$(j,0)\oplus(0,j)$ REPRESENTATION SPACE: MAJORANA-LIKE CONSTRUCT*,**

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This is second of the two invited lectures presented at the "XVII International School of Theoretical Physics: Standard Model and Beyond' 93." The text is essentially based on a recent publication by the present authors [*Mod. Phys. Lett.* **A** (in press)]. Here, after briefly reviewing the $(j,0)\oplus(0,j)$ Dirac-like construct in the front form, we present a detailed construction of the $(j,0)\oplus(0,j)$ Majorana-like fields.

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To facilitate the study of the massless limit, we work in the front-form [1] Weinberg–Soper formalism [2, 3] recently developed in Ref. [4]. The first lecture in this two-part series presented the *instant form* Dirac-like construction in the $(j,0)\oplus(0,j)$ representation space and is not ideally suited to study the massless limit. The formalism that we develop is valid for massive as well as massless particles.

1. Review of Dirac-like $(j,0)\oplus(0,j)$ spinors in the front form

The front-form Dirac-like $(j,0)\oplus(0,j)$ covariant spinors [4] in the Weinberg–Soper formalism (in the chiral representation) are defined as:

$$\psi\{p^\mu\} = \begin{bmatrix} \phi_{\rm R}(p^\mu) \\ \phi_{\rm L}(p^\mu) \end{bmatrix}. \tag{1}$$

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The argument p^μ of chiral-representation spinors will be enclosed in curly brackets $\{\}$. The Lorentz transformation of the front-form $(j,0)$ spinors is given [4] by

$$\phi_{\rm R}(p^\mu) = \Lambda_{\rm R}(p^\mu)\phi_{\rm R}(\overset{\circ}{p}{}^\mu) = \exp\left(\boldsymbol{\beta}\cdot\boldsymbol{J}\right)\phi_{\rm R}(\overset{\circ}{p}{}^\mu)\,, \tag{2}$$

and the front-form $(0,j)$ spinors transform as

$$\phi_{\rm L}(p^\mu) = \Lambda_{\rm L}(p^\mu)\phi_{\rm L}(\overset{\circ}{p}{}^\mu) = \exp\left(-\boldsymbol{\beta}^*\cdot\boldsymbol{J}\right)\phi_{\rm L}(\overset{\circ}{p}{}^\mu)\,. \tag{3}$$

The $\overset{\circ}{p}{}^\mu$ represents the front-form four momentum for a particle at rest: $\overset{\circ}{p}{}^\mu \equiv (p^+ = m, p^1 = 0, p^2 = 0, p^- = m)$. The $\boldsymbol{J}$ are the standard $(2j+1)\times(2j+1)$ spin matrices, and $\boldsymbol{\beta}$ is the boost parameter introduced in Ref. [4]

$$\boldsymbol{\beta} = \eta\left(\alpha v^r, -i\alpha v^r, 1\right)\,, \tag{4}$$

where $\alpha = [1-\exp(-\eta)]^{-1}$, $v^r = v_x + iv_y$ (and $v^\ell = v_x - iv_y$). In terms of the front-form variable $p^+ \equiv E + p_z$, one can show that

$$\cosh(\eta/2) = \Omega\left(p^+ + m\right)\,,\quad \sinh(\eta/2) = \Omega\left(p^+ - m\right)\,, \tag{5}$$

with $\Omega = [1/(2m)]\sqrt{m/p^+}$. The norm $\overline{\psi}\{p^\mu\}\psi\{p^\mu\}$, with

$$\overline{\psi}\{p^\mu\} \equiv \psi^\dagger\{p^\mu\}\Gamma^0,\quad \Gamma^0 \equiv \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}\,, \tag{6}$$

is so chosen that *in the massless limit* :

a. The Dirac-like $(j,0)\oplus(0,j)$ rest spinors identically vanish (there can be no massless particles at rest); and
b. Only the Dirac-like $(j,0)\oplus(0,j)$ spinors associated with $h=\pm j$ front-form helicity [4] degrees of freedom survive. ($\mathbb{1}$ is the $(2j+1)\times(2j+1)$ identity matrix.)

These requirements uniquely determine (up to a constant factor, which we choose to be $1/\sqrt{2}$) the $(2j+1)$-element-column form of $\phi_{\rm R}(\overset{\circ}{p}{}^\mu)$ and $\phi_{\rm L}(\overset{\circ}{p}{}^\mu)$ to be

$$\phi^R_j(\overset{\circ}{p}{}^\mu) = \frac{m^j}{\sqrt{2}}\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix},\quad \phi^R_{j-1}(\overset{\circ}{p}{}^\mu) = \frac{m^j}{\sqrt{2}}\begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix},\cdots\quad \phi^R_{-j}(\overset{\circ}{p}{}^\mu) = \frac{m^j}{\sqrt{2}}\begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}, \tag{7}$$

[with similar expressions for $\phi_{\rm L}(\overset{\circ}{p}{}^{\mu})$] in a representation in which J_z is diagonal. The subscripts $h = j, j-1, \cdots, -j$ on $\phi_h^R(\overset{\circ}{p}{}^{\mu})$ in Eq. (7) refer to the front-form helicity [4] degree of freedom. The reader should refer to Sec. 2.5 of Ref. [5] for an alternate discussion of the nontrivial nature (even though it appears as a "normalization factor") of the factor m^j in Eq. (7).

2. Majorana-like $(j,0)\oplus(0,j)$ spinors in the front form

Following Ramond's work [6] on spin-$1/2$, we define the front-form $(j,0)\oplus(0,j)$ *θ-conjugate spinor*

$$\psi^\theta\{p^\mu\} \equiv \begin{bmatrix} \left(\xi\Theta_{[j]}\right)\phi_{\rm L}^*(p^\mu) \\ \left(\xi\Theta_{[j]}\right)^*\phi_{\rm R}^*(p^\mu)\end{bmatrix}, \tag{8}$$

where ξ is a c-number, and $\Theta_{[j]}$ is the Wigner's time-reversal operator (see Refs: p. 61 of [7], Eqs (6.7) and (6.8) of the first reference in [2], and Ch. 26 of [8]),

$$\Theta_{[j]}\mathbf{J}\Theta_{[j]}^{-1} = -\boldsymbol{J}^*, \tag{9}$$

and * denotes the operation of algebraic complex conjugation. The parameter ξ is fixed by imposing the constraint:

$$\left[\psi^\theta\{p^\mu\}\right]^\theta = \psi\{p^\mu\}. \tag{10}$$

The time-reversal operator $\Theta_{[j]}$ is defined as: $\Theta_{[j]} = (-1)^{j+\sigma}\delta_{\sigma',-\sigma}$. It has the properties: $\Theta_{[j]}^*\Theta_{[j]} = (-1)^{2j}$, $\Theta_{[j]}^* = \Theta_{[j]}$. In the definition of $\Theta_{[j]}$, σ and σ' represent eigenvalues of $\boldsymbol{J}$. For $j=1/2$ and $j=1$, the $\Theta_{[j]}$ have the explicit forms:

$$\Theta_{[1/2]} = \begin{bmatrix} 0 & -1 \\ 1 & 0\end{bmatrix}, \quad \Theta_{[1]} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0\end{bmatrix}. \tag{11}$$

The properties of Wigner's $\Theta_{[j]}$ operator allow the parameter ξ involved in the definition of θ-conjugation to be fixed as $\pm i$ for fermions and ± 1 for bosons. However, without loss of generality, we can ignore the *minus* sign [which contributes an *overall* phase factor to the θ-conjugated spinors $\psi^\theta\{p^\mu\}$] and fix ξ as

$$\xi = \begin{cases} i, & \text{for fermions} \\ 1, & \text{for bosons}\end{cases}. \tag{12}$$

The existence of the Majorana spinors for the $(^1\!/_2,0)\oplus(0,^1\!/_2)$ representation space is usually (see, *e.g.*, p.16 of Ref. [6]) associated with the "magic of Pauli matrices," $\boldsymbol{\sigma}$. The reader may have already noticed that $i\Theta_{[1/2]}$ is identically equal to σ_y; and it is precisely this matrix that enters into the CP-conjugation of the $(^1\!/_2,0)\oplus(0,^1\!/_2)$ spinors.

The reason that the Majorana-like $(j,0)\oplus(0,j)$ representation spaces, as opposed to the $(j,0)\oplus(0,j)$ spaces spanned by *Dirac-like spinors* defined by Eq. (1), can be constructed for arbitrary spins hinges upon two observations:

1. Independent of spin, the front-form boosts for the $(j,0)$ and $(0,j)$ spinors have the property that
$[\Lambda_{\rm R}(p^\mu)]^{-1}=[\Lambda_{\rm L}(p^\mu)]^\dagger\,,\quad [\Lambda_{\rm L}(p^\mu)]^{-1}=[\Lambda_{\rm R}(p^\mu)]^\dagger\,;$
and
2. Existence of the Wigner's time-reversal matrix $\Theta_{[j]}$ for any spin.

These two observations, when coupled with the transformation properties of the right- and left-handed spinors, Eqs (2), (3), imply that if $\phi_{\rm R}(p^\mu)$ transforms as $(j,0)$, then $(\zeta\Theta_{[j]})^*\phi_{\rm R}^*(p^\mu)$ transforms as $(0,j)$ spinor. Similarly, if $\phi_{\rm L}(p^\mu)$ transforms as $(0,j)$, then $(\zeta\Theta_{[j]})^*\phi_{\rm L}^*(p^\mu)$ transforms as $(j,0)$ spinor. Here, $\zeta=\exp(i\vartheta)$ is an arbitrary phase factor. As such, we introduce $(j,0)\oplus(0,j)$ *Majorana-like spinors*

$$(j,0)\mapsto\ \rho\{p^\mu\}=\begin{bmatrix}\phi_{\rm R}(p^\mu)\\ \left(\zeta_\rho\Theta_{[j]}\right)^*\phi_{\rm R}^*(p^\mu)\end{bmatrix},$$

$$(0,j)\mapsto\ \lambda\{p^\mu\}=\begin{bmatrix}\left(\zeta_\lambda\Theta_{[j]}\right)\phi_{\rm L}^*(p^\mu)\\ \phi_{\rm L}(p^\mu)\end{bmatrix}. \tag{13}$$

For formal reasons, the operator multiplying $\phi_{\rm L}^*(p^\mu)$, in the definition of $\lambda\{p^\mu\}$, is written as $\zeta_\lambda\Theta$ rather than $\left(\zeta'_\lambda\Theta\right)^*$. What we have done, in fact, is to exploit the property $\Theta^*=\Theta$ and choose $\zeta_\lambda=\zeta_\lambda^{\prime *}$. Since ζ_λ is yet to be determined, this introduces no loss of generality. The advantage of all this is that $\rho\{p^\mu\}$ and $\lambda\{p^\mu\}$ can now be seen as nothing but Weyl spinors (in the $2(2j+1)$-element form)

$$\psi_{\rm R}\{p^\mu\}=\begin{bmatrix}\phi_{\rm R}(p^\mu)\\ 0\end{bmatrix},\quad \psi_{\rm L}\{p^\mu\}=\begin{bmatrix}0\\ \phi_{\rm L}(p^\mu)\end{bmatrix}, \tag{14}$$

added to their respective θ-conjugates. We now fix ζ_ρ and ζ_λ by demanding (the defining property of the Majorana-like spinors):

$$\rho^\theta\{p^\mu\}=\pm\rho\{p^\mu\}\quad\text{and}\quad \lambda^\theta\{p^\mu\}=\pm\lambda\{p^\mu\}\,, \tag{15}$$

and find:

$$\zeta_\rho = \pm\xi \quad \text{and} \quad \zeta_\lambda = \pm\xi\,. \tag{16}$$

The choice $\zeta_\rho = \zeta_\lambda = +\xi$ yields self-θ-conjugate spinors $\rho^{S_\theta}\{p^\mu\}$ and $\lambda^{S_\theta}\{p^\mu\}$; while $\zeta_\rho = \zeta_\lambda = -\xi$ corresponds to antiself-θ-conjugate spinors $\rho^{A_\theta}\{p^\mu\}$ and $\lambda^{A_\theta}\{p^\mu\}$. The condition (10) is satisfied not only by Majorana-like self-θ-conjugate spinors but also by antiself-θ-conjugate spinors.

It may be noted that the Dirac-like spinors, Eq. (1), and the Majorana-like spinors, Eqs (13), are two of the simplest choices of spinors that can be introduced in any P-covariant theory[1] in the $(j,0)\oplus(0,j)$ representation space. The former describe particles with a definite charge (which may be zero), while the latter are inherently for the description of neutral particles.

Before we proceed further, we make a few observations on the definition of θ-conjugation. For the $(^1\!/_2,0)\oplus(0,^1\!/_2)$ case, the definition (8) of θ-conjugation can be verified to coincide with CP-conjugation. The reader may wish to note that what Ramond (see Ref. [6], p. 20) calls a "charge conjugate spinor," in the context of spin-$\frac{1}{2}$, is actually a CP-conjugate spinor. This, we suspect, remains true for fermions of higher spins also. Surprisingly, for the $(1,0)\oplus(0,1)$ spinors (and presumably for bosons of higher spins also) θ-conjugation equals $\Gamma^5 C$ within a phase factor [2] of $-(-1)^{|h|}$. The mathematical origin of this fact may be traced to the constraint (10) and the property $\Theta^*_{[j]}\Theta_{[j]} = (-1)^{2j}$ of Wigner's time-reversal operator Θ. One may ask if the constraint (10) is changed to read $\left[\psi^\theta\{p^\mu\}\right]^\theta = -\psi^\theta\{p^\mu\}$ for bosons, whether one can obtain an alternate definition of θ-conjugation (so that θ-conjugation equals CP for bosons also) to construct self/antiself-θ-conjugate objects. A simple exercise reveals that no such construction yields self/antiself-θ-conjugate objects. The reader may wish to note parenthetically that when the result (12) is coupled with the definition of θ-conjugation, Eq. (8), we discover that the operation of θ-conjugation treats the right-handed and left-handed spinors in a fundamentally asymmetric fashion for *fermions*. This is readily inferred by studying the relative phases with which $\left(\xi\Theta_{[j]}\right)\phi^*_{\rm L}(p^\mu)$ and $\left(\xi\Theta_{[j]}\right)^*\phi^*_{\rm R}(p^\mu)$ enter in Eq. (8).

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[1] A theory that is covariant under the operation of parity is *not* necessarily a parity non-violating theory. See Sec. 7 for a brief discussion of this point.

[2] The charge conjugation operator C, along with P and T, for the $(1,0)\oplus(0,1)$ spinors and fields was recently obtained in Ref. [9].

3. Explicit construction of Majorana-like $(1,0)\oplus(0,1)$ spinors in the front form

We now cast these formal considerations into more concrete form by studying the $(1,0)\oplus(0,1)$ Majorana-like representation space as an example. As in Ref. [4], we introduce[3] the generalized canonical representation in the front form:

$$\psi[p^\mu] = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{bmatrix}\psi\{p^\mu\}\,. \tag{17}$$

The argument p^μ of canonical-representation spinors will be enclosed in square brackets []. The boost $M(p^\mu)$, which connects the rest-spinors $\psi[\overset{\circ}{p}{}^\mu]$ with the spinors associated with front-form four momentum p^μ, $\psi[p^\mu]$, is determined from Eqs (2), (3), and (17):

$$\psi[p^\mu] = M(p^\mu)\psi[\overset{\circ}{p}{}^\mu], \quad M(p^\mu) = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B} & \mathcal{A} \end{bmatrix}, \tag{18}$$

with $\mathcal{A} = \Lambda_{\rm R}(p^\mu) + \Lambda_{\rm L}(p^\mu)$, $\mathcal{B} = \Lambda_{\rm R}(p^\mu) - \Lambda_{\rm L}(p^\mu)$.

Using the identities (needed to evaluate $M(p^\mu)$ explicitly) given in Ref. [4], we first obtain the spin-1 $\rho^{S_\theta}[p^\mu]$ spinors. These are tabulated in Table I. The $\rho^{S_\theta}[p^\mu]$ spinors satisfy the following orthonormality relations: $\overline{\rho}_h^{S_\theta}[p^\mu]\rho_{h'}^{S_\theta}[p^\mu] = m^2\Theta_{hh'}$, where

$$\overline{\rho}_h[p^\mu] \equiv (\rho_h[p^\mu])^\dagger\,\Gamma^0, \quad \Gamma^0 \equiv \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}. \tag{19}$$

TABLE I

Spin-1 self-θ-conjugate Majorana-like $\rho^{S_\theta}[p^\mu]$ spinors. Here $p^\pm = E \pm p_z$, $p^r = p_x + ip_y$, and $p^\ell = p_x - ip_y$. The subscript $h = 0, \pm 1$ on $\rho_h^{S_\theta}[p^\mu]$ refers to the *front-form helicity* [4] degree of freedom. The remaining spinors $\lambda^{S_\theta}[p^\mu]$, $\rho^{A_\theta}[p^\mu]$, and $\lambda^{A_\theta}[p^\mu]$ are related to $\rho^{S_\theta}[p^\mu]$ via Eqs (21a) to (21c).

$\rho_{+1}^{S_\theta}[p^\mu]$	$\rho_0^{S_\theta}[p^\mu]$	$\rho_{-1}^{S_\theta}[p^\mu]$
$\frac{1}{2}\begin{bmatrix} p^+ + (p^{\ell 2}/p^+) \\ \sqrt{2}(p^r - p^\ell) \\ p^+ + (p^{r2}/p^+) \\ p^+ - (p^{\ell 2}/p^+) \\ \sqrt{2}(p^r + p^\ell) \\ -p^+ + (p^{r2}/p^+) \end{bmatrix}$	$\frac{m}{\sqrt{2}}\begin{bmatrix} p^\ell/p^+ \\ 0 \\ p^r/p^+ \\ -p^\ell/p^+ \\ \sqrt{2} \\ p^r/p^+ \end{bmatrix}$	$\frac{m^2}{2}\begin{bmatrix} 1/p^+ \\ 0 \\ 1/p^+ \\ -1/p^+ \\ 0 \\ 1/p^+ \end{bmatrix}$

[3] The generalized canonical representation is introduced here for no other reason than to be able to compare the results of the present work with our earlier work of Ref. [4].

The front-form $(1,0)\oplus(0,1)$ Majorana-like spinors, Table I, should be compared with the front-form $(1,0)\oplus(0,1)$ *Dirac-like* spinors obtained in our recent work [4]. For instance, in the massless limit for the Dirac-like spinors, the $h=\pm 1$ degrees of freedom are non-vanishing and the $h=0$ degree of freedom identically vanishes. On the other hand, in the massless limit, for the spin-1 Majorana-like spinors $\rho^{S_\theta}[p^\mu]$, it is only the $h=+1$ degree of freedom that is non-vanishing, while the $h=0$ and $h=-1$ degrees of freedom identically vanish.

The origin of the above observation lies in the fact[4] that the $(1,0)$ and $(0,1)$ boosts, $\Lambda_{\rm R}(p^\mu)$ and $\Lambda_{\rm L}(p^\mu)$, essentially become projectors of the $\phi^R_{+1}(p^\mu)$ and $\phi^L_{-1}(p^\mu)$ as $m\to 0$. To see this, introduce

$$\mathcal{Q}_{\rm R}(m)\equiv\left(\frac{m}{p^+}\right)\Lambda_{\rm R}(p^\mu)=\begin{bmatrix}1&0&0\\ \frac{\sqrt{2}p^r}{p^+}&\frac{m}{p^+}&0\\ \left(\frac{p^r}{p^+}\right)^2&\frac{\sqrt{2}mp^r}{(p^+)^2}&\frac{m^2}{(p^+)^2}\end{bmatrix},\tag{20a}$$

$$\mathcal{Q}_{\rm L}(m)\equiv\left(\frac{m}{p^+}\right)\Lambda_{\rm L}(p^\mu)=\begin{bmatrix}\frac{m^2}{(p^+)^2}&\frac{-\sqrt{2}mp_\ell}{(p^+)^2}&\left(\frac{p^\ell}{p^+}\right)^2\\ 0&\frac{m}{p^+}&\frac{-\sqrt{2}p^\ell}{p^+}\\ 0&0&1\end{bmatrix}.\tag{20b}$$

The quasi-projector nature of $\mathcal{Q}_{\rm R}(m\to 0)$ and $\mathcal{Q}_{\rm L}(m\to 0)$ is immediately observed by verifying that: $\mathcal{Q}^2_{\rm R}(m\to 0)=\mathcal{Q}_{\rm R}(m\to 0)$ and $\mathcal{Q}^2_{\rm L}(m\to 0)=\mathcal{Q}_{\rm L}(m\to 0)$; but in general $\mathcal{Q}_{\rm R}(m\to 0)+\mathcal{Q}_{\rm L}(m\to 0)\not\to \mathbb{1}$ and $\mathcal{Q}^\dagger_{R,L}(m\to 0)\neq\mathcal{Q}_{R,L}(m\to 0)$.

To incorporate the $h=-1$ degree of freedom in the massless limit, and to be able to treat the massive particles without introducing manifest parity violation, we now repeat the above procedure for the $\lambda^{S_\theta}[p^\mu]$ (and $\rho^{A_\theta}[p^\mu]$ and $\lambda^{A_\theta}[p^\mu]$ for the sake of completeness) spinors. We find:

$$\lambda^{S_\theta}_{-h}[p^\mu]=-(-1)^{|h|}\rho^{S_\theta}_h[p^\mu],\tag{21a}$$

$$\rho^{A_\theta}_h[p^\mu]=\Gamma^5\rho^{S_\theta}_h[p^\mu],\quad \Gamma^5=\begin{bmatrix}0&\mathbb{1}\\ \mathbb{1}&0\end{bmatrix},\tag{21b}$$

$$\lambda^{A_\theta}_{-h}[p^\mu]=(-1)^{|h|}\Gamma^5\rho^{S_\theta}_h[p^\mu]=(-1)^{|h|}\rho^{A_\theta}_h[p^\mu];\tag{21c}$$

with $\mathbb{1}=3\times 3$ identity matrix and

$$\overline{\rho}^{S_\theta}_h[p^\mu]\lambda^{S_\theta}_{h'}[p^\mu]=m^2\delta_{hh'}=\overline{\rho}^{A_\theta}_h[p^\mu]\lambda^{A_\theta}_{h'}[p^\mu],\tag{22a}$$

$$\overline{\rho}^{A_\theta}_h[p^\mu]\rho^{S_\theta}_{h'}[p^\mu]=0=\overline{\rho}^{S_\theta}_h[p^\mu]\rho^{A_\theta}_{h'}[p^\mu].\tag{22b}$$

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[4] Even though we make these observations for spin-1, results similar to those that follow are true for all spins (including spin-$\frac{1}{2}$).

As we will see in Sec. 6, the *bi-orthogonal* [16] nature of the $\rho[p^\mu]$ and $\lambda[p^\mu]$ spinors results in a rather unusual quantum field theoretic structure for the $(1,0)\oplus(0,1)$ Majorana-like field. Similar results hold true for other spins (including spin $\frac{1}{2}$). The bi-orthogonal nature of the Majorana-like spinors is forced upon us by self/antiself-θ-conjugacy condition (15) and cannot be changed as long as we require that the basis spinors correspond to definite spin projections (front form helicity-basis in our case).

It should now be recalled that for the Dirac-like $(1,0)\oplus(0,1)$ spinors $u_\sigma\{p^\mu\}$ and $v_\sigma\{p^\mu\}$, we know [9] from the associated wave equation that the $u_\sigma\{p^\mu\}$ spinors are associated with the forward-in-time propagating solutions (the "positive energy solutions") $u_\sigma\{p^\mu\}\exp[-i(Et-\boldsymbol{p}\cdot\boldsymbol{x})]$, and $v_\sigma\{p^\mu\}$ spinors are associated with the backward-in-time propagating[5] solutions (the "negative energy solutions") $v_\sigma\{p^\mu\}\exp[+i(Et-\boldsymbol{p}\cdot\boldsymbol{x})]$. Can one infer similar results by studying the wave equation associated with the front-form $(1,0)\oplus(0,1)$ Majorana-like spinors?

4. Wave equation for Majorana-like $(1,0)\oplus(0,1)$ spinors in the front form

Combining the Lorentz transformation properties for the $\phi_{\rm R}(p^\mu)$ and $\phi_{\rm L}(p^\mu)$, given by Eqs (2) and (3), with the definitions (13) of Majorana-like spinors, we obtain the wave equations satisfied by the $(1,0)\oplus(0,1)$ Majorana-like spinors. For the $\rho\{p^\mu\}$ spinors, the wave equation we obtain reads (in chiral representation, where it takes its simplest form):

$$\begin{bmatrix} -\zeta_\rho m^2\Theta_{[1]} & \mathcal{O}_1 \\ \mathcal{O}_2 & -\zeta_\rho m^2\Theta_{[1]} \end{bmatrix}\rho\{p^\mu\}=0\,, \tag{23}$$

where in the front form the operators $\mathcal{O}_1$ and $\mathcal{O}_2$ are defined as:

$$\mathcal{O}_1=g_{\mu\nu}p^\mu p^\nu\exp\left(-\boldsymbol{\beta}\cdot\boldsymbol{J}^*\right)\exp\left(\boldsymbol{\beta}^*\cdot\boldsymbol{J}\right)$$

and

$$\mathcal{O}_2=g_{\mu\nu}p^\mu p^\nu\exp\left(\boldsymbol{\beta}^*\cdot\boldsymbol{J}^*\right)\exp\left(-\boldsymbol{\beta}\cdot\boldsymbol{J}\right).$$

The nonzero elements of the front-form (the flat space-time) metric $g_{\mu\nu}$ are: $g_{+-}={}^1/_2=g_{-+}$ and $g_{11}=-1=g_{22}$. The wave equation for the $\lambda\{p^\mu\}$ spinors is the same as Eq. (23) with ζ_λ being replaced by ζ_ρ. The dispersion relations associated with the solutions of Eq. (23) are obtained by setting

[5] Recall that the usual interpretation of the "negative energy" states as *anti*particles fails (see p. 66 of Ref. [10]) for bosons. On the other hand, the Stückelberg-Feynman framework [11] applies equally to fermions and bosons.

the determinant of the square bracket in Eq. (23) equal to zero. A simple, though somewhat lengthy, algebra transforms the resulting equation into (true for all spin-1 Majorana-like spinors, hence all reference to a specific spinor is dropped below):

$$-\left(p^\ell p^r - p^+p^- - \zeta m^2\right)^3 \left(p^\ell p^r - p^+p^- + \zeta m^2\right)^3 = 0.$$

As a result, the associated dispersion relations read:

$$p^+ = \frac{p^\ell p^r + \zeta m^2}{p^-}, \quad p^+ = \frac{p^\ell p^r - \zeta m^2}{p^-}, \tag{24}$$

each with a multiplicity 3 (for a *given* ζ). Again, as seen in Refs [12–14], as in the case for the Dirac-like $(1,0)\oplus(0,1)$ spinors, the wave equation for the Majorana-like $(1,0)\oplus(0,1)$ spinors contains tachyonic degeneracy. For the Dirac-like $(1,0)\oplus(0,1)$ spinors, we find that the tachyonic solutions can be reinterpreted as physical solutions within the context of a quartic self interaction and spontaneous symmetry breaking [14]. Here, we concentrate on the physically acceptable dispersion relations $p^+ = (p^\ell p^r + m^2)/p^-$; or equivalently $E^2 = \mathbf{p}^2 + m^2$.

The wave equation satisfied by the plane-wave solutions

$$\rho\{x\} = \rho\{p^\mu\}\exp(-i\epsilon p^\mu x_\mu) \quad \text{and} \quad \lambda\{x\} = \lambda\{p^\mu\}\exp(-i\epsilon p^\mu x_\mu)$$

is obtained by first expanding the exponentials in Eq. (23), in accordance with the identities given in Appendix A of Ref. [4], and then letting $p^\mu \to i\partial^\mu$. Next, to determine ϵ, we study the resulting equation for the plane-wave solutions associated with the rest spinors. It is easily verified that for $\rho\{p^\mu\}$ as well as $\lambda\{p^\mu\}$, it is not ϵ (directly) but ϵ^2 that is constrained by the relation: $\epsilon^2 = 1$, giving $\epsilon = \pm 1$. This is consistent with the intuitive understanding in that we cannot distinguish between the forward-in-time propagating ("particles") and the backward-in-time propagating ("antiparticles") Majorana-like objects. The above arguments are independent of which representation we choose within the $(1,0)\oplus(0,1)$ representation space.

5. Majorana-like $(1,0)\oplus(0,1)$ field operator in the front form

We now exploit the above considerations on the Majorana-like spinors to construct the associated field operator. Generalizing the spin-$^1/_2$ definition for a Majorana particle of Ref. [15], we define a general $(j,0)\oplus(0,j)$ Majorana-like field operator $\Xi(x)$

$$U(C_\theta)\Xi(x)U^{-1}(C_\theta) = \pm\Xi(x). \tag{25}$$

In Eq. (25), the "+" sign defines the self-θ-conjugate and the "−" sign defines the antiself-θ-conjugate field operator. The explicit chiral-representation expression for θ-conjugation operator C_θ as contained in Eq. (8) is

$$C_\theta = \mathcal{C}_\theta K = \begin{bmatrix} 0 & \xi\Theta_{[j]} \\ \left(\xi\Theta_{[j]}\right)^* & 0 \end{bmatrix} K\,, \tag{26}$$

where, K complex conjugates (on the right) the objects in the Majorana-like $(j,0)\oplus(0,j)$ representation space. For the example case of spin-1, when Eqs (25) are coupled with the additional physical requirement that all helicity degrees of freedom be treated symmetrically for manifest rotational- and P-covariance, the field operator $\Xi(x)$ is determined to be

$$\begin{aligned}\Xi^{S_\theta}(x) = \sum_{h=0,\pm1} \int d^4p\Big[&\mathcal{S}_h^{(\rho)}(p^\mu)\rho_h^{S_\theta}[p^\mu]\exp(-ipx) \\ &+ \eta_{\rm GK}\mathcal{S}_h^{(\lambda)\dagger}(p^\mu)\lambda_h^{S_\theta}[p^\mu]\exp(+ipx)\Big]\,,\end{aligned} \tag{27a}$$

$$\begin{aligned}\Xi^{A_\theta}(x) = \sum_{h=0,\pm1} \int d^4p\Big[&\mathcal{A}_h^{(\rho)}(p^\mu)\rho_h^{A_\theta}[p^\mu]\exp(-ipx) \\ &+ \eta_{\rm GK}\mathcal{A}_h^{(\lambda)\dagger}(p^\mu)\lambda_h^{A_\theta}[p^\mu]\exp(+ipx)\Big],\end{aligned} \tag{27b}$$

where $\eta_{\rm GK}$ is the generalized[6] Goldhaber–Kayser phase factor; and

$$\left[\mathcal{S}_h^{(\rho)}(p^\mu), \mathcal{S}_{h'}^{(\rho)\dagger}(p'^\mu)\right] = -(-1)^{|h|}(2\pi)^3 2E(\vec{p})\delta_{h,-h'}\delta^3(\vec{p}-\vec{p}')\,, \tag{28a}$$

$$\left[\mathcal{S}_h^{(\lambda)}(p^\mu), \mathcal{S}_{h'}^{(\lambda)\dagger}(p'^\mu)\right] = -(-1)^{|h|}(2\pi)^3 2E(\vec{p})\delta_{h,-h'}\delta^3(\vec{p}-\vec{p}')\,, \tag{28b}$$

with similar expressions for the creation and annihilation operators of the $\Xi^{A_\theta}(x)$ field. Several unusual features of expressions for the field operators $\Xi(x)$, Eqs (27a) and (27b), and commutators, (28a) and (28b), should be explicitly noted:

I. The factor $-(-1)^{|h|}\delta_{h,-h'}$, rather than the usual $\delta_{hh'}$, in the *rhs* of Eqs (28a) and (28b) arises from the bi-orthogonal [16] nature of $\rho[p^\mu]$ and $\lambda[p^\mu]$ spinors.

II. The creation operator $\mathcal{S}_h^{(\lambda)\dagger}(p^\mu)$ for the plane wave $\lambda_h^{S_\theta}[p^\mu]\exp(ipx)$ is identical (within a phase factor) to the creation operator for the plane wave $\rho_{-h}^{S_\theta}[p^\mu]\exp(-ipx)$

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[6] See footnote 19 of Ref. [15].

$$\mathcal{S}_h^{(\lambda)\dagger}(p^\mu) = -(-1)^{|h|}\mathcal{S}_{-h}^{(\rho)\dagger}(p^\mu)\,, \tag{29}$$

with similar comments applicable to $\mathcal{A}_h^{(\lambda)\dagger}(p^\mu)$ and $\mathcal{A}_h^{(\rho)\dagger}(p^\mu)$.

In addition, in view of our results of Sec. 4, the association of the $\rho[p^\mu]$ spinors with the forward-in-time propagating solutions and $\lambda[p^\mu]$ spinors with backward-in-time propagating solutions in the explicit expressions of $\Xi(x)$ above is purely a convention.

Finally, we wish to emphasize that the field operators we arrive at differ from similar expressions found in literature for the $(^1/_2,0)\oplus(0,^1/_2)$ Majorana field. Unlike the field operators $\Xi(x)$, these expressions[7] [even though they satisfy Eq. (25)] do not exploit the Majorana-construction in the $(j,0)\oplus(0,j)$ representation space *and* as a result cannot be expected to contain the full physical content of a truly neutral particle.

6. Concluding remarks

We have succeeded in extending the Majorana-construction for the $(^1/_2,0)\oplus(0,^1/_2)$ representation space to all $(j,0)\oplus(0,j)$ representation spaces despite the general impression that Majorana's original construction was due to a certain "magic of Pauli matrices." We studied the $(1,0)\oplus(0,1)$ Majorana-like representation space in some detail and presented an associated wave equation. Since nature has a host of neutral "fundamental particles" of spin-1 and -2 and composite hadronic structures of even higher spins, the existence of the Majorana-like $(j,0)\oplus(0,j)$ representation spaces introduced in this work may have some physical relevance for the unification beyond the electroweak theory and hadronic phenomenologies. In the massless limit, $(j,0)\oplus(0,j)$ fields, independent of spin and independent of whether they are Dirac-like or Majorana-like, contain only two helicity degrees of freedom. This observation allows the construction of higher-spin field theories without introducing or imposing any auxiliary fields, negative-norm states, or constraints. This fact may have some significance for theories involving supersymmetric transformations, which transform between fermions and bosons, and which are normally rife with nonphysical, additional fields. It should be explicitly noted that even though the construction of the $(j,0)\oplus(0,j)$ Majorana-like fields is *manifestly* covariant under parity, in general, massive Majorana-like particles carry *imaginary* intrinsic parity [18] and hence these particles in interactions with Dirac-like/Dirac particles (like charged leptons and quarks) naturally lead to non-conservation of parity.

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[7] See, for example, Eq. (3.25) of Ref. [15]; and Eq. (2.5) of Gluza and Zrałek's paper in Ref. [17].

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[18] For *spin*-$^1/_2$ see, for instance, P.C. Carruthers, *Spin and Isospin in Particle Physics*, Gordon and Breach Science Publishers, New York 1971; and for *spins* $j \geq 1$, results will be reported in a forthcoming publication.