TECHNIQUES FOR THE ANALYTICAL CALCULATION OF THE ELECTRON g–2*

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The present status of the calculation of the electron $g-2$, the anomalous magnetic moment, is reviewed. An outline of the methods used in analytical calculations is given, together with a detailed exposition of the "natural variables" required to rationalize square roots of integrals required.

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1. Premise and general problems

Electromagnetic vertex for a mass shell electron and a photon of arbitrary momentum Δ is written[1]

$$\mathcal{V}_\mu(p,\Delta) = (-e)\bar{u}_2(p-\tfrac{1}{2}\Delta)\Big(F_1(-\Delta^2)\gamma_\mu - \frac{i}{4m^2}F_2(-\Delta^2)(\gamma_\mu\not{\Delta} - \not{\Delta}\gamma_\mu)\Big)u_1(p+\tfrac{1}{2}\Delta), \tag{1.1}$$

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[1] Metric is $\delta_{\mu\nu}$ $\mu,\nu = 1,2,3,4$. For instance, an electron on mass shell has momentum $p_\mu = (\vec{p}, p_4) = (\vec{p}, ip_0)$, $p_\mu p_\mu = |\vec{p}|^2 - p_0^2 = -m^2$ and its propagator is: $(-i)\dfrac{-i\not{p}+m}{p^2+m^2}$.

where $-e, m$ are the electron charge and mass, $\bar{u}_2, u_1$ the spinor wave function of final and initial electron, and $p \mp {}^1\!/_2 \Delta$ their four momenta. $F_1(-\Delta^2)$ and $F_2(-\Delta^2)$ are the Dirac and Pauli form factors, depending only on $-\Delta^2 = \Delta_0^2 - |\vec{\Delta}|^2$, the moment transfer.
Renormalized perturbative QED expansion gives them as power series in α/π, α being the fine-structure constant [1]:

$$F_1(-\Delta^2) = 1 + \sum_{n=1}^{\infty} \left(\frac{\alpha}{\pi}\right)^n F_1^{(2n)}(-\Delta^2),$$
$$F_2(-\Delta^2) = \sum_{n=1}^{\infty} \left(\frac{\alpha}{\pi}\right)^n F_2^{(2n)}(-\Delta^2), \tag{1.2}$$

hence to zero th order in α one has simply:

$$F_1(-\Delta^2) = 1 + O(\alpha), \quad F_2(-\Delta^2) = 0 + O(\alpha).$$

Conservation of electric charge requires $F_1(0) = 1$, F_1 being the whole, physical, renormalized Dirac form factor at zero momentum transfer, *i.e.* $-\Delta^2 = 0$. A practical rule for renormalizing Dirac form factor is to subtract from it its infinite value at zero momentum transfer; the form factor so obtained satisfies $F_1(0) = 1$. On the contrary $F_1(-\Delta^2)$, Pauli form factor, needs no renormalization: it vanishes at zero order, but not at zero momentum transfer, and it is a pure radiative correction.

As it is well known, the anomalous magnetic moment of the electron is: $\mu_e = {}^1\!/_2 g \mu_B$, where μ_B is Bohr magneton. One has from QED: $g = 2\big(F_1(0) + F_2(0)\big)$. At zero th order in α due to (1.2) one has $g = 2$, but, at $-\Delta^2 = 0$, according to radiative corrections and the condition $F_1(0) = 1$, one has $g = 2(1 + F_2(0))$. It is preferred to define the "electron anomaly" a_e so that

$$a_e \equiv F_2(0) = \tfrac{1}{2}(g - 2). \tag{1.3}$$

From (1.2) one has then that also the "electron anomaly" a_e is expressed as a power series in α/π :

$$a_e^{\text{th}} = \sum_{n=1}^{\infty} a_n \left(\frac{\alpha}{\pi}\right)^n. \tag{1.4}$$

2. Experimental and theoretical values

a) Experimental Values:

The anomalous magnetic moment of the electron and of the positron have been measured by Van Dyck, Jr., Schwinger and Dehmelt up to 0.004 p.p.m. [2–4], using Penning trap technique:

$$\begin{aligned} a_{e^-}^{\rm exp} &= 1\ 159\ 652\ 188.4\ (4.3) \times 10^{-12}\,, \\ a_{e^+}^{\rm exp} &= 1\ 159\ 652\ 187.9\ (4.3) \times 10^{-12}\,. \end{aligned} \qquad (2.1)$$

Experimental error in (2.1) derives from several sources. Statistical error: 0.62×10^{-12}, error due to microwave power shift: 1.3×10^{-12}, error due to cavity shift: 4×10^{-12}.

The same experimental group obtained a new better result constructing a hyperboloid cavity to avoid strong coupling of electron with the resonances of the cavity:

$$a_e^{\rm exp} = 1\ 159\ 652\ 185.5\ (4.0) \times 10^{-12}\,. \qquad (2.2)$$

A precision of 0.0005 p.p.m. is expected.

It is worth while to note that the equivalence of the values for e^- and e^+ is one of the best proofs of charge conjugation. Such an accuracy demands comparable or better theoretical precision to become perhaps the most precise test of QED, or the QED definition of α (which would be now the most accurate) to be compared with the determinations offered by the new solid–state effects.

b) Theoretical Values:

In perturbative QED a_e is written [4]:

$$a_e^{\rm th} = A_1 + A_2\left(\frac{m_e}{m_\mu}\right) + A_3\left(\frac{m_e}{m_\tau}\right) + A_4\left(\frac{m_e}{m_\mu}, \frac{m_e}{m_\tau}\right) \qquad (2.3)$$

and each term A_k, $k = 1,2,3,4$, can then be expanded as a power series in α/π (1.4):

$$A_k = a_k^{(1)}\left(\frac{\alpha}{\pi}\right) + a_k^{(2)}\left(\frac{\alpha}{\pi}\right)^2 + a_k^{(3)}\left(\frac{\alpha}{\pi}\right)^3 + a_k^{(4)}\left(\frac{\alpha}{\pi}\right)^4 + \ldots\,, \qquad k = 1,2,3,4\,, \qquad (2.4)$$

where $a_k^{(n)}$, $k = 1,2,3,4$ and $n = 1,2,\ldots$ are numerical coefficients free from infrared divergences.

First and second order coefficients of A_1, $a_1^{(1)}$ and $a_1^{(2)}$, have been known analytically since a long time [1, 5]:

$$\begin{aligned} a_1^{(1)} &= \tfrac{1}{2}\,, \\ a_1^{(2)} &= \frac{197}{144} + \tfrac{1}{2}\zeta(2) + \tfrac{3}{4}\zeta(3) - 3\zeta(2)\,{\rm lg}(2) = -0.328\ 478\ 965\,. \end{aligned} \qquad (2.5)$$

Riemann $\zeta(n)$ function is used: $\zeta(n) = \sum_{p=1}^{\infty} \frac{1}{p^n}, \quad n = 2, 3, \ldots$
whose values are (analytical values are known only for even integers arguments):

$$\zeta(2) = 1.644\,934\,067\ \ldots = \frac{\pi^2}{6}, \qquad \zeta(3) = 1.202\,056\,903\ \ldots,$$
$$\zeta(4) = 1.082\,323\,234\ \ldots = \frac{\pi^4}{90}, \qquad \zeta(5) = 1.036\,927\,755\ \ldots,$$
$$\ldots$$
$$\zeta(2n) = \text{const}\ \pi^{2n}.$$

The values of third and fourth order coefficients a_3 and a_4 are known only numerically [3, 4]:

$$a_1^{(3)} = 1.176\ 13\ (42) \qquad (\text{error } 0.04\%),$$
$$a_1^{(4)} = -1.434\ (138) \qquad (\text{error } 10\ \%). \tag{2.6}$$

Third order coefficient $a_1^{(3)}$ consists of 72 Feynman Graphs (40 of them different) belonging to 3 different sets:

1) 6 *light light* graphs, recently evaluated in close analytic form [11],
2) 16 *vacuum polarization* graphs, known analytically [6],
3) 50 *three photon exchange* graphs ([3, 4, 7–9] and references therein) subdivided into three topological families:

3a) twice reducible diagrams (no crossed photon lines), known analytically,
3b) once reducible diagrams (2 crossed photon lines): many are known analytically, but the whole family is known only numerically,
3c) irreducible diagrams (3 crossed photon lines), known only numerically.

Fourth order coefficient $a_1^{(4)}$ consists of 891 Feynman graphs, and its calculation can be done only numerically, not only due to the number of the graphs, but also for the essentially elliptical nature of required integrals [3, 4]. Some simple graphs have been calculated analytically mainly to check numerical calculation of the whole [10].
Higher terms in α/π, $a_1^{(n)}, n \geq 5$, are too small to be significative at present. It seems impossible as far to now to calculate them also numerically, nevertheless some simple graphs have been calculated (analytically or only numerically) [10].

Summing up the results (2.5) (2.6) and using the most precise available value of α from Quantum Hall Effect (QHE) [12]:

$$\alpha^{-1}(\text{QHE}) = 137.035\,997\,9\ (32)\,, \qquad (\text{error } 0.024 \text{ p.p.m.}) \tag{2.7}$$

one has [4]:

$$A_1 \;=\; 1\,159\,652\,136.2\ (5.3)(4.1)(27.1)\times 10^{-12} \tag{2.8}$$

Error in the value (2.8) of A_1 derives [3, 4]:

$$\begin{array}{rl} 5.3\times 10^{-12} & \text{from } a_1^{(3)}\,, \\ 4.1\times 10^{-12} & \text{from } a_1^{(4)}\,, \\ 27.1\times 10^{-12} & \text{from } \alpha\,. \end{array}$$

It is worth while to note that the error is dominated by the error of α.

Other QED contributions to electron anomaly are [3, 4]:

$$\begin{aligned} A_2(\ \mu \text{ meson}) &= +\frac{1}{45}\Big(\frac{m_e}{m_\mu}\Big)^2\left(\frac{\alpha}{\pi}\right)^2 \simeq 2.804\times 10^{-12}\,, \\ A_3(\ \tau \text{ meson}) &\simeq 0.010\times 10^{-12}\,. \end{aligned} \tag{2.9}$$

The most significative not QED contributions are

$$\begin{aligned} a(\text{hadrons}) &\simeq 1.6(2)\times 10^{-12}\,, \\ a(\text{weak}) &\simeq 0.05\times 10^{-12}\,. \end{aligned} \tag{2.10}$$

The importance of the hadronic contribution has to be noted.
Other terms, A_4 and further not QED contributions, are to small to be to be significative at present.

Summing up the results of (2.8), (2.9), (2.10), the theoretical value of a_{e^-} becomes [4]:

$$a_{e^-}^{\text{th}} \;=\; 1\,159\,652\,140.7\ (28)\times 10^{-12} \tag{2.11}$$

which agree within 1.7 standard deviation with (2.1) and (2.2). It is worth to note that the theoretical error is larger than the experimental error: it becomes then necessary to improve the theoretical value!

As the error is still dominated by the error of α, to have a better test of QED it is necessary to improve the value of α. Conversely it is possible to obtain the value of α comparing the experimental and theoretical values of a_e to get [4]:

$$\alpha^{-1}(\text{QED}) = 137.035\,992\,22\,(94) \qquad (\text{error } 0.0069 \text{ p.p.m.}) \tag{2.12}$$

which is better than previous non QED results (2.7).

3. Techniques and algorithms

3.1. Form factor decomposition

To calculate electron anomaly it is enough to know the off mass shell vertex amplitude up to the first order in the photon momentum and to expand it with respect to Δ for an arbitrary p [7]:

$$\mathcal{W}_\mu(p,\Delta) = \mathcal{V}_\mu(p) + \Delta_\nu \mathcal{T}_{\mu\nu}(p) + \dots \tag{3.1.1}$$

On invariance grounds the following decomposition hold:

$$\begin{aligned}\mathcal{V}_\mu(p) &= V_1\gamma_\mu + V_2\tfrac{1}{2}\{\gamma_\mu, i\not{p}+1\} + V_3 i p_\mu(i\not{p}+1) + V_4\tfrac{1}{2}[\gamma_\mu, i\not{p}+1]\,,\\ \mathcal{T}_{\mu\nu}(p) &= T_1\tfrac{1}{2}i[\gamma_\mu,\gamma_\nu] + T_2\tfrac{1}{4}\{[\gamma_\mu,\gamma_\nu],\not{p}\} + T_3 i[p_\mu,\gamma_\nu]\not{p} + T_4\tfrac{1}{2}i[\{\not{p}_\mu,\gamma_\nu\},\not{p}]\\ &\quad + T_5 i\delta_{\mu\nu} + T_6\delta_{\mu\nu}\not{p} + T_7 i p_\mu p_\nu + T_8 p_\mu p_\nu \not{p} + T_9[p_\mu,\gamma_\nu] + T_{10}\{p_\mu,\gamma_\nu\}\,,\end{aligned} \tag{3.1.2}$$

where $[A,B] \equiv AB - BA$, $\{A,B\} \equiv AB + BA$, and electron mass $= 1$ everywhere.
Having already taken the $\Delta \to 0$ limit, all the above factors depend only on the variable $u \equiv -p^2$:

$$\begin{aligned} V_i &\equiv V_i(u) \qquad && i = 1,\dots,4\,,\\ T_j &\equiv T_j(u) \qquad && j = 1,\dots,10\,.\end{aligned} \tag{3.1.3}$$

Moreover, due to charge conjugation, $V_4 = 0$ and $T_j = 0, j \geq 5$.
Closing $\mathcal{W}_\mu$ between spinor states and comparing with usual expression for the vertex in the case of mass-shell electrons and arbitrary photon momentum (1.1), one finds that the anomaly is:

$$a \equiv F_2(0) = V_2(1) - 2T_1(1) - 2T_2(1)\,. \tag{3.1.4}$$

To obtain V_i, T_j to be used in (3.1.4) one has to work out, by a straightforward but cumbersome calculation, projectors $P^i_\mu(p), P^j_{\mu\nu}(p)$ to obtain form factors by a suitable trace:

$$\begin{aligned} V_i &= \text{Tr}\,\{P^i_\mu(p)\mathcal{V}_\mu(p)\} \qquad && i = 1,\dots,4\,;\\ T_j &= \text{Tr}\,\{P^j_{\mu\nu}(p)\mathcal{T}_{\mu\nu}(p)\} \qquad && j = 1,\dots,10\,.\end{aligned} \tag{3.1.5}$$

These projectors are found to be:

Vectorial Projectors for $\mathcal{V}_\mu(p)$

$$
\begin{aligned}
P^1_\mu(p) &= \frac{1}{12p^2}\left((p^2+1)\gamma_\mu + 3ip_\mu - (\frac{4}{p^2}+1)p_\mu \not{p}\right),\\
P^2_\mu(p) &= \frac{1}{12p^2}\left(\frac{4}{p^2}p_\mu \not{p} - 3ip_\mu - \gamma_\mu\right),\\
P^3_\mu(p) &= \frac{1}{12p^2}\left(\gamma_\mu - \frac{4}{p^2}p_\mu \not{p}\right),\\
P^4_\mu(p) &= \frac{i}{12p^2}\left(p_\mu - \not{p}\gamma_\mu\right).
\end{aligned}
\tag{3.1.6}
$$

Tensorial Projectors for $\mathcal{T}_{\mu\nu}(p)$

$$
\begin{aligned}
P^1_{\mu\nu}(p) &= \frac{-i}{24p^2}\left(-\tfrac{1}{2}p^2[\gamma_\mu,\gamma_\nu] + \not{p}[p_\mu,\gamma_\nu]\right),\\
P^2_{\mu\nu}(p) &= \frac{1}{24p^2}\left(-\tfrac{1}{2}\not{p}[\gamma_\mu,\gamma_\nu] + [p_\mu,\gamma_\nu]\right),\\
P^3_{\mu\nu}(p) &= \frac{-i}{24p^2}\left(-\tfrac{1}{2}p^2[\gamma_\mu,\gamma_\nu] + 2\not{p}[p_\mu,\gamma_\nu]\right),\\
P^4_{\mu\nu}(p) &= \frac{-i}{24p^2}\left(-2p_\mu p_\nu + \not{p}\{p_\mu,\gamma_\nu\}\right),\\
P^5_{\mu\nu}(p) &= \frac{-i}{12p^2}\left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right),\\
P^6_{\mu\nu}(p) &= \frac{1}{12p^4}\left(p^2\delta_{\mu\nu} + p_\mu p_\nu\right)\not{p},\\
P^7_{\mu\nu}(p) &= \frac{-i}{12p^4}\left(-p^2\delta_{\mu\nu} + 4p_\mu p_\nu\right),\\
P^8_{\mu\nu}(p) &= \frac{1}{12p^6}\left(-p^2\not{p}\delta_{\mu\nu} + 6\not{p}p_\mu p_\nu - p^2[p_\mu,\gamma_\nu]\right),\\
P^9_{\mu\nu}(p) &= \frac{1}{24p^2}[p_\mu,\gamma_\nu],\\
P^{10}_{\mu\nu}(p) &= \frac{1}{24p^2}\left(-2\not{p}p_\mu p_\nu + p^2[p_\mu,\gamma_\nu]\right).
\end{aligned}
\tag{3.1.7}
$$

As already stated, once the $\Delta \to 0$ limit is taken, all the above factors V, T (3.1.4) depend only on the variable $u \equiv -p^2$, like the form factors

$$S_k(u),\ k = 1,2 \tag{3.1.8}$$

of the self–mass amplitude:

$$i\Sigma(p) = i(i\not{p} + 1)[S_1(u) + (i\not{p} + 1)S_2(u)]\,, \tag{3.1.9}$$

where mass renormalization has been carried out, but not wave function renormalization. Of course also self-mass form factors can be obtained like (3.1.4).

$$S_k(u) = Tr\ \{P^k(p)i\Sigma(p)\}, \qquad k = 1,2 \tag{3.1.10}$$

and respectively:

Scalar Projectors for $i\Sigma(p)$

$$P^1(p) = -\frac{i}{4p^2}\left((p^2-1)\frac{-i\not{p}+1}{p^2+1}+1\right),$$
$$P^2(p) = -\frac{i}{4p^2}\left(\frac{-i\not{p}+1}{p^2+1}-1\right). \tag{3.1.11}$$

In the $\Delta \to 0$ limit the only differences between vertex and self–mass are:

i) strings of γ–matrices in the numerator, giving different polynomials in the external and loop momenta when the form factors are projected;

ij) square denominators in the vertex amplitudes. As the main difficulties in analytical calculation come from denominators, the "core" of the analytical calculation is to handle them, and it important to be able to handle them in a simpler form. One has, for instance, that the denominator of the electron–propagator just before and after the emission of the external photon can be expanded in Δ:

$$\frac{1}{(p\pm\frac{1}{2}\Delta)^2+1} = \frac{1}{p^2+1} \mp \frac{p\Delta}{(p^2+1)^2} + O(\Delta^2) \Rightarrow$$
$$\frac{1}{[(p+\frac{1}{2}\Delta)^2+1]\,[(p-\frac{1}{2}\Delta)^2+1]} = \frac{1}{(p^2+1)^2} + O(\Delta^2)\,.$$

Integrals with square denominators can be calculated as derivatives of suitable single denominators

$$\int d^4p\ \ldots\ \frac{1}{(p^2+1)^2} = -\frac{\partial}{\partial\mu^2}\int d^4p\ \ldots\left(\frac{1}{p^2+\mu^2}-\frac{1}{p^2+M^2}\right)\Bigg|_{\mu^2=1},$$

where the exchange between integration and derivation is allowed, and Pauli–Villars counter term is only for convergence, and does not play any other role.

When a suitable analytic expression for the self-mass form factors is obtained, vertex form factors can then be derived by "straightforward" operations, such as rearranging terms and taking derivatives with respect to intermediate masses.

The knowledge of the "off mass shell" amplitudes permits also their insertion in more complicated graphs, being careful of infrared divergences. It becomes then possible to calculate separate single "subgraphs" and to "assemble" them in the complete graph. This technique allows, for instance, a rather simple evaluation of "multiple-ladder" graphs [10], which were the most difficult using old methods [1].

3.2. Integration

In principle one has the standard dispersion relations:

$$F(-q^2) = \frac{1}{\pi} \int \frac{du}{q^2 + u} \, \mathrm{Im}\, F(u) \,, \tag{3.2.1}$$

where F may be any of V, T, S (3.1.3) (3.1.8) and discontinuities are taken according Cutkosky–Veltman rule [13] before performing the $(-\partial/\partial\mu^2)$, if necessary. Subtracted relations are used if necessary, and, as usual, infrared singularities are parameterized giving fictitious mass λ to photon.

If the number of graphs increases too much and the most complicated structure can be isolated, it may be convenient to Wick rotate dp_0 and use hyperspherical integration $\int dp_1 dp_2 dp_3 dp_4$ for peripheric propagators [9].

These techniques permit to control ellipticity and divergences. About ellipticity, although there is a lot of radicals involving integration variables, they are all of the kind:

$$R(a, b, c) = \sqrt{a^2 + b^2 + c^2 - 2ab - 2bc - 2ac}, \tag{3.2.2}$$

which is the usual "phase space root". The integrations can be conveniently done by introducing the "natural variables" [1, 7, 9] which allow their analytical evaluation (if the roots are not too many!). The choice of the suitable "natural variable" depends on the integration range, determined by the values of a, b. For instance, given the complete root of eq. (3.2.2), when $a \in [(\sqrt{b} + \sqrt{c})^2, \infty[$, *i.e.* over the threshold, one has to perform the change of variable:

$$a = +\frac{\sqrt{bc}}{x}\left(x + \sqrt{\frac{b}{c}}\right)\left(x + \sqrt{\frac{c}{b}}\right) \iff x = \frac{+(+a - b - c) - R(+a, +b, +c)}{2\sqrt{bc}} \tag{3.2.3}$$

to have:

$$R(a,b,c) = \frac{\sqrt{bc}}{x}(1-x^2). \tag{3.2.4}$$

Typical integrals can then be rationalized in the following way:

$$\int\limits_{(\sqrt{b}+\sqrt{c})^2}^{+\infty} \frac{da}{R(+a,+b,+c)} f(a) = \int\limits_0^1 \frac{dx}{x} f(a(x)), \tag{3.2.5}$$

$$R(+q,+b,+c) \int\limits_{(\sqrt{b}+\sqrt{c})^2}^{+\infty} \frac{da}{R(+a,+b,+c)} \frac{1}{a-q} f(a) =$$
$$+ \int\limits_0^1 \frac{dx}{x} \left(\frac{1}{x-q_+} - \frac{1}{x-q_-} f(a(x)) \right), \tag{3.2.6}$$

where:

$$q_{\pm} = +\frac{q-b-c \pm R(+q,+b,+c)}{2\sqrt{bc}}, \qquad q_+ q_- = +1.$$

Such changes of variables must be done as late as possible, because due to them the number of terms increases enormously also for computer, and moreover it becomes extremely hard to control fictitious divergences arising from separate calculation of integrals. It is then essential to manipulate integrals in their elliptic aspect as far as it is possible, and to find relations to calculate analytically groups of integrals which would be one by one elliptic.

When the suitable method has been chosen, analytical calculations [1, 6–10] are performed using powerful and fast algebraic computer programs: SCHOONSCHIP [14], by Veltman, on CDC and (now) on SUN computers, ASHMEDAI by Levine on VAX computers and now FORM by Vermaseren [15] on VAX computers (an PC's too). Numerical calculations [3, 4] are done using the adaptative Monte Carlo integration routines RIWIAD and VEGAS [16].

3.3. Nielsen polylogarithms

Multiple integration of logarithms required by the calculation of Feynman graphs requires the knowledge of a special class of functions, the *Nielsen*

Polylogarithms [1, 9, 17], defined as:

$$S_{n,p}(x) = \frac{(-1)^{(n+p-1)}}{(n-1)!p!} \int_0^1 \frac{dt}{t} \lg^{n-1} t \lg^p(1-xt) \qquad n,p \geq 1\,. \quad (3.3.1)$$

For $p = 1$ one defines:

$$Li_n(x) \equiv S_{n-1,1}(x)\,, \quad (3.3.2)$$

that when $n = p = 1$ reduces to the *Euler Dilogarithm*:

$$S_{1,1}(x) = Li_2(x) = -dt \int_0^1 \frac{\lg(1-xt)}{t} = -dt \int_0^x \frac{\lg(1-t)}{t} \quad (3.3.3)$$

which is strictly related to the so-called Spence function, defined usually as:

$$\Phi(x) \equiv \int_1^x \frac{\lg(1+t)}{t} dt = -Li_2(-x) + Li_2(-1)\,. \quad (3.3.4)$$

It has been introduced (see, for example, [18] and references therein) also the *Rogers Dilogarithm*:

$$\mathcal{L}(x) = -\tfrac{1}{2} \int_0^x dt \left(\frac{\lg(1-t)}{t} + \frac{\lg(t)}{1-t} \right), \quad (3.3.5)$$

that, using Euler Dilogarithm, after a simple partial integration reduces to

$$\mathcal{L}(x) = Li_2(x) + \tfrac{1}{2} \lg x \lg(1-x).$$

One has the most important rules (often not enough pointed out)

$$\frac{d}{dx} S_{n,p}(x) = \frac{1}{x} S_{n-1,p}(x)\,, \qquad \int_0^x \frac{dt}{t} S_{n,p}(t) = S_{n+1,p}(x)\,. \quad (3.3.6)$$

The possibility to extend to usual logarithms these rules, defined above only for $n, p \geq 1$, also to $n = 0, p \geq 1$ defining

$$S_{0,p}(x) \equiv \frac{(-1)^p}{p!} \lg^p(1-x)\,, \quad (3.3.7)$$

suggests the introduction of a classification of the "trascendentality", the Logarithmic Degree (LD):

$$LD\ \{S_{n,p}\} = n + p\,, \tag{3.3.8}$$

giving LD to products of logarithms according to the usual rule

$$LD\ \{\lg^{p_1}(\ldots)\lg^{p_2}(\ldots)\} = p \quad \text{if} \quad p_1 + p_2 + \ldots\,,$$

extended to

$$LD\ \{S_{n_1,p_1}(\ldots)\ S_{n_2,p_2}(\ldots)\} = n_1 + p_1 + n_2 + p_2$$

and to

$$LD = 0$$

for pure algebraic expressions.
LD is conserved in algebraic transformations of the argument, like $x \to 1 - x$, $x \to 1/x$, $\ldots$, integrations by parts or sums and decomposition, but, due to (3.3.6), LD *is increased* by 1 by non trivial integrations (*i.e.* not by part) and *is decreased* by 1 by differentiation:

$$\begin{aligned} &LD \left\{ \int dx \left(\frac{1}{x}, \frac{1}{x-1}, \frac{1}{x+1}, \ldots \right) \ S(x, LD = n) \right\} = n + 1\,, \\ &LD \left\{ \frac{d}{dx} S(x, LD = n) \right\} = n - 1\,. \end{aligned} \tag{3.3.9}$$

This structure suggests to handle "pure logarithmic expressions", *i.e.* expressions which are sum of addenda having the same LD (logarithmic degree), to obtain simpler and more ordered calculations.

3.4. Algorithms for analytical calculation

As stated above, it is better to handle "pure logarithmic expressions" [9, 19]. After some algebraic manipulations it is possible to write required integrals in one of the "standard" forms:

$$\begin{aligned} F_1(q,b,c) &= \int_{a_1}^{a_2} \frac{da}{R(a,b,c)} f(a,b,c)\,, \\ F_2(q,b,c) &= R(q,b,c) \int_{a_1}^{a_2} \frac{da}{R(a,b,c)} \frac{1}{(a-q)} f(a,b,c)\,, \end{aligned} \tag{3.4.1}$$

where R is the usual phase space root (3.2.2) and $f(a,b,c)$ is either a rational function or

$$f(a,b,c) = \lg \frac{\alpha(a,b) + \beta(a,b)R(a,b,c)}{\alpha(a,b) - \beta(a,b)R(a,b,c)}$$

and α, β may be either radicals or functions having logarithmic degree lower than $F(q,b,c)$.

Performing the change of integration variable from a to x (the "natural variable"), one obtains:

$$F_1(q,b,c) = \int_{x_1(a_1)}^{x_2(a_2)} \frac{dx}{x} f(a(x),b,c)\,,$$

$$F_2(q,b,c) = \int_{x_1(a_1)}^{x_2(a_2)} \frac{dx}{x} \left(\frac{1}{x-q_+} - \frac{1}{x-q_-} f(a(x),b,c) \right)\,.$$

It is clear due to (3.3.6) that if $f(a,b,c)$ has pure logarithmic degree, integrals (3.4.1) have themselves pure logarithmic degree, and precisely:

$$LD\,\{F_1\} = LD\,\{F_1\} = f(a,b,c)\,,$$
$$LD\,\{F_2\} = LD\,\{F_2\} = f(a,b,c)\,.$$

It is worth to differentiate them to have lower LD expressions:

$$\frac{\partial}{\partial b} F_1(q,b,c) = \frac{\partial}{\partial b}\left\{ \int_{a_1}^{a_2} \frac{da}{R(a,b,c)} f(a,b,c) \right\} = \left[\left(\frac{\partial a}{\partial b} - \frac{a+b-c}{2b} \right) \right]_{a=a_1}^{a=a_2}$$
$$+ \int_{a_1}^{a_2} \frac{da}{R(a,b,c)} \left\{ \frac{\partial f(a,b,c)}{\partial b} + \frac{a+b-c}{2b} \, \frac{\partial f(a,b,c)}{\partial a} \right\},$$

$$\frac{\partial}{\partial b} F_2(q,b,c) = \frac{\partial}{\partial b}\left\{ R(q,b,c) \int_{a_1}^{a_2} \frac{da}{R(a,b,c)} \frac{1}{(a-q)} f(a,b,c) \right\} =$$
$$\left[\left(\frac{\partial a}{\partial b} \frac{R(q,b,c)}{(a-q)} - \frac{a(-b-q+c) + 3b^2 - 2bc - qb + qc - c^2}{2bR(q,b,c)} \right) \frac{f(a,b,c)}{R(a,b,c)} \right]_{a=a_1}^{a=a_2}$$
$$+ \int_{a_1}^{a_2} \frac{da}{R(a,b,c)} \left\{ \frac{R(q,b,c)}{(a-q)} \frac{\partial f(a,b,c)}{\partial b} \right.$$
$$\left. - \frac{a(-b-q+c) + 3b^2 - 2bc - qb + qc - c^2}{2bR(q,b,c)} \, \frac{\partial f(a,b,c)}{\partial a} \right\}. \tag{3.4.2}$$

Performing the change of variable (3.2.5) (3.2.6): this must be done several times to obtain a "simple" expression, *i.e.* which can be integrated by means of the changes of variable of the kind of Eq. (3.2.3). Repeated integrations give then the final result, that, as usual in QED calculations, is quite simple, despite the amount of work. Is there a more compact approach to QED? The problem stands still open, also if many improvements have been made since thirty years!

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Appendix A

"Natural variables" to rationalize phase space roots

As pointed out in Sect. 3.4 radicals involved are always of the kind of Eq. (3.4.1). The changes of variable required to rationalize them belong to the same family, although slight differences occur depending on the values of the arguments. Due to their wide use in such and similar calculations, it seems worth to write them in detail.

Case 1: $R(+a,+b,+c) = \sqrt{a^2+b^2+c^2-2ab-2ac-2bc}$.
One searches to rationalize it in the following way:

$$\begin{aligned} R^2(+a,+b,+c) &= R^2(-a,-b,-c) \\ &= a^2 - 2(b+c)a + (b-c)^2 \\ &= a^2+b^2+c^2-2ab-2ac-2bc \\ &= \left[a-(\sqrt{b}+\sqrt{c})^2\right]\left[a-(\sqrt{b}-\sqrt{c})^2\right] \\ &= \frac{bc}{x^2}|1-x^2|^2 . \end{aligned} \tag{A.1}$$

This can be performed using the change of variable:

$$\begin{aligned} a_{\pm}(x) &= \pm\sqrt{bc}\frac{\left(x\pm\sqrt{\frac{b}{c}}\right)\left(x\pm\sqrt{\frac{c}{b}}\right)}{x} \\ &= \pm\frac{\sqrt{bc}}{x}\left(x^2 \pm \frac{+b+c}{\sqrt{bc}}x + 1\right) \\ &= \pm\sqrt{bc}\left(x+\frac{1}{x}\right) + b + c . \end{aligned} \tag{A.2}$$

Solving Eq. (A.2) with respect to x, one has the solutions $x_+^{\pm}$ for a_+ and $x_-^{\pm}$ for a_- :

$$x_+^{\pm}(a_+) = \frac{+(a-b-c) \pm R(+a,+b,+c)}{2\sqrt{bc}},$$
$$x_-^{\pm}(a_-) = \frac{-(a-b-c) \pm R(+a,+b,+c)}{2\sqrt{bc}}, \tag{A.3}$$

having the properties:

$$x_+^+ = 1/x_+^-\,, \quad x_-^+ = 1/x_-^-\,, \quad x_+^+ = -x_-^-\,, \quad x_+^- = -x_-^+\,, \tag{A.4}$$

$$a \in [(\sqrt{b}+\sqrt{c})^2, +\infty[\quad \Rightarrow$$
$$x_+^+ \in [+1,+\infty[; \quad x_+^- \in [+1,0]; \quad x_-^+ \in [-1,0]; \quad x_-^- \in [-1,-\infty]$$

$$a \in]-\infty, (\sqrt{b}-\sqrt{c})^2] \quad \Rightarrow$$
$$x_+^+ \in [0,-1]; \quad x_+^- \in]-\infty,-1]; \quad x_-^+ \in [+\infty,+1]; \quad x_-^- \in [0,+1]$$
$$a \in [(\sqrt{b}-\sqrt{c})^2, (\sqrt{b}+\sqrt{c})^2] \quad \Rightarrow \quad R(+a,+b,+c) \notin \mathrm{Re}\,. \tag{A.5}$$

The order of the extremes in the intervals of Eq. (A.9) shows the direction of the variation of x with respect to a. As it is required for convenience of calculation that $x \in [0,+1]$, one has to choose:
x_+^- for the values of a over the threshold and x_-^- for the values of a under the threshold.
Summarizing the results, one has:

- $$a \in [(\sqrt{b}+\sqrt{c})^2, +\infty[\quad \Rightarrow \quad x_+^- \in [0,1]\,, \tag{A.6}$$

$$a = a_+(x_+^-) = +\sqrt{bc}\frac{\left(x+\sqrt{\frac{b}{c}}\right)\left(x+\sqrt{\frac{c}{b}}\right)}{x}$$
$$= +\frac{\sqrt{bc}}{x}\left(x^2 + \frac{+b+c}{\sqrt{bc}}x + 1\right) = +\sqrt{bc}\left(x+\frac{1}{x}\right) + b + c\,,$$

$$x = \frac{+(a-b-c) - R(+a,+b,+c)}{2\sqrt{bc}}\,,$$

$$R(+a,+b,+c) = \sqrt{a^2+b^2+c^2-2ab-2ac-2bc} = \frac{\sqrt{bc}}{x}(1-x^2)$$

and, for the typical integrals:

$$\int_{(\sqrt{b}+\sqrt{c})^2}^{+\infty} \frac{da}{R(+a,+b,+c)} f(a) = \int_0^1 \frac{dx}{x} f(a(x)),$$

$$R(+q,+b,+c) \int_{(\sqrt{b}+\sqrt{c})^2}^{+\infty} \frac{da}{R(+a,+b,+c)} \frac{1}{(a-q)} f(a) =$$

$$+ \int_0^1 \left(\frac{1}{x-q_+} - \frac{1}{x-q_-} \right) f(a(x)),$$

$$q_\pm = + \frac{q-b-c \pm R(+q,+b,+c)}{2\sqrt{bc}}, \qquad q_+ q_- = +1.$$

- $$a \in]-\infty, (\sqrt{b}-\sqrt{c})^2] \quad \Rightarrow \quad x_-^- \in [0,1], \tag{A.7}$$

$$a = a_-(x_-^-) = -\sqrt{bc} \frac{\left(x - \sqrt{\frac{b}{c}}\right)\left(x - \sqrt{\frac{c}{b}}\right)}{x}$$

$$= -\frac{\sqrt{bc}}{x}\left(x^2 - \frac{+b+c}{\sqrt{bc}} x + 1\right)$$

$$= -\sqrt{bc}\left(x + \frac{1}{x}\right) + b + c,$$

$$x = \frac{-(a-b-c) - R(+a,+b,+c)}{2\sqrt{bc}},$$

$$R(+a,+b,+c) = \sqrt{a^2+b^2+c^2-2ab-2ac-2bc} = \frac{\sqrt{bc}}{x}(1-x^2),$$

$$\int_{-\infty}^{(\sqrt{b}-\sqrt{c})^2} \frac{da}{R(+a,+b,+c)} f(a) = \int_0^1 \frac{dx}{x} f(a(x)),$$

$$R(+q,+b,+c) \int_{-\infty}^{(\sqrt{b}-\sqrt{c})^2} \frac{da}{R(+a,+b,+c)} \frac{1}{(a-q)} f(a) =$$

$$- \int_0^1 \left(\frac{1}{x-q_+} - \frac{1}{x-q_-} \right) f(a(x)),$$

$$q_\pm = -\frac{q - b - c \pm R(+q,+b,+c)}{2\sqrt{bc}}\,, \quad q_+ q_- = +1\,.$$

- $$a \in [(\sqrt{b}-\sqrt{c})^2, (\sqrt{b}+\sqrt{c})^2] \quad \Rightarrow R(+a,+b,+c) \notin \text{Re}\,. \qquad \text{(A.8)}$$

A most common special case occurs when $b, c = 1$, *i.e.* are on mass shell. Referring to case 1, one has:

Case 1a: $R(+a,+1,+1) = \sqrt{a^2 - 4a}$

$$R^2(+a,+1,+1) = a^2 - 4a = a(a-4) = \frac{1}{x^2}|1 - x^2|^2\,, \qquad \text{(A.9)}$$

$$a_\pm(x) = \pm\frac{(x \pm 1)^2}{x} = \pm\frac{x^2 \pm 2x + 1}{x} = \pm(x + \frac{1}{x}) + 2\,, \qquad \text{(A.10)}$$

$$x^\pm_+(a_+) = \frac{+(a-2) \pm \sqrt{a(a-4)}}{2}\,, \quad x^\pm_-(a_-) = \frac{-(a-2) \pm \sqrt{a(a-4)}}{2}\,, \qquad \text{(A.11)}$$

$$x^+_+ = 1/x^-_+\,, \quad x^+_- = 1/x^-_-\,, \quad x^+_+ = -x^-_-\,, \quad x^-_+ = -x^+_-\,. \qquad \text{(A.12)}$$

$$\begin{aligned}
&a \in [+4, +\infty[\quad \Rightarrow \\
&x^+_+ \in [+1, +\infty[; \quad x^-_+ \in [+1, 0]; \quad x^+_- \in [-1, 0]; \quad x^-_- \in [-1, -\infty[\\
&a \in]-\infty, 0] \quad \Rightarrow \\
&x^+_+ \in [0, -1]; \quad x^-_+ \in]-\infty, -1]; \quad x^+_- \in [+\infty, +1]; \quad x^-_- \in [0, +1] \\
&a \in [0, +4] \quad \Rightarrow \quad R(+a,+1,+1) = \sqrt{a(a-4)} \quad \notin \text{Re}\,. \qquad \text{(A.13)}
\end{aligned}$$

- $$a \in [+4, +\infty[\quad \Rightarrow \quad x^-_+ \in [0, 1]\,. \qquad \text{(A.14)}$$

$$a = +\frac{(x+1)^2}{x} = +\left(x + \frac{1}{x}\right) + 2\,, \quad x = \frac{+(a-2) - \sqrt{a(a-4)}}{2}\,,$$

$$\sqrt{a(a-4)} = \frac{1}{x}(1 - x^2)\,.$$

$$\int_4^{+\infty} \frac{da}{\sqrt{a(a-4)}} f(a) = \int_0^1 \frac{dx}{x} f(a(x))\,,$$

$$\sqrt{q(q-4)}\int_4^{+\infty}\frac{da}{\sqrt{a(a-4)}}\frac{1}{(a-q)}f(a)=+\int_0^1\left(\frac{1}{x-q_+}-\frac{1}{x-q_-}\right)f(a(x)),$$

$$q_\pm=+\frac{q-2\pm\sqrt{q(q-4)}}{2},\qquad q_+q_-=+1.$$

$$\bullet\qquad a\in]-\infty,0]\quad\Rightarrow\quad x_-^-\in[0,1], \tag{A.15}$$

$$a=-\frac{(x-1)^2}{x}=-\left(x+\frac{1}{x}\right)+2,$$

$$x=\frac{-(a-2)-\sqrt{a(a-4)}}{2},$$

$$\sqrt{a(a-4)}=\frac{1}{x}(1-x^2),$$

$$\int_{-\infty}^0\frac{da}{\sqrt{a(a-4)}}f(a)=\int_0^1\frac{dx}{x}f(a(x)),$$

$$\sqrt{q(q-4)}\int_{-\infty}^0\frac{da}{\sqrt{a(a-4)}}\frac{1}{(a-q)}f(a)=-\int_0^1\left(\frac{1}{x-q_+}-\frac{1}{x-q_-}\right)f(a(x)),$$

$$q_\pm=-\frac{q-2\pm\sqrt{q(q-4)}}{2},\qquad q_+q_-=+1.$$

$$\bullet\qquad a\in[0,+4]\quad\Rightarrow\sqrt{a(a-4)}\notin\mathrm{Re}\,. \tag{A.16}$$

Appendix B

In a similar way one has the following cases:

Case 2: $R(+a,-b,-c)\sqrt{a^2+2(b+c)a+(b-c)^2}$.

$$\begin{aligned}R^2(+a,-b,-c)&=R^2(-a,+b,+c)=a^2+2(b+c)a+(b-c)^2\\&=a^2+b^2+c^2+2ab+2ac-2bc\\&=\left[a+(\sqrt{b}+\sqrt{c})^2\right]\left[a+(\sqrt{b}-\sqrt{c})^2\right]\\&=\frac{bc}{x^2}|1-x^2|^2,\end{aligned} \tag{B.1}$$

$$a_\pm(x) = \pm\sqrt{bc}\frac{\left(x \mp \sqrt{\frac{b}{c}}\right)\left(x \mp \sqrt{\frac{c}{b}}\right)}{x}$$
$$= \pm\frac{\sqrt{bc}}{x}\left(x^2 \mp \frac{b+c}{\sqrt{bc}}x + 1\right)$$
$$= \pm\sqrt{bc}(x + \frac{1}{x}) - b - c\,, \tag{B.2}$$

$$x_+^\pm(a_+) = \frac{+(a+b+c) \pm R(+a,-b,-c)}{2\sqrt{bc}}\,,$$
$$x_-^\pm(a_-) = \frac{-(a+b+c) \pm R(+a,-b,-c)}{2\sqrt{bc}}\,. \tag{B.3}$$

$$x_+^+ = 1/x_+^-\,, \quad x_-^+ = 1/x_-^-\,, \quad x_+^+ = -x_-^-\,, \quad x_+^- = -x_-^+\,. \tag{B.4}$$

$$a \in [-(\sqrt{b} - \sqrt{c})^2, +\infty[\quad \Rightarrow$$
$$x_+^+ \in [+1, +\infty[\quad x_+^- \in [+1, 0] \quad x_-^+ \in [-1, 0] \quad x_-^- \in [-1, -\infty[$$
$$a \in]-\infty, -(\sqrt{b} + \sqrt{c})^2] \quad \Rightarrow$$
$$x_+^+ \in [0, -1] \quad x_+^- \in]-\infty, -1] \quad x_-^+ \in [+\infty, +1] \quad x_-^- \in [0, +1]$$
$$a \in [-(\sqrt{b} + \sqrt{c})^2, -(\sqrt{b} - \sqrt{c})^2] \quad \Rightarrow \quad R(+a,-b,-c) \notin \mathrm{Re}\,. \tag{B.5}$$

- $$a \in [-(\sqrt{b} - \sqrt{c})^2, +\infty[\quad \Rightarrow \quad x_+^- \in [0, 1]\,. \tag{B.6}$$

$$a = a_+(x_+^-) = +\sqrt{bc}\frac{\left(x - \sqrt{\frac{b}{c}}\right)\left(x - \sqrt{\frac{c}{b}}\right)}{x}$$
$$= +\frac{\sqrt{bc}}{x}\left(x^2 - \frac{b+c}{\sqrt{bc}}x + 1\right)$$
$$= +\sqrt{bc}(x + \frac{1}{x}) - b - c\,,$$
$$x = \frac{+(a+b+c) - R(+a,-b,-c)}{2\sqrt{bc}}\,,$$

$$R(+a,-b,-c) = \sqrt{a^2 + b^2 + c^2 + 2ab + 2ac - 2bc} = \frac{\sqrt{bc}}{x}(1 - x^2)\,,$$

$$\int_{(\sqrt{b}+\sqrt{c})^2}^{+\infty} \frac{da}{R(+a,-b,-c)} f(a) = \int_0^1 \frac{dx}{x} f(a(x)) \,,$$

$$R(+q,-b,-c) \int_{(\sqrt{b}+\sqrt{c})^2}^{+\infty} \frac{da}{R(+a,-b,-c)} \frac{1}{(a-q)} f(a) =$$

$$+ \int_0^1 \left(\frac{1}{x-q_+} - \frac{1}{x-q_-} \right) f(a(x)) \,,$$

$$q_\pm = + \frac{q-b-c \pm R(+q,-b,-c)}{2\sqrt{bc}} \,, \qquad q_+ q_- = +1 \,.$$

$$\bullet \qquad a \in]-\infty, -(\sqrt{b}+\sqrt{c})^2] \quad \Rightarrow \quad x_-^- \in [0,1] \,. \tag{B.7}$$

$$a = a_-(x_-^-) = -\sqrt{bc} \frac{\left(x + \sqrt{\frac{b}{c}}\right)\left(x + \sqrt{\frac{c}{b}}\right)}{x}$$

$$= -\frac{\sqrt{bc}}{x}\left(x^2 + \frac{b+c}{\sqrt{bc}} x + 1\right)$$

$$= -\sqrt{bc}(x + \frac{1}{x}) - b - c \,,$$

$$x = \frac{-(a+b+c) - R(+a,-b,-c)}{2\sqrt{bc}} \,.$$

$$R(+a,-b,-c) = \sqrt{a^2+b^2+c^2+2ab+2ac-2bc} = \frac{\sqrt{bc}}{x}(1-x^2) \,,$$

$$\int_{-\infty}^{-(\sqrt{b}+\sqrt{c})^2} \frac{da}{R(+a,-b,-c)} f(a) = \int_0^1 \frac{dx}{x} f(a(x)) \,,$$

$$R(+q,-b,-c) \int_{-\infty}^{(\sqrt{b}-\sqrt{c})^2} \frac{da}{R(+a,-b,-c)} \frac{1}{(a-q)} f(a) =$$

$$- \int_0^1 \left(\frac{1}{x-q_+} - \frac{1}{x-q_-} \right) f(a(x)) \,,$$

$$q_\pm = -\frac{q - b - c \pm R(+q,-b,-c)}{2\sqrt{bc}}, \qquad q_+q_- = +1\,.$$

- $$a \in [-(\sqrt{b}+\sqrt{c})^2, -(\sqrt{b}-\sqrt{c})^2] \quad \Rightarrow R(+a,-b,-c) \notin \mathrm{Re}\,. \quad \text{(B.8)}$$

Case 2a: $R(+a,-1,-1)\sqrt{a^2+4a}$.

$$R^2(+a,-1,-1) = a^2 + 4a = a(a+4) = \frac{1}{x^2}|1-x^2|^2\,, \tag{B.9}$$

$$a_\pm(x) = \pm\frac{(x\mp 1)^2}{x} = \pm\frac{x^2 \mp 2x + 1}{x} = \pm(x + \frac{1}{x}) - 2\,, \tag{B.10}$$

$$x_+^\pm(a_+) = \frac{+(a-2) \pm \sqrt{a(a+4)}}{2}, \qquad x_-^\pm(a_-) = \frac{-(a-2) \pm \sqrt{a(a+4)}}{2}, \tag{B.11}$$

$$x_+^+ = \frac{1}{x_+^-}, \quad x_-^+ = \frac{1}{x_-^-}, \quad x_+^+ = -x_-^-, \quad x_+^- = -x_-^+, \tag{B.12}$$

$$\begin{array}{l} a \in [0,+\infty[\quad \Rightarrow \\ x_+^+ \in [+1,+\infty[\quad x_+^- \in [+1,0] \quad x_-^+ \in [-1,0] \quad x_-^- \in [-1,-\infty[\\ a \in]-\infty,-4] \quad \Rightarrow \\ x_+^+ \in [0,-1] \quad x_+^- \in]-\infty,-1] \quad x_-^+ \in [+\infty,+1] \quad x_-^- \in [0,+1] \\ a \in [-4,0] \quad \Rightarrow \quad R(+a,-1,-1) = \sqrt{(a(a+4)} \quad \notin \mathrm{Re}\,. \end{array} \tag{B.13}$$

- $$a \in [0,+\infty[\quad \Rightarrow \quad x_+^- \in [0,1]\,. \tag{B.14}$$

$$a = +\frac{(x-1)^2}{x} = +\left(x - \frac{1}{x}\right) - 2\,, \quad x = \frac{+(a+2) - \sqrt{(a(a+4)}}{2},$$

$$\sqrt{a(a+4)} = \frac{1}{x}(1-x^2)\,,$$

$$\int_4^{+\infty} \frac{da}{\sqrt{a(a+4)}} f(a) = \int_0^1 \frac{dx}{x} f(a(x))\,,$$

$$\sqrt{q(q+4)}\int_4^{+\infty}\frac{da}{\sqrt{a(a+4)}}\,\frac{1}{(a-q)}\,f(a)=+\int_0^1\left(\frac{1}{x-q_+}-\frac{1}{x-q_-}\right)f(a(x))\,,$$

$$q_\pm=+\frac{q-2\pm\sqrt{q(q+4)}}{2}\,,\qquad\qquad q_+q_-=+1\,.$$

$$\bullet\qquad a\in]-\infty,-4]\quad\Rightarrow\quad x_-^-\in[0,1]\,. \tag{B.15}$$

$$a=-\frac{(x-1)^2}{x}=-(x+\frac{1}{x})-2\,,\quad x=\frac{-(a+2)-\sqrt{(a(a+4)}}{2}\,,$$

$$\sqrt{a(a+4)}=\frac{1}{x}(1-x^2)\,,$$

$$\int_{-\infty}^0\frac{da}{\sqrt{a(a+4)}}f(a)=\int_0^1\frac{dx}{x}f(a(x))\,,$$

$$\sqrt{q(q+4)}\int_{-\infty}^0\frac{da}{\sqrt{a(a+4)}}\,\frac{1}{(a-q)}f(a)=-\int_0^1\left(\frac{1}{x-q_+}-\frac{1}{x-q_-}\right)f(a(x))\,,$$

$$q_\pm=-\frac{q-2\pm\sqrt{q(q+4)}}{2}\,,\qquad\qquad q_+q_-=+1\,.$$

$$\bullet\qquad a\in[-4,0]\quad\Rightarrow\sqrt{a(a+4)}\notin\mathrm{Re}\,. \tag{B.16}$$

Appendix C

Case 3: $R(+a,-b,+c)\sqrt{a^2+2(b-c)a+(b+c)^2}$

$$\begin{aligned}R^2(+a,-b,+c)=R^2(-a,+b,-c)&=a^2+2(b-c)a+(b+c)^2\\&=a^2+b^2+c^2+2ab-2ac+2bc\\&=\left[a-(\sqrt{c}+i\sqrt{b})^2\right]\left[a-(\sqrt{c}-i\sqrt{b})^2\right]\\&=\frac{bc}{x^2}|1+x^2|^2\,,\end{aligned} \tag{C.1}$$

$$a_\pm(x) = \pm\sqrt{bc}\frac{\left(x \mp \sqrt{\frac{b}{c}}\right)\left(x \pm \sqrt{\frac{c}{b}}\right)}{x}$$
$$= \pm\frac{\sqrt{bc}}{x}\left(x^2 \pm \frac{-b+c}{\sqrt{bc}}x - 1\right) = \pm\sqrt{bc}\left(x - \frac{1}{x}\right) - b + c\,, \quad \text{(C.2)}$$

$$x_+^\pm(a_+) = \frac{+(a+b-c) \pm R(+a,-b,+c)}{2\sqrt{bc}}\,,$$
$$x_-^\pm(a_-) = \frac{-(a+b-c) \pm R(+a,-b,+c)}{2\sqrt{bc}}\,, \quad \text{(C.3)}$$

$$x_+^+ = -\frac{1}{x_+^-}\,, \quad x_-^+ = -\frac{1}{x_-^-}\,, \quad x_+^+ = -x_-^-\,, \quad x_+^- = -x_-^+\,, \quad \text{(C.4)}$$

$$a \in [-b+c, +\infty[\quad \Rightarrow$$
$$x_+^+ \in [+1, +\infty[; \quad x_+^- \in [-1, 0]; \quad x_-^+ \in [+1, 0]; \quad x_-^- \in [-1, -\infty[$$
$$a \in]-\infty, -b+c] \quad \Rightarrow$$
$$x_+^+ \in [0, +1]; \quad x_+^- \in]-\infty, -1]; \quad x_-^+ \in]+\infty, +1]; \quad x_-^- \in [0, -1]$$
$$R(+a,-b,+c) \in \mathrm{Re} \quad \forall a \in \mathrm{Re};$$
$$R(+a,-b,+c) \neq 0 \quad \forall a \in \mathrm{Re}\,. \quad \text{(C.5)}$$

- $a \in [-b+c, +\infty[\quad \Rightarrow \quad x_-^+ \in [0,1]\,.$ (C.6)

$$a = a_-(x_-^+) = -\sqrt{bc}\frac{\left(x + \sqrt{\frac{b}{c}}\right)\left(x - \sqrt{\frac{c}{b}}\right)}{x}$$
$$= -\frac{\sqrt{bc}}{x}\left(x^2 - \frac{-b+c}{\sqrt{bc}}x - 1\right)$$
$$= -\sqrt{bc}\left(x - \frac{1}{x}\right) - b + c\,,$$

$$x = \frac{-(a+b-c) + R(+a,-b,+c)}{2\sqrt{bc}}\,,$$

$$R(+a,-b,+c) = \sqrt{a^2+b^2+c^2+2ab-2ac+2bc} = \frac{\sqrt{bc}}{x}(1-x^2)\,,$$

$$\int_{-b+c}^{+\infty} \frac{da}{R(+a,-b,+c)} f(a) = \int_0^1 \frac{dx}{x} f(a(x)),$$

$$R(+q,+b,+c) \int_{(\sqrt{b}+\sqrt{c})^2}^{+\infty} \frac{da}{R(+a,+b,+c)} \frac{1}{(a-q)} f(a) =$$

$$+ \int_0^1 \left(\frac{1}{x-q_+} - \frac{1}{x-q_-} \right) f(a(x)),$$

$$q_\pm = + \frac{q-b-c \pm R(+q,+b,+c)}{2\sqrt{bc}}, \qquad q_+ q_- = +1.$$

$$\bullet \qquad a \in]-\infty, -b+c] \quad \Rightarrow \quad x_+^+ \in [0,1]. \tag{C.7}$$

$$\begin{aligned} a = a_+(x_+^+) &= +\sqrt{bc} \frac{\left(x - \sqrt{\frac{b}{c}}\right)\left(x + \sqrt{\frac{c}{b}}\right)}{x} \\ &= + \frac{\sqrt{bc}}{x} \left(x^2 + \frac{-b+c}{\sqrt{bc}} x - 1 \right) \\ &= +\sqrt{bc} \left(x - \frac{1}{x} \right) - b + c, \end{aligned}$$

$$x = \frac{+(a+b-c) + R(+a,-b,+c)}{2\sqrt{bc}},$$

$$R(+a,-b,+c) = \sqrt{a^2+b^2+c^2+2ab-2ac+2bc} = \frac{\sqrt{bc}}{x}(1-x^2),$$

$$\int_{-\infty}^{-b+c} \frac{da}{R(+a,-b,+c)} f(a) = \int_0^1 \frac{dx}{x} f(a(x)),$$

$$R(+q,+b,+c) \int_{-\infty}^{(\sqrt{b}-\sqrt{c})^2} \frac{da}{R(+a,+b,+c)} \frac{1}{(a-q)} f(a) =$$

$$- \int_0^1 \left(\frac{1}{x-q_+} - \frac{1}{x-q_-} \right) f(a(x)),$$

$$q_\pm = -\frac{q-b-c\pm R(+q,+b,+c)}{2\sqrt{bc}}\,, \qquad q_+q_- = +1\,.$$

$$\bullet \qquad \forall a \in \mathrm{Re} \quad R(+a,-b,+c) \in \mathrm{Re}\,. \tag{C.8}$$

Case 3a: $R(+a,-1,+1)\sqrt{a^2+4}$

$$R^2(+a,-1,+1) = a^2+4 = \frac{1}{x^2}|1+x^2|^2\,, \tag{C.9}$$

$$a_\pm(x) = \pm\frac{x^2-1}{x} = \pm\left(x-\frac{1}{x}\right)\,, \tag{C.10}$$

$$x_+^\pm(a_+) = \frac{+a\pm\sqrt{a^2+4}}{2}\,, \qquad x_-^\pm(a_-) = \frac{-a\pm\sqrt{a^2+4}}{2}\,, \tag{C.11}$$

$$x_+^+ = -\frac{1}{x_+^-}\,, \quad x_-^+ = -\frac{1}{x_-^-}\,, \quad x_+^+ = -x_-^-\,, \quad x_+^- = -x_-^+\,, \tag{C.12}$$

$$\begin{aligned}
&a \in [0,+\infty[\quad \Rightarrow \\
&x_+^+ \in [+1,+\infty[; \quad x_+^- \in [-1,0]; \quad x_-^+ \in [+1,0]; \quad x_-^- \in [-1,-\infty[\\
&a \in]-\infty,0] \quad \Rightarrow \\
&x_+^+ \in [0,+1]; \quad x_+^- \in]-\infty,-1]; \quad x_-^+ \in]+\infty,+1]; \quad x_-^- \in [0,-1] \\
&R(+a,-b,+c) \in \mathrm{Re} \quad \forall a \in \mathrm{Re}; \\
&R(+a,-b,+c) \neq 0 \quad \forall a \in \mathrm{Re}\,.
\end{aligned} \tag{C.13}$$

$$\bullet \qquad a \in [0,+\infty[\quad \Rightarrow \quad x_-^+ \in [0,1]\,. \tag{C.14}$$

$$a = -\frac{x^2-1}{x} = -\left(x-\frac{1}{x}\right)\,, \quad x = \frac{-a+\sqrt{a^2+4}}{2}\,,$$

$$\sqrt{a^2+4} = \frac{1}{x}(1-x^2)\,,$$

$$\int_4^{+\infty} \frac{da}{\sqrt{a^2+4}} f(a) = \int_0^1 \frac{dx}{x} f(a(x))\,,$$

$$\sqrt{q^2+4}\int_4^{+\infty} \frac{da}{\sqrt{a^2+4}}\frac{1}{(a-q)} f(a) = +\int_0^1 \left(\frac{1}{x-q_+} - \frac{1}{x-q_-}\right) f(a(x))\,,$$

$$q_{\pm} = +\frac{q-2\pm\sqrt{q^2+4}}{2}, \qquad q_+q_- = +1 .$$

$$\bullet \qquad a \in]-\infty,0] \quad \Rightarrow \quad x_+^+ \in [0,1] . \tag{C.15}$$

$$a = -\frac{x^2-1}{x} = -\left(x+\frac{1}{x}\right) - 2, \quad x = \frac{+a+\sqrt{a^2+4}}{2},$$

$$\sqrt{a^2+4} = \frac{1}{x}(1-x^2),$$

$$\int_{-\infty}^{0} \frac{da}{\sqrt{a^2+4}} f(a) = \int_{0}^{1} \frac{dx}{x} f(a(x)),$$

$$\sqrt{q^2+4}\int_{-\infty}^{0} \frac{da}{\sqrt{a^2+4}}\frac{1}{(a-q)} f(a) = -\int_{0}^{1}\left(\frac{1}{x-q_+} - \frac{1}{x-q_-}\right) f(a(x)),$$

$$q_{\pm} = -\frac{q-2\pm\sqrt{q^2+4}}{2}, \qquad q_+q_- = +1 .$$

$$\bullet \qquad \forall a \in \mathrm{Re} \quad R(+a,-b,+c) \in \mathrm{Re} . \tag{C.16}$$

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