BILOCAL THEORY REVISITED*

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To the memory
of Jerzy Rayski

It is shown that the *ad hoc* assumptions of Yukawa's bilocal theory can be derived from first principles if one takes into account the full symmetry group of internal and external symmetries represented in a complex space.

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1. Introduction

Many years ago Professor Jerzy Rayski was deeply engaged in Yukawa's so called "bilocal theory" [1]. It was another one of many preceding attempts (*cf. e.g.* [2–4]) to introduce internal structure of particles in order to avoid all the trouble connected with the pointlike idealization.

The basic assumption of Yukawa's theory [5] is that the particle is described by two four-vectors x and y, one of them (x) representing the space-time coordinates of the "centre" of the particle, the other (y) being the distance from the centre. To make this distance space-like and bounded in a covariant way it was necessary to introduce *ad hoc* a mysterious time-like unit vector r pointing towards the future

$$r_\mu r^\mu = r_1^2 + r_2^2 + r_3^2 - r_0^2 = -1\,, \qquad r_0 > 0 \tag{1.1}$$

and to restrict y by means of the two relativistic conditions

$$y_\mu y^\mu = c_1 > 0\,, \qquad y_\mu r^\mu = c_2\,. \tag{1.2}$$

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In a particular frame of reference in which $r_i = 0$, $i = 1, 2, 3$; $r_0 = 1$, equation (1.2) takes the form

$$y_1^2 + y_2^2 + y_3^2 = c_1 + c_2^2 . \tag{1.3}$$

For $c_1 + c_2^2 > 0$ this equation represents a sphere S^2 and the internal degrees of freedom of the particle are restricted to this sphere.

The next step was to consider fields depending on the two four-vectors x and y, the second restricted by the two conditions (1.2). The second order covariant differential operator consisted of the two d'Alembert operators $\Box_x$ and $\Box_y$, the second restricted by means of equations (1.2) to essentially its spherical part. On the basis of this equation a field theory was developed.

This theory, which was very popular in the early fifties, was later abandoned. One of the reasons was of technical character. The constant vector r distinguished a direction in space-time and thus spoiled the invariance of the theory. Another reason was of general character. The introduction of bilocality had no justification in basic physical principles.

In this paper we shall show that the *ad hoc* assumptions of the bilocal theory can be derived as unique consequences of simple postulates if one takes into account internal symmetries. This is a rather curious fact because the original version of bilocal theory is based on Poincaré symmetry only. Let us state the postulates:

A. The physical symmetry is the direct product of external conformal or Poincaré and internal unitary symmetry $P_4 \times \mathrm{SU}(m) \subset \mathrm{SU}(2,2) \times \mathrm{SU}(m)$.
B. The configuration space of the particle is that orbit in $\mathbf{C}^{4m}$ of the physical symmetry $P_4 \times \mathrm{SU}(m)$ or $\mathrm{SU}(2,2) \times \mathrm{SU}(m)$ which contains Minkowski space.

The second postulate is a consequence of two independent ideas. The first is a generalization to both kinds of symmetries of an old idea (*cf. e.g.* [6], [7]) to consider the other homogeneous spaces

$$P_4/H = P_4/\mathrm{SO}(3.1) \times \mathrm{SO}(3.1)/H , \qquad H \subset \mathrm{SO}(3.1)$$

as possible candidates for the configuration space of particles in analogy to the conventional case $M_4 = P_4/\mathrm{SO}(3.1)$. The second is the idea to describe physical laws in the domain of elementary particle physics in a complex space, providing in this way a natural geometrical basis for both kinds of symmetries, external and internal [8].

These two ideas combined in postulate B select, as we shall see, one particular out of the many homogeneous spaces of the direct product $P_4 \times \mathrm{SU}(m)$. This particular space is a $P_4 \times \mathrm{SU}(m)$-invariant generalization of the bilocal model in which the mysterious vector r is an integral part of the theory.

The essential results concerning bilocal theory do not depend on the integer m in SU (m). We keep, therefore, m arbitrary in order to cover such particular cases as U(1)×SU(2), SU(3)×SU(2)×U(1), *etc.*

The first postulate does not need justification. It is just a statement of experimental facts.

2. Homogeneous spaces of GL(4,C)×GL(m),C)

In order to find the homogeneous spaces of the direct products $P_4 \times \mathrm{SU}(m) \subset \mathrm{SU}(2,2) \times \mathrm{SU}(m)$, we need, according to postulate B, a common geometrical basis for internal as well as external symmetries. Such a common basis is provided by the space of all complex $4 \times m$ matrices $\xi = \{\xi_{a\alpha}\}$, $a = 1,\ldots,4$; $\alpha = 1,\ldots,m$; $\xi \in \mathbf{C}^{4m}$. This space is the smallest linear irreducible representation space of $\mathrm{GL}(4,\mathbf{C}) \times \mathrm{GL}(m,\mathbf{C})$ on which this direct product acts effectively according to

$$\xi \to \xi' = g\xi h^{-1}, \quad g \in \mathrm{GL}(4,\mathbf{C}), \quad h \in \mathrm{GL}(m,\mathbf{C}). \tag{2.1}$$

(The first factor acts on the columns, the second on the rows of ξ.)

The physical symmetries $P_4 \times \mathrm{SU}(m) \subset \mathrm{SU}(2,2) \times \mathrm{SU}(m) \subset \mathrm{GL}(4,\mathbf{C}) \times \mathrm{GL}(m,\mathbf{C})$ appear in this picture as subgroups of $\mathrm{GL}(4,\mathbf{C}) \times \mathrm{GL}(m,\mathbf{C})$ and, in particular, the external Poincaré symmetry appears as a subgroup $P_4 \subset \mathrm{SU}(2,2) \subset \mathrm{GL}(4,\mathbf{C})$ of the conformal group SU(2,2).

With respect to $\mathrm{GL}(4,\mathbf{C}) \times \mathrm{GL}(m,\mathbf{C})$ the space $\mathbf{C}^{4m}$ decomposes into submanifolds consisting of matrices of equal rank

$$\mathbf{C}^{4m} = \bigcup_{k=0}^{4} O_k^{(4,m)}, \quad \cap O_k^{(4,m)} O_l^{(4,m)} = \delta_{kl} O_k^{(4,m)}, \tag{2.2}$$

where

$$O_k^{(4,m)} = \{\xi \in \mathbf{C}^{4m} : \ \mathrm{rank}\ \xi = k\}, \quad k = 0,1,2,3,4. \tag{2.3}$$

These submanifolds are homogeneous spaces of $\mathrm{GL}(4,\mathbf{C}) \times \mathrm{GL}(m,\mathbf{C})$

$$O_k^{(4,m)} = \frac{\mathrm{GL}(4,\mathbf{C}) \times \mathrm{GL}(m,\mathbf{C})}{H_k^{(4,m)}}, \tag{2.4}$$

where $H_k^{(4,m)}$ is the isotropy group of an arbitrary point in $O_k^{(4,m)}$ *e.g.* the point

$$\begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.5}$$

Another description of the manifolds $O_k^{(4,m)}$ is as (non-trivial) fibre bundles with $G_k^4 \times G_{m-k}^m$ as base and fibres homeomorphic to $\mathrm{GL}(k, \mathbf{C})$, where G_{m-k}^m and G_k^4 are Grassmann manifolds (G_k^m is the set of all k-dimensional hypersurfaces in $\mathbf{C}^m$, $G_{m-k}^m \sim G_k^m$). This description arises if one considers $\mathbf{C}^{4m}$ as the set of homomorphisms $\mathrm{Hom}(\mathbf{C}^m \to \mathbf{C}^4)$ of $\mathbf{C}^m$ into $\mathbf{C}^4$ and is very covenient if we search for those homogeneous spaces of $\mathrm{GL}(4, \mathbf{C}) \times \mathrm{GL}(m, \mathbf{C})$ which admit a projection onto Minkowski space (for details *cf.* [9]). Indeed, it is well known that the complex Grassmann manifold G_2^4 is homeomorphic with the complex compactified Minkowski space $M_4^{\mathbf{C}}$. Thus we have shown that the only submanifold of $\mathbf{C}^{4m}$ which is a homogeneous manifold of $\mathrm{GL}(4, \mathbf{C}) \times \mathrm{GL}(m, \mathbf{C})$ and, at the same time, admits a projection onto the complex compactified Minkowski space $M_4^{\mathbf{C}}$ is the submanifold $O_2^{(4,m)}$.

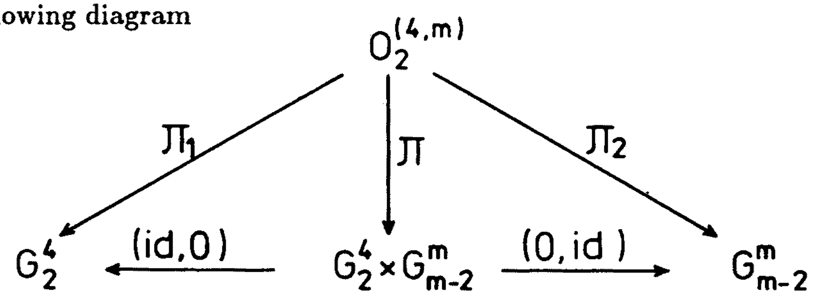
To express these facts in local coordinates, we remind that on any $\xi \in O_2^{(4,m)}$ there exists a 2×2 matrix k with non-vanishing determinant $\det K \neq 0$ and that all 3×3 submatrices have vanishing determinants. Without loss of generality we consider a neighbourhood in which K stands in the upper left corner of ξ

$$\xi = \begin{pmatrix} K & B \\ A & Y \end{pmatrix} . \tag{2.6}$$

If $\xi \in O_2^{(4,m)}$ *i.e.* if ξ is of rank 2 we have, according to elementary formulas from linear algebra, $A = aK$, $Y = aB$, $B = Kb$, $Y = Ab$ (two last rows are linear combinations of the first two and $m-2$ last columns are linear combinations of the first two). From these relations it follows that

$$Y = aKb = aB = Ab = AK^{-1}B . \tag{2.7}$$

Here the elements of the 2×2 matrix a and of the $2 \times (m-2)$ matrix b are local coordinates of the Grassmann manifolds G_2^4 and G_{m-2}^m of the base and the elements of K are coordinates of the fibre. Two alternative descriptions of the fibre bundle are possible if we consider G_2^4 as base and B as fibre or G_{m-2}^m as base and A as fibre. The three possibilities are illustrated on the following diagram



(2.8)

The homeomorphism of G_2^4 and $M_4^{\mathbb{C}}$ can now be expressed as

$$z_\mu = x_\mu + iy_\mu = \frac{i\lambda}{2}\,\mathrm{Tr}\,a\sigma_\mu$$
$$a = \frac{1}{i\lambda} z_\mu \tilde{\sigma}^\mu\,, \tag{2.9}$$

where σ_i, $i = 1,2,3$ are Pauli matrices, $\sigma_0 = \mathbb{1}_2$ is the 2×2 unit matrix, $\tilde{\sigma}_0 = -\sigma_0$, $\tilde{\sigma}_i = \sigma_i$. The parameter λ with dimension of length must be introduced from dimensional reasons because $a = AK^{-1}$ is dimensionless.

An important fact may be noted here that the Grassmann coordinates a do not depend on the columns (*i.e.* on the Greek index α) and the Grassmann coordinates b do not depend on the rows of $\xi = \{\xi_{a\alpha}\}$ (*i.e.* on the Latin index a). This is obvious if we rewrite relations (2.7) explicitly

$$\xi_{a''\alpha} = a^{a'}_{a''}\xi_{a'\alpha}\,, \quad \xi_{a\alpha''} = \xi_{a\alpha'}b^{\alpha'}_{\alpha''}\,. \tag{2.10}$$

Here a' and α' are indices of the rows and columns of the 2×2 matrix K with determinant $\det K \neq 0$ (the choice of this matrix determines the coordinate neighbourhood). The remaining indices are denoted by a'' and α''. In the particular neighbourhood illustrated by the matrix (2.6) we have a', $\alpha' = 1,2$, $a'' = 3,4$, $\alpha'' = 3,4,\ldots,m$.

3. Restriction of symmetry to SU(2,2)×SU(m) and P$_4$×SU(m)

If we restrict the general GL(4, $\mathbb{C}$)$\times$GL(m, $\mathbb{C}$) symmetry to the physical SU$(2,2) \times$ SU(m) and $P_4 \times$ SU(m) symmetries there appear invariant and the homogeneous manifold $O_2^{(4,m)}$ of GL(4, $\mathbb{C}$)$\times$GL(m, $\mathbb{C}$) decomposes into subdomains being homogeneous spaces of the restricted groups.

There are two independent SU$(2,2) \times$ SU(m) invariants namely Tr r and Tr r^2, where

$$r_{\alpha\beta} := \xi^*_{\alpha a}\gamma_4^{\dot{a}b}\xi_{b\alpha}\,, \tag{3.1}$$

and

$$\gamma_4 = -\begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} \tag{3.2}$$

is the metric tensor of SU$(2,2)$ (the metric tensor od SU(m) is $\mathbb{1}_m$). If we express in (3.1) the dependent elements Y of the matrix ξ by the independent elements z, K, B we obtain

$$\mathrm{Tr}\; r = \frac{2}{\lambda} y_\mu r^\mu\,,$$
$$\mathrm{Tr}\; r^2 = \frac{4}{\lambda^2}\{-\tfrac{1}{2}y_\mu y^\mu r_\nu r^\nu + (y_\mu r^\mu)^2\}\,, \tag{3.3}$$

where

$$r_\mu = -\xi^*_{\alpha a'}(\tilde{\sigma}_\mu)^{\dot{a}'\dot{\beta}'}\xi_{b'\alpha}\,, \qquad a', b' = 1,2\,. \tag{3.4}$$

From (3.3) it is clear that instead of Tr r and Tr r^2 we can, equivalently, consider the two independent invariants $y_\mu r^\mu$ and $y_\mu y^\mu r_\nu r^\nu$ and decompose the manifold $O_2^{(4,m)}$ into a two parameter family of homogeneous manifolds of SU(2,2) $\times$ SU (m) described by the equations

$$y_\mu r^\mu = \text{const}\,, \quad y_\mu y^\mu r_\nu r^\nu = \text{const}\,. \tag{3.5}$$

If we further restrict the symmetry to $P_4 \times$ SU (m) another invariant appears (*cf.* (4.5)) namely $y_\mu y^\mu$ and the manifold $O_2^{(4,m)}$ decomposes into a three parameter family of homogeneous spaces of $P_4 \times$ SU (m)

$$y_\mu y^\mu + c_{11} = y_\mu r^\mu + c_{12} = r_\mu r^\mu + c_{22} = 0\,. \tag{3.6}$$

It is seen that the complex variables $\xi_{a'\alpha}$, $a' = 1,2$; $\alpha = 1,\ldots,m$, enter equations (3.5) and (3.6) only by the intermediary of the vector r_μ defined by equation (3.4). This vector is time-like and points towards the future. Indeed, from (3.4) we have ($\tilde{\sigma}_0 = -\mathbb{1}_2$)

$$r_0 = \sum_{a'=1}^{2}\sum_{\alpha=1}^{m} \mid \xi_{a'\alpha} \mid^2 \tag{3.7}$$

and one also easily shows that

$$r_\mu r^\mu = -\sum_{\alpha=1}^{m}\sum_{\beta=1}^{m}\left|\det\begin{pmatrix}\xi_{1\alpha} & \xi_{1\beta}\\ \xi_{2\alpha} & \xi_{2\beta}\end{pmatrix}\right|^2. \tag{3.8}$$

It follows

$$r_\mu r^\mu < 0\,, \qquad r_0 > 0\,. \tag{3.9}$$

The equality signs in (3.9) are excluded on $O_2^{(4,m)}$.

It is seen that conditions (1.2) of the bilocal theory follow from the two postulates A and B described in Section 1. Moreover, the "mysterious" vector r of this theory finds his natural interpretation in equation (3.4) and is an integral part of the scheme based on postulates A and B. This vector is normalized according to (1.1) on the homogeneous spaces corresponding to the constant $c_{22} = 1$. One easily checks that the three constants appearing in (3.6) must satisfy the condition

$$\det c = \det\begin{pmatrix}c_{11} & c_{12}\\ c_{12} & c_{22}\end{pmatrix} = c_{11}c_{22} - c_{12}^2 < 0 \tag{3.10}$$

in order that the three equations (3.6) are compatible. Due to $c_{22} > 0$, this is always the case if $c_{11} < 0$ *i.e.* if y is a space-like vector.

It is a curious fact that relations (1.2) of the bilocal theory which involve only the external Poincaré symmetry follow from postulates involving internal symmetries. The internal symmetries manifest themselves in the dependence of the Minkowski four-vector r on the complex variables $\xi_{a'\alpha}$, $a' = 1, 2$, $\alpha = 1, \ldots, m$.

4. The transformation properties

It remains to consider the transformation character of the quantities x, y and r which follows from the assumed transformation character of ξ. It is clear that all three quantities are SU(m)-scalars. Let us consider their transformation character with respect to SU$(2,2)$. It is sufficient to consider infinitesimal transformations. The generators of SU$(2,2)$ in $\mathbf{C}^4$ are

$$d = -\tfrac{i}{2}\gamma^5\,, \quad p_\mu = i\lambda^{-1}\gamma_-\gamma_\mu\,, \quad k_\mu = -i\lambda\gamma_+\gamma_\mu$$
$$m_{\mu\nu} = \tfrac{i}{4}[\gamma_\mu, \gamma_\nu]\,, \tag{4.1}$$

where d, p_μ, k_μ and $m_{\mu\nu}$ denote dilatations, translations, special conformal transformations and rotations. The γ_μ, $\mu = 0, 1, 2, 3$ are Dirac matrices satisfying $\gamma_i^* = \gamma_i$, $\gamma_0^* = -\gamma_0$ and $\gamma_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma_5)$, $\gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_0$. According to the relation $A = aK$, the elements of $a = AK^{-1}$ are represented in the neighbourhood det $K \neq 0$ as ratios of determinants

$$a^1_{a''} = (\det K)^{-1} \det \begin{pmatrix} \xi_{a''\alpha} & \xi_{a''\beta} \\ \xi_{2\alpha} & \xi_{2\beta} \end{pmatrix},$$
$$a^2_{a''} = (\det K)^{-1} \det \begin{pmatrix} \xi_{1\alpha} & \xi_{1\beta} \\ \xi_{a''\alpha} & \xi_{a''\beta} \end{pmatrix},$$
$$K = \begin{pmatrix} \xi_{1\alpha} & \xi_{1\beta} \\ \xi_{2\alpha} & \xi_{2\beta} \end{pmatrix}, \tag{4.2}$$

where α and β are indices of two arbitrary columns of ξ. It may be noted that, on $O_2^{(4,m)}$, the ratios (4.2) do not depend upon the choice of these columns (*cf.* *e.g.* the remark after formula (2.9)).

One can easily convince oneself that transformations (4.1), acting, according to (2.1), on the first, Latin, index of $\xi_{a\alpha}$, induce the following infinitesimal transformations of the local complex coordinates of z

$$\begin{aligned} dz_\mu &= -iz_\mu\,, \\ p_\mu z_\lambda &= -ig_{\mu\lambda}\,, \\ k_\mu z_\lambda &= ig_{\mu\lambda}z_\nu z^\nu - 2iz_\mu z_\lambda\,, \\ m_{\mu\nu} z_\lambda &= -ig_{\mu\lambda}z_\nu + ig_{\nu\lambda}z_\mu\,. \end{aligned} \tag{4.3}$$

Similarly we obtain for r, which is a bilinear hermitean form in the $\xi_{a'\alpha}$, $a' = 1,2$, $\alpha = 1,\dots,m$,

$$\begin{aligned} dr_\mu &= ir_\mu\,, \\ p_\mu r_\lambda &= 0\,, \\ k_\mu r_\lambda &= 2i(g_{\mu\lambda}x_\nu r^\nu + x_\mu r_\lambda - x_\lambda r_\mu - \varepsilon_{\mu\lambda\rho\nu}y^\nu r^\lambda)\,, \\ m_{\mu\nu}r_\lambda &= -ig_{\mu\lambda}r_\nu + ig_{\nu\lambda}r_\mu\,, \end{aligned} \tag{4.4}$$

For interpretation and comparison with bilocal theory it is convenient to split (4.3) into the real and imaginary parts. Due to the fact that the infinitesimal generators act in the same way on $z_\mu = x_\mu + iy_\mu$ as on the complex conjugate $\bar{z}_\mu = x_\mu - iy_\mu$, we obtain

$$\begin{aligned} dx_\mu &= -ix_\mu\,, \\ p_\mu x_\lambda &= -ig_{\mu\lambda}\,, \\ k_\mu x_\lambda &= ig_{\mu\lambda}(x_\nu x^\nu - y_\nu y^\nu) - 2i(x_\mu x_\lambda - y_\mu y_\lambda)\,, \\ m_{\mu\nu}x_\lambda &= -ig_{\mu\lambda}x_\nu + ig_{\nu\lambda}x_\mu\,, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} dy_\mu &= -iy_\mu\,, \\ p_\mu y_\lambda &= 0\,, \\ k_\mu y_\lambda &= 2ig_{\mu\lambda}x_\nu y^\nu - 2i(x_\mu y_\lambda + x_\lambda y_\mu)\,, \\ m_{\mu\nu}y_\lambda &= -ig_{\mu\lambda}y_\nu + ig_{\nu\lambda}y_\mu\,. \end{aligned} \tag{4.6}$$

From (4.5) and (4.6) it is seen that the real part x of z transforms like a vector with respect to dilatations and Poincaré transformations whereas the complex part y of z transforms like a vector difference — is translationally invariant ($p_\mu y_\lambda = 0$). From (4.4) it is seen that also r is translationally invariant ($p_\mu r_\lambda = 0$). Thus we obtain all the assumptions of the bilocal theory as consequences of the assumptions A and B.

Moreover it is seen that with respect to special conformal transformations k_μ the real and imaginary parts of z are mixed. Also the special conformal transformations of r depend on x and y. The bilocal theory must be reformulated from the point of view of transformation properties if we wish to enlarge the symmetry from P_4 to SU(2,2).

We note, finally, that all coordinates of the upper strip $\xi_{a'\alpha}$, $a' = 1,2$, $\alpha = 1,\dots,m$ are translationally invariant. Indeed, we can write these coordinates as $\xi_{a'\alpha} = (\gamma_+\xi)_{a'\alpha}$ and we obtain, according to (4.1),

$$p_\mu\xi_{a'\alpha} = p_\mu(\gamma_+\xi)_{a'\alpha} = (\gamma_+ p_\mu\xi)_{a'\alpha} = i\lambda^{-1}(\gamma_+\gamma_-\gamma_\mu\xi)_{a'\alpha} = 0\,.$$

5. Coordinate independent formulation

For practical calculations, description by means of local coordinates is necessary. In more complicated systems like (3.5) and (3.6) this is not so easy. It is not at all clear how to introduce convenient coordinates (in analogy to hyperbolic coordinates on the three-dimensional hyperboloid $y_\mu y^\mu$ = const) on the six- and five-dimensional manifolds described by equations (3.5) and (3.6). It is also not clear how to introduce convenient coordinates on the submanifold of $\mathbb{C}^{2m}$ described by the four relations (3.4) with fixed r. In such cases a coordinate free formulation may be of some use. Therefore, we shall now describe the manifolds (3.4)–(3.6) as homogeneous spaces of the corresponding symmetry groups.

Let us first consider the $P_4 \times \mathrm{SU}(m)$ invariant case. With respect to this symmetry the local coordinates x and y do not mix (*cf.* (4.5)). The coordinate x does not appear in the conditions (3.5) and (3.6). The configuration space is, therefore, the direct product of Minkowski space M_4 with the global coordinate system x_μ, $\mu = 0,1,2,3$ and an internal space M^{int} with local coordinates y_μ, $\xi_{a'\alpha}$, $\mu = 0,1,2,3$; $a' = 1,2$; $\alpha = 1,\dots,m$; subject to conditions (3.6) with r given by equation (3.4). This space is a homogeneous space of the group $\mathrm{SO}(3,1) \times \mathrm{SU}(m)$. Indeed, consider the point

$$\begin{aligned}
\overset{\circ}{m}_{1\alpha} &= 0\,, \quad \alpha = 2,3,\dots,m\,,\\
\overset{\circ}{m}_{2\alpha} &= 0\,, \quad \alpha = 3,4,\dots,m\,,\\
\overset{\circ}{y}_1 &= \overset{\circ}{y}_2 = 0\,,\\
\overset{\circ}{r}_i &= -\overset{\circ}{m}{}^*_{\alpha a}(\tilde\sigma_i)^{\dot a b}\,\overset{\circ}{m}_{b\alpha} = 0\,,\\
\overset{\circ}{r}_0 &= -\overset{\circ}{m}{}^*_{\alpha a}(\tilde\sigma_0)^{\dot a b}\,\overset{\circ}{m}_{b\alpha} = \sqrt{c_{22}}\,,\\
\overset{\circ}{y}{}_3^2 - \overset{\circ}{y}{}_0^2 &= -c_{11}\,,\ \overset{\circ}{y}_0\overset{\circ}{r}_0 = c_{12}\,,\ \det c < 0\,.
\end{aligned}\tag{5.1}$$

It is easily seen that this point satisfies conditions (3.6) and that its isotropy group is $\mathrm{SO}(2) \times \mathrm{SU}(m-2) \sim \mathrm{U}(1)\times\mathrm{SU}(m-2)$. Moreover, every point satisfying (3.6) can be reached from the point (5.1) by a transformation of $\mathrm{SO}(3,1) \times \mathrm{SU}(m)$. We can write, therefore, globally,

$$\begin{aligned}
M = M_4 \times M^{\mathrm{int}} &= \frac{P_4}{\mathrm{SO}(3,1)} \times \frac{\mathrm{SO}(3,1)\times\mathrm{SU}(m)}{\mathrm{SO}(2)\times\mathrm{SU}(m-2)}\\
&= \frac{P_4 \times \mathrm{SU}(m)}{\mathrm{U}(1)\times\mathrm{SU}(m-2)}\,,
\end{aligned}\tag{5.2}$$

where $P_4/\mathrm{SO}(3,1)$ represents the Minkowski space M_4.

In a similar way one proceeds in the $SU(2,2) \times SU(m)$ invariant case. The configuration space has now one real dimension more and can be represented globally as the homogeneous space

$$M = \frac{SU(2,2) \times SU(m)}{K_4 \times U(1) \times SU(m-2)}, \tag{5.3}$$

where K_4 stands for the special conformal group considered as a subgroup of SU(2,2).

It may be noted that, due to the fact that conformal transformations mix the real and imaginary parts of z, the homogeneous space (5.3) cannot be considered as a direct product of Minkowski an internal space. Such a splitting is possible only locally.

Equation (1.3) shows that one can consider, locally, the space

$$\frac{SO(3.1)}{SO(2)} \overset{\text{loc}}{\sim} \frac{SO(3,1)}{SO(3)} \times \frac{SO(3)}{SO(2)} = H^3 \times S^2$$

as the direct product of a three-dimensional hyperboloid and a two-dimensional sphere. Similarly

$$\frac{SU(m)}{SU(m-2)} \overset{\text{loc}}{=} \frac{SU(m)}{SU(m-1)} \times \frac{SU(m-1)}{SU(m-2)} = S^{2m-1} \times S^{2m-3}$$

can be considered as the product of two spheres of dimensions $2m-1$ and $2m-3$. Such a local description, although not $SO(3,1) \times SU(m)$ invariant, might help in visualizing the local properties of the geometric structures considered in this paper.

6. Outlook

We have shown that the bilocal theory is not just another attempt to introduce *ad hoc* internal structure of the particles but that it uniquely follows from two first principles. One principle (Postulate A) states that the physical symmetry is the direct product of external and internal symmetries ($SU(2,2) \times SU(m)$ for massless, $P_4 \times SU(m)$ for massive particles) in conformity with experimental evidence. The other principle (Postulate B) is of purely esthetical origin and expresses the desire to have a common geometrical basis for all physical symmetries.

The structure of the configuration space, in particular the structure of the space of internal parameters of the particle, follows uniquely from these principles. In the case of $P_4 \times SU(m)$ symmetry this structure goes over into the two-point configuration space of bilocal theory if one neglects the dependence of the vector r (*cf.* (3.4)) on the internal parameters $\xi_{a'\alpha}$; $a' = 1,2$; $\alpha = 1,\ldots,m$. Such an approximation spoils relativistic invariance because, in this case, the vector r distinguishes a direction in spacetime. However, the full structure with r given by Eq. (3.4) is not only relativistically invariant but also invariant with respect to the group of internal symmetries SU(m). One can say that the internal structure of an $P_4 \times \mathrm{SU}(m)$ invariant particle model manifests itself on two levels. The first level is the homogeneous space $\mathrm{SO}(3,1)/\mathrm{SO}(2) \overset{\mathrm{loc}}{\sim} H^3 \times S^2$ and corresponds to bilocal theory. The second level is the homogeneous space $\mathrm{SU}(m)/\mathrm{SU}(m-2)$. It provides the full relativistic and SU(m) invariance of the model.

It might be interesting to continue and follow Yukawa's and Rayski's ideas and to develop a field theory on the basis of the general model. Due to the rather complicated character of the geometric structures this seems to be a quite difficult task but we hope to come back to it in future.

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