FROM SELF-DUAL YANG–MILLS FIELDS TO SELF-DUAL GRAVITY

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Reduction of $sl(\infty;C)$ self-dual Yang–Mills equations to the heavenly equations is proposed.

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Recently a great deal of interest has been devoted to symmetry reductions of the self-dual Yang–Mills (SDYM) equations.

Ward [1–4] made a conjecture that many (perhaps all) of integrable systems in physics were symmetry reductions of the SDYM equations. Therefore the latter ones constitute in a sense a "master system". Ward [1, 2] was able to show how some symmetry reductions of the SDYM equations lead to the Toda lattice equation, the sigma model equations, the Ernst equation, the Sine-Gordon equation *etc.*

Then Mason and Sparling [5] performed the symmetry reductions to obtain the Korteweg-de Vries and the Nonlinear Schrödinger equations.

It has been shown that the Euler-top and the Kovalesvskaya-top equations are also symmetry reductions of the SDYM equations [6] (see also [7] for some other interesting reductions). Mason and Newman [8] found the reduction of the SDYM equations to the self-dual Einstein equations.

Then it has been demonstrated that the appropiate symmetry reductions of the $sl(\infty)$ SDYM equations lead to the self-dual Einstein equations in the form of the first or the second heavenly equation [9–12].

The purpose of our note is to show how the natural generalization of the $sl(n;C)$ SDYM equations to the $sl(\infty;C)$ equations and then symmetry

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reductions of these new equations yield the first or the second heavenly equations as well as the evolution heavenly equations given in Refs [13–15]. Our considerations seem to justify the result of the previous paper [15], *i.e.* the statement that the heavenly equation has the Painlevé property.

The philosophy of the present note is very closely related to the one given in Refs [10–12].

[*Remark*: It seems that the approach proposed by Castro [12] leads not only to the second heavenly equation but also to the linear equation

$$\Omega_{,pq} - \frac{1}{2}(\Omega_{,p\tilde{z}} + \Omega_{,qz}) = 0$$

which can be obtained by substracting Eqs (4a) and (4b) of Ref. [12].]

We deal with $SL(n;C)$ SDYM fields in a flat (4-dimensional) complex space-time $\mathcal{M}$. It is well known [16] that one can choose the coordinates $p, q, \bar{p}, \bar{q}$ so that the SDYM equations read (the bar doesn't mean the complex conjugation!)

$$F_{pq} = 0\,, \quad F_{\bar{p}\bar{q}} = 0\,, \quad F_{p\bar{p}} + F_{q\bar{q}} = 0\,, \tag{1}$$

where $F_{ij} \in sl(n;C)$, $i,j \in \{p,q,\bar{p},\bar{q}\}$, is the Yang–Mills field strength defined by the Yang–Mills potential $A_i \in sl(n;C)$ according to the well known formulas

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]\,, \tag{2}$$

where the square bracket $[\cdot,\cdot]$ stands for the Lie bracket; ∂_i denotes the partial derivative. The system of partial differential equations (1) is known to follow from the Lax pair [7, 17]

$$\begin{aligned}(\partial_p + \lambda\partial_{\bar{q}})\Psi &= -(A_p + \lambda A_{\bar{q}})\Psi\,,\\ (\partial_q - \lambda\partial_{\bar{p}})\Psi &= -(A_q - \lambda A_{\bar{p}})\Psi\,,\end{aligned} \tag{3}$$

where λ is the spectral parameter.

The compatibility condition of Eqs (3) reads

$$[\partial_p + \lambda\partial_{\bar{q}}, A_q - \lambda A_{\bar{p}}] - [\partial_q - \lambda\partial_{\bar{p}}, A_p + \lambda A_{\bar{q}}] = [A_q - \lambda A_{\bar{p}}, A_p + \lambda A_{\bar{q}}]\,. \tag{4}$$

It is an easy matter to show that equating independent powers of λ gives (1). Now we consider the large n limit of the SDYM equations. It is known [10–12], [18, 19] that

$$sl(n \to \infty, C) \cong sdiff\ \mathcal{N}^2\,, \tag{5}$$

where $sdiff\ \mathcal{N}^2$ denotes the Lie algebra of the area preserving group of diffeomorphims of the 2-surface $\mathcal{N}^2$. Consequently if $n \to \infty$ then the potentials $A_i \in sdiff\ \mathcal{N}^2$ take the form

$$A_i = \Phi_{i,s}\frac{\partial}{\partial r} - \Phi_{i,r}\frac{\partial}{\partial s}\,, \tag{6}$$

where r, s are the coordinates on $\mathcal{N}^2$, $\Phi_i = \Phi_i(p, q, \bar{p}, \bar{q}, r, s)$ are some functions on $\mathcal{M} \times \mathcal{N}^2$ and

$$\Phi_{i,r} \equiv \frac{\partial \Phi_i}{\partial r}, \quad \Phi_{i,s} \equiv \frac{\partial \Phi_i}{\partial s}.$$

Inserting (6) into (4) and comparing the terms of the same power of λ one gets the $sl(n \to \infty, \mathcal{C})$ limit of the SDYM equations to be

$$\Phi_{p,q} - \Phi_{q,p} + (\Phi_{p,r}\Phi_{q,s} - \Phi_{p,s}\Phi_{q,r}) + \mathcal{F}(p, q, \bar{p}, \bar{q}) = 0\,, \tag{7a}$$
$$\Phi_{\bar{p},\bar{q}} - \Phi_{\bar{q},\bar{p}} + (\Phi_{\bar{p},r}\Phi_{\bar{q},s} - \Phi_{\bar{p},s}\Phi_{\bar{q},r}) + \bar{\mathcal{F}}(p, q, \bar{p}, \bar{q}) = 0\,, \tag{7b}$$
$$\Phi_{p,\bar{p}} - \Phi_{\bar{p},p} + \Phi_{q,\bar{q}} - \Phi_{\bar{q},q} + (\Phi_{p,r}\Phi_{\bar{p},s} - \Phi_{p,s}\Phi_{\bar{p},r} + \Phi_{q,r}\Phi_{\bar{q},s} - \Phi_{q,s}\Phi_{\bar{q},r}) + \mathcal{G}(p, q, \bar{p}, \bar{q}) = 0\,, \tag{7c}$$

where $\mathcal{F} = \mathcal{F}(p, q, \bar{p}, \bar{q})$, $\bar{\mathcal{F}} = \bar{\mathcal{F}}(p, q, \bar{p}, \bar{q})$ and $\mathcal{G} = \mathcal{G}(p, q, \bar{p}, \bar{q})$ are arbitrary holomorphic functions of their arguments.

We now show how Eqs (7a), (7b), (7c) can be reduced to the first and the second heavenly equations.

(i) The first heavenly equation.
Assume

$$\Phi_p = \Omega_{,p}\,, \quad \Phi_q = \Omega_{,q}\,, \quad \Phi_{\bar{p}} = 0 = \Phi_{\bar{q}}\,,$$
$$\bar{\mathcal{F}} = 0 = \mathcal{G}\,, \qquad \mathcal{F} = \mathcal{F}(p, q)\,, \tag{8}$$

where $\Omega = \Omega(p, q, r, s)$ is some holomorphic function of their arguments. Then one finds that Eqs (7b) and (7c) are satisfied trivially, and Eq. (7a) gives

$$\Omega_{,pr}\Omega_{,qs} - \Omega_{,ps}\Omega_{,qr} + \mathcal{F}(p, q) = 0\,. \tag{9}$$

If $\mathcal{F} \neq 0$ then without any loss of generality (changing eventually the coordinates p, q) one gets for (9)

$$\Omega_{,pr}\Omega_{,qs} - \Omega_{,ps}\Omega_{,qr} = 1\,. \tag{10}$$

i.e. the first heavenly equation [9].

(ii) The second heavenly equation.
Here we assume

$$\Phi_p = \theta_{,s}\,, \quad \Phi_q = -\theta_{,r}\,, \quad \Phi_{\bar{p}} = 0 = \Phi_{\bar{q}}\,,$$
$$\bar{\mathcal{F}} = 0 = \mathcal{G}\,, \quad \mathcal{F} = \mathcal{F}(p, q)\,, \quad \theta = \theta(p, q, r, s)\,. \tag{11}$$

Then Eqs (7b), (7c) are satisfied trivially and Eq. (7a) yields

$$\theta_{,rr}\theta_{,ss} - \theta^2_{,rs} + \theta_{,rp} + \theta_{,sq} + \mathcal{F}(p, q) = 0\,. \tag{12}$$

Substituting the function $\Theta = \Theta(p,q,r,s)$ by

$$\Theta = \theta + r \cdot f\,, \quad f = f(p,q) \quad \text{and} \quad f_{,p} = \mathcal{F}\,, \tag{13}$$

we arrive at the second heavenly equation for Θ [9]

$$\Theta_{,rr}\Theta_{,ss} - \Theta^2_{,rs} + \Theta_{,rp} + \Theta_{,sq} = 0\,. \tag{14}$$

Now we are going to show that the $sl(n \to \infty; C)$ limit of the SDYM equations can be reduced to the evolution heavenly equations found in Refs [13, 14]. To this end we write down Eqs (7a), (7b), (7c) in terms of the differential forms:

$$\begin{aligned}
&(d\Phi_p \wedge dp \wedge dr \wedge ds + d\Phi_q \wedge dq \wedge dr \wedge ds \\
&\quad - d\Phi_p \wedge d\Phi_q \wedge dp \wedge dq - \mathcal{F} dp \wedge dq \wedge dr \wedge ds) \wedge d\bar{p} \wedge d\bar{q} = 0\,, \quad &(15\text{a})\\
&(d\Phi_{\bar{p}} \wedge d\bar{p} \wedge dr \wedge ds + d\Phi_{\bar{q}} \wedge d\bar{q} \wedge dr \wedge ds \\
&\quad - d\Phi_{\bar{p}} \wedge d\Phi_{\bar{q}} \wedge d\bar{p} \wedge d\bar{q} - \bar{\mathcal{F}} d\bar{p} \wedge d\bar{q} \wedge dr \wedge ds) \wedge dp \wedge dq = 0\,, \quad &(15\text{b})\\
&(d\Phi_p \wedge dp \wedge dr \wedge ds + d\Phi_{\bar{p}} \wedge d\bar{p} \wedge dr \wedge ds - d\Phi_p \wedge d\Phi_{\bar{p}} \wedge dp \wedge d\bar{p}) \wedge dq \wedge d\bar{q} \\
&+ (d\Phi_q \wedge dq \wedge dr \wedge ds + d\Phi_{\bar{q}} \wedge d\bar{q} \wedge dr \wedge ds - d\Phi_q \wedge d\Phi_{\bar{q}} \wedge dq \wedge d\bar{q}) \wedge dp \wedge d\bar{p} \\
&\quad + \mathcal{G} dp \wedge dq \wedge d\bar{p} \wedge d\bar{q} \wedge dr \wedge ds = 0\,. \quad &(15\text{c})
\end{aligned}$$

Consider the following cases:

(iii) The Grant equation
Here we assume

$$p = -h_{,\Phi_p}\,, \quad \Phi_q = h_{,q}\,, \quad \Phi_{\bar{p}} = 0 = \Phi_{\bar{q}}\,,$$
$$\bar{\mathcal{F}} = 0 = \mathcal{G}\,, \quad \mathcal{F} = \mathcal{F}(p,q)\,, \tag{16}$$

where $h = h(\Phi_p, q, r, s)$ is a holomorphic function of their arguments such that $h_{,\Phi_p\Phi_p} \neq 0$. Then Eqs (15b), (15c) hold trivially and Eq. (15a) yields

$$\mathcal{F} \cdot h_{,tt} + h_{,qs}h_{,tr} - h_{,qr}h_{,ts} = 0\,, \quad t \equiv -\Phi_p\,. \tag{17}$$

For $\mathcal{F} = -1$ Eq. (17) appears to be exactly Grant's equation [13].

(iv) The evolution heavenly equation of Ref. [14]
Now we put

$$r = H_{,\Phi_q}\,, \quad \Phi_p = H_{,s}\,, \quad \Phi_{\bar{p}} = 0 = \Phi_{\bar{q}}\,,$$
$$\bar{\mathcal{F}} = 0 = \mathcal{G}\,, \quad \mathcal{F} = \mathcal{F}(p,q)\,, \tag{18}$$

where $H = H(p, q, \Phi_q, s)$, $H_{,\Phi_q \Phi_q} \neq 0$. Here, as before, Eqs (15b), (15c) are satisfied trivially. Eq. (15a) gives

$$H_{,tt} - H_{,pz} + H_{,zq} H_{,tz} - H_{,zz} H_{,tq} - \mathcal{F} \cdot H_{,zz} = 0\,, \tag{19}$$

$$t \equiv s\,, \qquad z \equiv \Phi_q\,.$$

For $\mathcal{F} = 0$ one gets the heavenly equation found in the previous paper (see Eq. (20) of Ref. [14]).

It is evident that our cases *(i)-(iv)* can be equivalently characterized by imposing the following integrable constraints

(i)

$$\Phi_i = \Phi_i(p, q, r, s)\,, \quad \Phi_{p,q} - \Phi_{q,p} = 0\,, \quad \Phi_{\bar{p}} = 0 = \Phi_{\bar{q}}\,,$$
$$\bar{\mathcal{F}} = 0 = \mathcal{G}\,, \quad \mathcal{F} = \mathcal{F}(p, q)\,. \tag{20}$$

(ii)

$$\Phi_i = \Phi_i(p, q, r, s)\,, \quad \Phi_{p,r} + \Phi_{q,s} = 0\,, \quad \Phi_{\bar{p}} = 0 = \Phi_{\bar{q}}\,,$$
$$\bar{\mathcal{F}} = 0 = \mathcal{G}\,, \quad \mathcal{F} = \mathcal{F}(p, q)\,. \tag{21}$$

(iii)

$$\Phi_i = \Phi_i(p, q, r, s)\,, \quad \Phi_{p,q} - \Phi_{q,p} = 0, \quad \Phi_{\bar{p}} = 0 = \Phi_{\bar{q}}\,, \quad \Phi_{p,p} \neq 0\,,$$
$$\bar{\mathcal{F}} = 0 = \mathcal{G}\,, \quad \mathcal{F} = \mathcal{F}(p, q)\,. \tag{22}$$

(iv)

$$\Phi_i = \Phi_i(p, q, r, s)\,, \quad \Phi_{p,r} + \Phi_{q,s} = 0\,, \quad \Phi_{\bar{p}} = 0 = \Phi_{\bar{q}}\,, \quad \Phi_{q,r} \neq 0\,,$$
$$\bar{\mathcal{F}} = 0 = \mathcal{G}\,, \quad \mathcal{F} = \mathcal{F}(p, q)\,. \tag{23}$$

[*Remark*: Without any loss of generality one can put in Eqs (7a), (7b), (7c) and (15a), (15b), (15c) $\mathcal{F} = \bar{\mathcal{F}} = \mathcal{G} = 0$. However, the cases *(i)* and *(iii)* suggest that it is convenient to keep arbitrary $\mathcal{F}$, $\bar{\mathcal{F}}$ and $\mathcal{G}$.]

We consider some interesting generalization of the case *(i)*:

(i)'

$$\Phi_i = \Phi_i(p, q, r, s)\,, \quad c \cdot \Phi_{p,q} - \Phi_{q,p} = 0\,, \quad \Phi_{\bar{p}} = 0 = \Phi_{\bar{q}}\,,$$
$$\bar{\mathcal{F}} = 0 = \mathcal{G}\,, \quad \mathcal{F} = \mathcal{F}(p, q)\,, \quad 0 \neq c \in \mathcal{C}\,. \tag{24}$$

Then there exists a function $\Omega = \Omega(p, q, r, s)$ such that

$$\Phi_p = \Omega_{,p}\,, \quad \Phi_q = c \cdot \Omega_{,q}\,. \tag{25}$$

Eqs (7b), (7c) hold trivially and Eq. (7a) yields

$$(1-c)\cdot \Omega_{,pq} + c\cdot(\Omega_{,pr}\Omega_{,qs} - \Omega_{,ps}\Omega_{,qr}) + \mathcal{F}(p,q) = 0\,. \tag{26}$$

This is exactly the equation obtained by Q-Han Park [11] (see Eq. (2.1.2) of [11]). In particular for $c = 1$ we get Eq. (9) and it is related to the topological two-dimensional sigma model; for $c = -1$ Eq. (26) is closely related to the principal chiral two-dimensional sigma model (for details see Ref. [11]).

It is of some interest to bring Eq. (26) to the evolution form (Grant's form). To this end we perform the Legendre transformation [14]

$$t = -\,\Omega_{,p} \to p = p(t,q,r,s)\,,$$

$$h = h(t,q,r,s) = \Omega(p(t,q,r,s),q,r,s) + t\cdot p(t,q,r,s)\,. \tag{27}$$

Then from (26) and (27) one gets the following evolution equation

$$\mathcal{F}\cdot h_{tt} - (1-c)h_{,tq} + c\cdot(h_{,qs}h_{,tr} - h_{,qr}h_{,ts}) = 0\,. \tag{28}$$

Of course our general considerations are, *mutatis mutandi*, also true in the case of 4-dimensional real flat Riemannian space of signature $(+,+,-,-)$.

The most fundamental question which arises in connection with our note can be stated as follows: Does there exist any continuous procedure leading from $sl(n;C)$ SDYM equations to our $sl(\infty;C)$ SDYM equations (7a), (7b), (7c)?

We intend to consider this question in the subsequent paper. [Compare this also with Refs [12, 18–20].]

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