CRITICAL PROPERTIES OF THE TWO-DIMENSIONAL GRAVITY WITH THE R^2 ACTION*

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We study the critical properties of the two-dimensional simplicial gravity with the R^2 term in the action. Changing the value of the coupling constant of this term we observe a sharp transition between the surfaces with low and high Haussdorff dimension. This transition seems not to be leading to a phase transition. For the whole range of the coupling constant a value of γ_{str} is consistent with the pure gravity value $-1/2$.

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1. Introduction

The Einstein action for pure gravity with cosmological term reads

$$S(g) = \int d^d x \sqrt{g}(KR + \Lambda) \tag{1}$$

in which g_{ij} is the metric tensor, R is the scalar curvature and K, Λ are two coupling constants. The cosmological constant Λ multiplies the volume

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element. In two dimensions classical gravity is trivial because the scalar curvature term $\int \sqrt{g}R$ is topological (Gauss–Bonnet theorem) and equal $4\pi\chi$, χ is the Euler characteristic of the universe. In quantum case, however, even two dimensional gravity is non trivial because large quantum fluctuations may change the genus of the surface and partition function hence involves a sum over all genera. Thus for the partition function one must take :

$$Z(\beta,\gamma) = \sum_h \int Dg e^{(-\beta A+\gamma\chi)}, \tag{2}$$

where sum over topologies is represented by the summation over h, the number of handles of the surface, A is the area of the surface, equal to $\int \sqrt{g}$, and the Euler characteristic $\chi = 2 - 2h$ is equal to $(1/4\pi) \int \sqrt{g}R$.

It is convenient in the following discussion to consider $2d$ quantum gravity with an ultraviolet cut-off and we will consider surfaces entering in the path integral as triangulated surfaces built out of equilateral triangles (see [1, 2]). Let $o(i)$ be the number of triangles which meet at vertex i. In this case the discrete analog of $\sqrt{g}$ is $o(i)/3$ and the discrete counterpart to the Ricci scalar R is $R_i = 2\pi(6 - o(i))/o(i)$ and is negative (positive) at vertex i depending on whether the number $o(i)$ is more (less) then six. If we call N_0, N_1, N_2 the total number of vertices, edges and faces respectively, due to topological relations $N_2 - N_1 + N_0 = \chi$, $3N_2 = 2N_1$ and $2N_1 = \sum_i o(i)$, one has ([3]):

$$\begin{aligned} \int Dg\sqrt{g} \quad &\to N_2\,, \\ \int Dg\sqrt{g}R &\to \sum_i 4\pi\left(1 - \frac{o(i)}{6}\right) = 4\pi\chi\,, \end{aligned} \tag{3}$$

The last equation coincides with the continuum definition. The summation over all random triangulations is thus a discrete analog of the integral $\int Dg$ over all possible geometries

$$\sum_{\text{genus } h} \int Dg \to \sum_{\text{random triangulations}} \tag{4}$$

The $2d$ quantum gravity in the discretized formulation is equivalent to the large-N matrix models and can be solved analytically (*cf.* [6]) and the summation over genera in 2 can be given a non–perturbative meaning.

In $4d$ the classical gravity, when quantized with only the Einstein action, is nonrenormalizable. It is possible to make the theory renormalizable by adding the term $\lambda \int d^dx\sqrt{g}R^2$ to the pure gravity action, paying the price of appearance of unphysical effects in such a theory (see [4] and references therein). In $2d$ this term corresponds to the irrelevant operator and naively one would expect that it should have no effect from the renormalization group point of view. It is not clear if this remains true on the non–perturbative level. One can expect dramatic changes of behaviour for relatively big values of the coupling constant λ. For large positive values of the coupling constant the system will probably favour configuration with close to zero local curvature, with the Haussdorff dimension [5] tending to 2. By the entropy arguments one can expect in this limit a phase similar to that of branched polymers. For large negative values of the coupling constant one expects to get "black hole" configurations with large negative local curvature. The Haussdorff dimension should be also large. The problem we try to address in this paper is whether this complicated behaviour can be attributed to one phase of the model or if there is a non-perturbative phase transition.

Discrete analog of the R^2 term is [3]:

$$\lambda \int \sqrt{g} R^2 \to S(t) = \frac{4\pi^2\lambda}{3} \sum_i \frac{(6 - o(i))^2}{o(i)} = t\left(\sum_i \frac{36}{o(i)} - 3N_2 - 12\chi \right), \quad (5)$$

where $t = 4\pi^2\lambda/3$. With such a term the model is no longer solvable analytically and the numerical methods remain the only source of information about its properties.

In the following we shall restrict ourselves to surfaces with a topology of a sphere. The numerical study will be made for surfaces with a fixed area, *i.e.* with a fixed number of faces N_2. The thermodynamic properties of such a system can be obtained from the partition function

$$Z_N(t) = \sum_{\text{triangulations}} \exp(-S(t)), \quad (6)$$

where the sum extends over all triangulations of the spherical surface with N faces.

The easiest quantity to measure is $\langle S \rangle = \langle S(t) \rangle / t N_2$. As usual we expect a signal of a possible phase transition to come from the maximum of the specific heat $C = -d\langle S \rangle / dt$. On a random lattice we measure also the distribution of geodesic distances d. The geodesic distance is defined as the length of the shortest path between two points following the edges of triangulation. The distribution of d will give us information about the shape of the surface. The mean distance is related to the volume of the universe through a relation [5]:

$$\langle d\rangle \sim (N_2)^{1/H}, \tag{7}$$

H is the Haussdorff dimension of the surface, one of critical exponents for our system.

The fractal and selfsimilar structure of $2d$ quantum gravity is connected with the string susceptibility exponent $\gamma_{\rm str}$, which is responsible for the continuum properties of the model. One expects that in the continuum limit the model can be described as a conformal field theory. In this case the entropy function for surfaces with the area A and the Euler character χ is [7]:

$$\mathcal{N}_\chi(A) = A^{\chi(\gamma_{\rm str}-2)/2-1} e^{\mu_0 A}(1+\cdots), \tag{8}$$

where

$$\gamma_{\rm str} = \frac{1}{12}\left(c - 1 - \sqrt{(c-1)(c-25)}\right), \tag{9}$$

c is the central charge of the matter sector and A is the area of the surface. In the case of pure $2d$ quantum gravity the total area of the surface is proportional to N_2 and $c = 0$. (8) gives for the $2d$ gravity canonical partition function:

$$Z(N_2,\chi) \sim e^{\mu_c N_2} N_2^{\gamma_{\rm str}(\chi)-3}. \tag{10}$$

For $\chi = 2$ (sphere) $\gamma_{\rm str} = -\frac{1}{2}$. Value of $\gamma_{\rm str}$ can be used as a probe of the matter content of the theory. We measure this quantity for all studied values of the coupling constant t using the method developed in [8].

2. Numerical simulations

The simulations were done on lattices of size ranging from 500 to 5000 triangles and using a standard "link-flip" [2] algorithm to update the geometry. The runs were of the order 10^5 sweeps long, each sweep consisted of N_2 performed link flips. We were doing measurements for each $10^{\rm th}$ sweep. We used a standard Metropolis algorithm with the Boltzmann factor $\exp(-S(t))$, where t was the coupling constant. The range of a coupling constant, where we investigated the structure of the surfaces extended from about -1.5 to 1.5. Outside this range the acceptance of the algorithm fell down and we were not able to get results of good quality.

2.1. Measurements of $\langle S\rangle$

The dependence of $\langle S\rangle$ on t is shown in Fig. 1. There is a clear maximum of the derivative of this quantity for $t \propto -0.25$. For increasing volume we observe very small size dependence. In fact for all values of t we were able to fit the size dependence with the formula

$$\langle S\rangle(N_2) = \langle S\rangle(\infty) + \frac{\text{const.}}{N_2}. \quad (11)$$

Fig. 1 shows the extrapolated values $\langle S\rangle(\infty)$ *vs.* t. Similar dependence was observed for the specific heat C. The extrapolated curve $C(N_2 = \infty)$ is shown in Fig. 2.

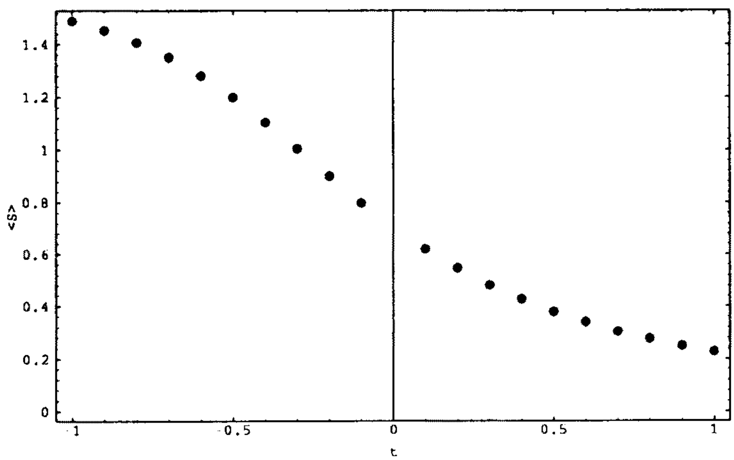

Fig. 1. The coupling constant dependence of the extrapolated to the infinite volume limit mean squared curvature $\langle S(N_2 = \infty)\rangle$.

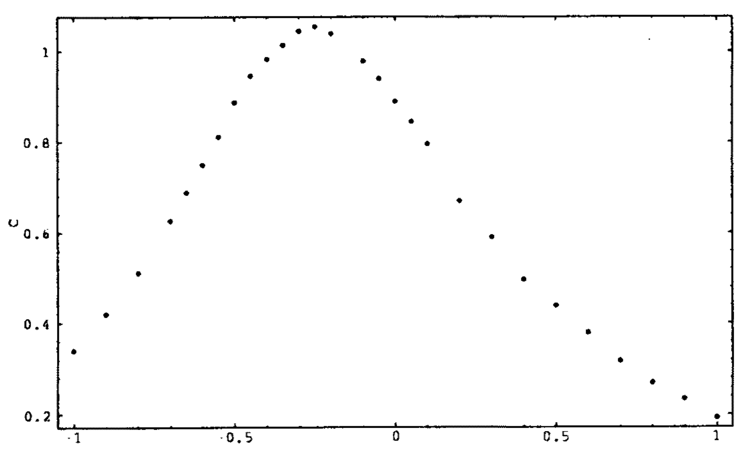

Fig. 2. The coupling constant dependence of the specific heat $C(N_2 = \infty)$.

From our measurements we cannot exclude the existence of a singularity for small negative values of $t \propto -0.25$. In the range of volumes we studied the specific heat peak moved very little and its height remained almost constant. This can mean that if there is a phase transition near this point it would have to be of high order. To clarify this point it would be necessary to study much larger systems.

2.2. Measurements of the mean distance

To find the mean distance in the universe we randomly choose a sample set of vertices and we find the distribution of geodesic distances around each vertex. From such distribution we extract the first and the second moments of the distribution. These are just the mean distance and the mean dispersion of the distance. Figs 3 and 4 show the dependence of the mean distance $\langle d \rangle$ and the dispersion of distances $D(d)$ as a function of the coupling t.

Also here we observe a change in the behaviour of the measured quantity. Both the mean distance and the dispersion of mean distance change particularly fast in the vicinity of $t = -0.3$. For positive values of the coupling constant we get cigar-like surfaces, with big mean distance and big dispersion of the mean distance. For negative values of the coupling we have surfaces with small mean distance and small dispersion of the mean distance. Vertices are close together in this case. From the volume and the coupling constant dependence on the mean distance we can extract using (7) the coupling constant dependence of the Haussdorff dimension H, which is one of critical exponents of the model. The dependence of H on t is plotted in Fig. 5. It shows a clear change in behaviour close to -0.3 but it looks to be rather continuous than singular. For large positive values of the coupling constant H seems to approach 2, which is the Haussdorff dimension of flat surfaces. For negative values of the coupling H grows rapidly. For the pure gravity we get the Haussdorf dimension $H = 3.05 \pm 0.05$.

2.3. Measurements of the $\gamma_{\rm str}$ exponent

The first attempts to measure the exponent $\gamma_{\rm str}$ used a grand canonical updating [10], which generated directly surfaces with varying area, according to the distribution (8). The disadvantage of this method is that one had to perform simulations for a whole range of N_2 and $\gamma_{\rm str}$ appeared as a subleading correction to the determination of the critical point μ_c. In this paper we used another method proposed and developed in [8]. In the case of spherical topology a closed, non-intersecting loop along the links will separate the surface in two parts. The smallest such loop will be of length 3. If the smaller part is different from a single triangle we call it a "minimum

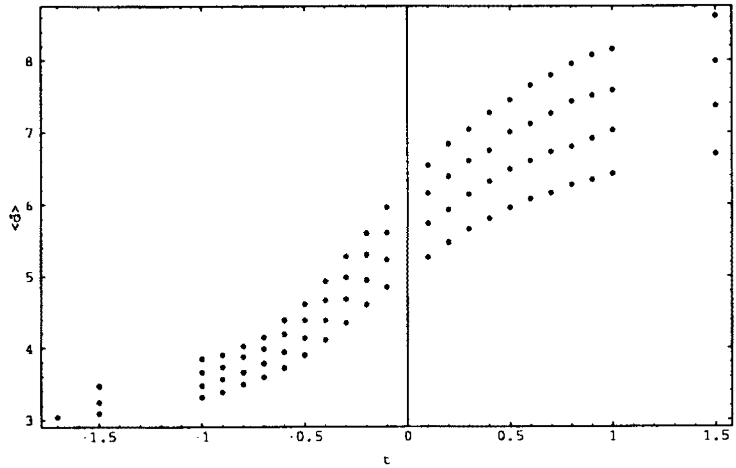

Fig. 3. The coupling constant dependence of the mean distance for four volumes of configuration (550, 700, 850 and 1000 triangles).

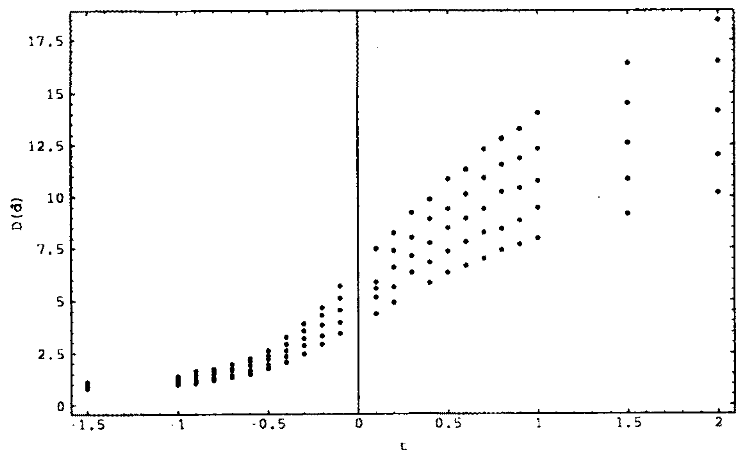

Fig. 4. The coupling constant dependence of the mean dispersion of the distance for five volumes of configurations (550, 700, 850, 1000 and 1200 triangles).

neck baby universe", abbreviated "minbu". If we assume the distribution (8) for surfaces, one can prove ([8]) that the average number of minbu's of area B on a closed surface of spherical topology and with area N is given by :

$$n_N(B) \sim (N-B)^{\gamma_{str}-2} B^{\gamma_{str}-2} + O\left(\frac{1}{B}\right) . \tag{12}$$

The above formalism is well suited for numerical simulations. Now γ_{str} does not appear as a subleading correction to μ_c and one can use the canonical

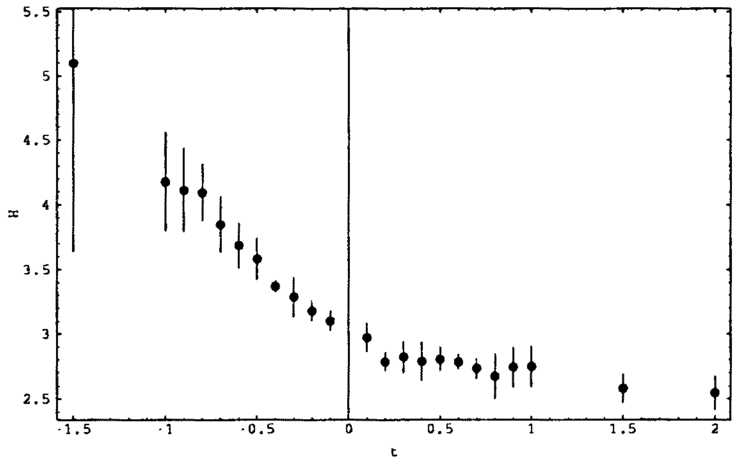

Fig. 5. The coupling constant dependence of the Haussdorff dimension.

Monte Carlo simulations, performing measurements of the distribution of minbu's in one very long run for each value of t.

In order to find if there is a minbu built on a given edge we scan all the links leaving the two ends of the chosen link and look for a common vertex of a pair of links. If we find such a vertex, which together with the two vertices of the chosen link forms a triangle not belonging to the surface, we succeeded in finding the neck of a minbu. We measure then the area of the two subsurfaces and call the smaller one the area B of the minbu.

In order to extract γ_{str} from the distribution of minbus we follow the method used in [8]. We introduce a lower cut-off B_0 in the data, because (12) is only asymptotically correct. We look for the range of B_0, where γ_{str} does not depend on the lower cut-off. We fitted the distribution of minbus to the formula:

$$\log(n_{N_2}) = A + (\gamma_{str} - 2)\log\left(B\left(1 - \frac{B}{N_2}\right)\right) + \frac{\mathrm{const}}{\mathrm{B}}\,. \tag{13}$$

In the case of near zero and negative values of the coupling constant the term $O(1/B)$ does not change significantly the value of γ_{str}. When the coupling constant grows, $O(1/B)$ becomes more important. This can be illustrated in Fig. 6, where we plot logarithm of distributions of the baby universe sizes as a function of $\log(B(1-B/N_2))$ for a system with $N = 5000$ triangles. We can observe the qualitative change of the distributions for bigger positive t. The Table I shows the coupling constant dependence on γ_{str}. In the whole studied range the fitted value of γ_{str} is consistent with the value $-\frac{1}{2}$. It seems that γ_{str} becomes more negative, about -0.55 in the neighbourhood of $t = -0.2$ but this effect is on the edge of the measurement error.

TABLE I

The coupling constant dependence of γ_{str}

t	γ_{str}
0.9	-0.46 ± 0.08
0.6	-0.50 ± 0.06
0.3	-0.48 ± 0.06
0.1	-0.52 ± 0.06
0.0	-0.51 ± 0.04
-0.1	-0.55 ± 0.04
-0.2	-0.57 ± 0.02
-0.4	-0.52 ± 0.02
-0.6	-0.50 ± 0.02
-0.8	-0.50 ± 0.02

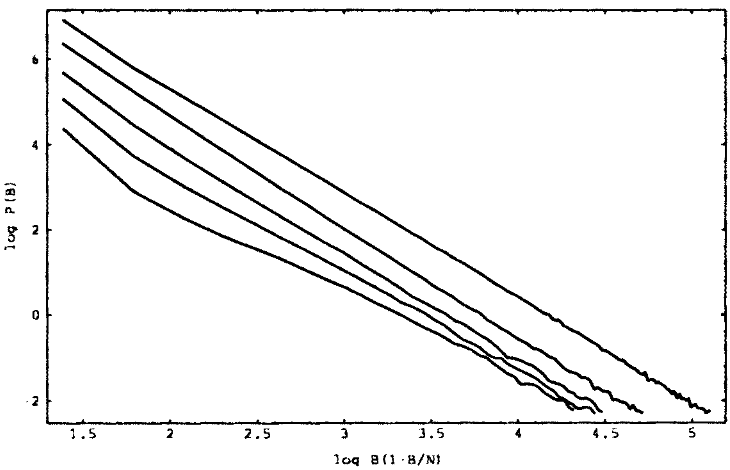

Fig. 6. The distributions of baby universes for four values of the coupling constant (from the top of the figure these are -0.7, -0.1, 0.3, 0.6, 0.9.) for $N_2 = 5000$.

Distributions in Fig. 6 are normalized by the number of measurements. Their relative position represents the dependence of the number of minbus on t. We see from the plot that for large t this number drops significantly and the determination of γ_{str} becomes more difficult.

3. Discussion

We investigated a model of the $2d$ dynamically triangulated gravity with the discretized version of the R^2 term added to the pure gravity action. We studied the model with varying value of the coupling constant t of this term. We observe a sharp transition between the large positive t, where the Haussdorff dimension of the system approaches 2, to the large negative t, where the fluctuations of the local curvature become large and H becomes large too. For all the studied values the measured value of the critical exponent $\gamma_{\rm str}$ is consistent with the pure gravity value $-\frac{1}{2}$.

We observe a sharp change of the behaviour at $t = -0.25$. We tried to check if this change is connected with a phase transition, but for the range of the sizes studied the specific heat seems to approach a limiting distribution without a singularity. This leads us to interpret our result as indicating that for all values of t we get a system, belonging to the same universality class as pure gravity. This is in agreement with the standard renormalization group perturbative arguments, where the R^2 term corresponds to the irrelevant operator. Other critical indices, like a Haussdorff dimension H are, however, non-trivial functions of t.

For large values of t one would nevertheless expect a transition to phases with different value of $\gamma_{\rm str}$. Unfortunately we were not able to study this limit because

(i) the acceptance rate of the standard "flip" algorithm falls down in this limit and

(ii) the total number of minbus becomes small for large t.

In effect we were not able to perform numerical simulations with reasonable accuracy.

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