lipkin})}$ is called the *zilch* of the electromagnetic field. Given infinitesimal operator $X_{\hat{a}\hat{c}}$ in the *$\vec{p}$-representation* (see (2.25)) one finds the respective operator $\tilde{X}_{\hat{a}\hat{c}}$ in the *$\vec{x}$-representation* to be

$$\tilde{X}_{\hat{a}\hat{c}} = \tilde{Y}_k \frac{\partial}{\partial E_k} + \tilde{Z}_k \frac{\partial}{\partial A_k}\,, \tag{3.38}$$

where

$$\tilde{Y}_k = (2\pi)^{-3/2}\int d^3p\, Y_k\, e^{i\vec{p}\cdot\vec{x}} = a_{kl}(-i\nabla)E_l + c_{kl}(-i\nabla)\nabla^2 A_l\,,$$
$$\tilde{Z}_k = (2\pi)^{-3/2}\int d^3p\, Z_k\, e^{i\vec{p}\cdot\vec{x}} = c_{kl}(-i\nabla)E_l + a_{kl}(-i\nabla)\nabla^2 A_l\,. \tag{3.39}$$

Of course, by (2.18)

$$a_{kl}(-i\nabla) = (a_{kl}(-i\nabla))^*\,, \quad c_{kl}(-i\nabla) = (c_{kl}(-i\nabla))^*\,. \tag{3.40}$$

Thus we arrive at the conclusion that the formulas (3.38)–(3.40) define a *non-local symmetry* or a general *Lie-Bäcklund symmetry* [10, 11]. In particular the conserved quantities $Z_{\mu\nu}$ are generated by the Lie-Bäcklund symmetry. In the next section we consider Z_{00} in some details.

Now we observe that the examples (3.d) and (3.e) show that the conservation laws for the energy and the momentum of source-free Maxwell field can be considered to follow from some general Lie-Bäcklund symmetries. Indeed, by (3.d), (3.38) and (3.39) one finds that the energy conservation law is generated by the infinitesimal operator $\tilde{X}$

$$\tilde{X} = \tfrac{1}{2}(\nabla^2 A_k)\frac{\partial}{\partial E_k} + \tfrac{1}{2}E_k\frac{\partial}{\partial A_k} \tag{3.41}$$

which leads to the following Lie-Bäcklund (formal) transformation group [2, 10]

$$\vec{E}' = \cos\left(\frac{\tau}{2}\sqrt{-\nabla^2}\right)\cdot\vec{E} - \sqrt{-\nabla^2}\sin\left(\frac{\tau}{2}\sqrt{-\nabla^2}\right)\cdot\vec{A}\,,$$
$$\vec{A}' = \left(\sqrt{-\nabla^2}\right)^{-1}\sin\left(\frac{\tau}{2}\sqrt{-\nabla^2}\right)\cdot\vec{E} + \cos\left(\frac{\tau}{2}\sqrt{-\nabla^2}\right)\cdot\vec{A}\,, \tag{3.42}$$

where $\tau \in R$ is the group parameter. (Note that to find (3.42) we integrate the Lie equations in the $\vec{p}$*-representation*

$$\frac{d\vec{\mathcal{E}}'}{d\tau} = -\tfrac{1}{2}p^2\vec{\mathcal{A}}', \quad \frac{d\vec{\mathcal{A}}'}{d\tau} = \tfrac{1}{2}\vec{\mathcal{E}}';$$
$$\vec{\mathcal{E}}'_{|\tau=0} = \vec{\mathcal{E}}, \qquad \vec{\mathcal{A}}'_{|\tau=0} = \vec{\mathcal{A}}. \tag{3.43}$$

and then we perform the inverse Fourier transformation). Similarly one can proceed in the case of the momentum conservation law given by the example (3.e).

4. Remarks on Z_{00}

As it has been shown in the example (3.a) of the preceding section the one-parameter group of transformations G_1 defined by the infinitesimal operator

$$X_1 = -\frac{i}{2}p^2\epsilon_{kjl}p_j\mathcal{A}_l\frac{\partial}{\partial\mathcal{E}_k} + \frac{i}{2}\epsilon_{kjl}p_j\mathcal{E}_l\frac{\partial}{\partial\mathcal{A}_k} \tag{4.1}$$

leads, according to the Noether theorem, to the conserved quantity Z_{00} given by (3.19). It is quite obvious that for the infinitesimal operator X_1 the conditions (2.27) are satisfied. Thus G_1 appears to be the symmetry group of the source-free Maxwell equations (2.13), (2.14) *i.e.* $G_1 \subset G_0$. Now we can find the explicit form of G_1. The Lie equations read

$$\frac{d\mathcal{E}'_k}{d\tau} = -\frac{i}{2}p^2\epsilon_{kjl}p_j\mathcal{A}'_l, \quad \frac{d\mathcal{A}'_k}{d\tau} = \frac{i}{2}\epsilon_{kjl}p_j\mathcal{E}'_l,$$
$$\mathcal{E}'_{k|\tau=0} = \mathcal{E}_k, \quad \mathcal{A}'_{k|\tau=0} = \mathcal{A}_k; \quad \tau \in R. \tag{4.2}$$

We assume that $\vec{p}\cdot\vec{\mathcal{E}} = 0$ and $\vec{p}\cdot\vec{\mathcal{A}} = 0$. Then one can easily integrate Eqs (4.2) to get

$$\vec{\mathcal{E}}' = \vec{\mathcal{E}}\cos(p^2\varphi) - i\vec{p}\times\vec{\mathcal{A}}\sin(p^2\varphi),$$
$$\vec{\mathcal{A}}' = ip^{-2}\vec{p}\times\vec{\mathcal{E}}\sin(p^2\varphi) + \vec{\mathcal{A}}\cos(p^2\varphi); \quad \varphi \doteq \frac{\tau}{2}. \tag{4.3}$$

Employing (3.15) we write down (4.3) in terms of $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$:

$$\vec{\mathcal{E}}' = \vec{\mathcal{E}}\cos(p^2\varphi) - \vec{\mathcal{B}}\sin(p^2\varphi),$$
$$\vec{\mathcal{B}}' = \vec{\mathcal{E}}\sin(p^2\varphi) + \vec{\mathcal{B}}\cos(p^2\varphi). \tag{4.4}$$

Now it is an easy matter to present (4.1), (4.3) and (4.4) in the $\vec{x}$*-representation.* Thus we obtain

$$\tilde{X}_1 = \tfrac{1}{2}\nabla^2(\nabla\times\vec{A})_k\frac{\partial}{\partial E_k} + \tfrac{1}{2}(\nabla\times\vec{E})_k\frac{\partial}{\partial A_k}, \tag{4.5}$$

$$\vec{E}' = \cos(\varphi\nabla^2)\vec{E} + \sin(\varphi\nabla^2)\nabla \times \vec{A},$$
$$\vec{A}' = \sin(\varphi\nabla^2)\nabla^{-2}\nabla \times \vec{E} + \cos(\varphi\nabla^2)\vec{A}, \tag{4.6}$$

and

$$\vec{E}' = \cos(\varphi\nabla^2)\vec{E} + \sin(\varphi\nabla^2)\vec{B},$$
$$\vec{B}' = -\sin(\varphi\nabla^2)\vec{E} + \cos(\varphi\nabla^2)\vec{B}. \tag{4.7}$$

Consequently, $\tilde{X}_1$ determines the *generalized symmetry* which is also called the *general Lie-Bäcklund symmetry* [10, 11]. The formulas (4.6) or (4.7) define one-parameter Lie-Bäcklund symmetry given by (4.7). (In physics such a symmetry is also called the *hidden symmetry*). As it has been mentioned above the Lie-Bäcklund transformation group (4.6) or (4.7) leave the source-free Maxwell equations invariant. Finally, one can write (4.7) in terms of $F_{\mu\nu}$ and $\check{F}_{\mu\nu}$ as follows

$$F'_{\mu\nu} = \cos(\varphi\nabla^2)F_{\mu\nu} + \sin(\varphi\nabla^2)\check{F}_{\mu\nu}. \tag{4.8}$$

The formulas (4.7) and (4.8) resemble the duality rotation [1–3] but in contrary to the latter one they are the formal transformation groups on the relevant *infinite jet space* [10, 11, 13]. It is of some interest to calculate the Z_{00} density, *i.e.* Z_{000} (see (3.19), (3.27) and (3.28))

$$Z_{000} = \tfrac{1}{2}[\vec{E}\cdot(\nabla \times \vec{E}) + \vec{B}\cdot(\nabla \times \vec{B})], \tag{4.9}$$

in some special and physically important cases. First consider the partially polarized electromagnetic wave which propagates along the z axis. The wave is defined by

$$\vec{E} = (a\sin(\omega_1 t - k_1 z) + \eta b\sin(\omega_2 t - k_2 z),$$
$$b\sin(\omega_1 t - k_1 z + \delta) - \eta a sin(\omega_2 t - k_2 z + \delta), 0),$$

$$\vec{B} = (-b\sin(\omega_1 t - k_1 z + \delta) + \eta a\sin(\omega_2 t - k_2 z + \delta),$$
$$a\sin(\omega_1 t - k_1 z) + \eta b sin(\omega_2 t - k_2 z), 0), \tag{4.10}$$

where $a, b, \omega_1 \neq \omega_2, k_1 \neq k_2$ and δ are constant parameters and η is the parameter defining the polarization. Substituting (4.10) into (4.9) one gets

$$Z_{000} = ab\cdot(\eta^2 k_2 - k_1)\sin\delta + \eta\{a^2[k_1\sin(\omega_2 t - k_2 z + \delta)\cos(\omega_1 t - k_1 z)$$
$$- k_2\cos(\omega_2 t - k_2 z + \delta)\sin(\omega_1 t - k_1 z)] + b^2[k_1\cos(\omega_1 t - k_1 z + \delta)$$
$$\sin(\omega_2 t - k_2 z) - k_2\sin(\omega_1 t - k_1 z + \delta)\cos(\omega_2 t - k_2 z)]\}. \tag{4.11}$$

Denote the time average of Z_{000} by $\bar{Z}_{000}$. Then from (4.11) we quickly find

$$\bar{Z}_{000} = ab\cdot(\eta^2 k_2 - k_1)\sin\delta. \tag{4.12}$$

This formula generalizes the result of Lipkin [4] on the case of the partially polarized wave (compare with (26a) of [4]). In particular for $\eta = 0$ (the polarized wave) (4.12) yields

$$\bar{Z}_{000} = -abk_1 \sin\delta\,. \tag{4.13}$$

Thus for the linear polarization when $\delta = 0$ or $\delta = \pi$ one gets $\bar{Z}_{000} = 0$. Another simple case one can consider is the dipole electromagnetic radiation of the electric charge moving with the constant acceleration $\vec{a} = \text{const}$. Then straightforward calculations show that in this case $Z_{000} = 0$.

5. Conclusions

In the paper we have found an infinite number of conserved quantities in the vacuum Maxwell electrodynamics. These quantities are shown to be generated according to the Noether theorem. The fundamental problem which now arises concerns the physical interpretation of our results and this is evidently connected with a deeper understanding the symmetries generating the conserved quantities. We intend to consider this problem in the subsequent paper.

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