Vol. 26(1995) *ACTA PHYSICA POLONICA B* No 1

POSITIVE FREQUENCY SOLUTIONS OF THE KLEIN–GORDON EQUATION IN THE N-DIMENSIONAL DE SITTER SPACETIME

Ewa L. Lokas

Copernicus Astronomical Center
Bartycka 18, 00-716 Warsaw, Poland
e-mail: lokas@camk.edu.pl

(Received September 5, 1994)

A massless scalar field in the $(1+n)$-dimensional Minkowski spacetime is considered. The d'Alembert equation for the field is solved in pseudospherical coordinates. The positive frequency part of the solutions is found using the symmetry between coordinates and momenta. The solutions are projected onto the n-dimensional de Sitter hyperboloid embedded in the flat spacetime. The results for the special $n = 4$ case are easily reproduced.

PACS numbers: 04.60.+n, 03.70.+k, 11.10.Kk

1. Introduction

It is generally believed that all known physical fields should be described on a fundamental level by the principles of a quantum theory. However in the absence of a viable theory of quantum gravity we are led to the natural subject of formulating a theory of a quantum field in a classically describable curved background spacetime [1].

The de Sitter spacetime, which we will consider in this paper, is one of the most frequently studied examples and has been applied to many physical problems [2]. The reason for that stems from the fact that it is the unique maximally symmetric curved spacetime with the same degree of symmetry (*i.e.* the same number of Killing vectors) as the Minkowski spacetime. Still the presence of curvature and non-trivial global properties introduce new aspects to the quantization of fields in the de Sitter spacetime.

Let us now state some fundamental concepts of a scalar field theory. The Lagrangian density of a classical scalar field $\phi(x)$ in a $(1+n)$-dimensional

Minkowski spacetime is given by

$$\mathcal{L}(x) = \tfrac{1}{2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2)$$
$$\mu, \nu = 0, 1, \ldots, n, \tag{1}$$

where m denotes the mass of the field and $g^{\mu\nu}$ is the metric tensor of signature $(+1, -1, \ldots, -1)$. By requiring the variation of the action

$$S = \int \mathcal{L}(x) d^{n+1}x, \tag{2}$$

to vanish we obtain the Klein–Gordon equation

$$(\Box_{n+1} + m^2)\phi = 0. \tag{3}$$

An analogous procedure may be applied to an n-dimensional curved spacetime. The Lagrangian density is then given by

$$\mathcal{L}(x) = \tfrac{1}{2}\left\{g^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x) - \left[m^2 + \zeta R(x)\right]\phi^2(x)\right\}$$
$$\mu, \nu = 0, 1, \ldots, n-1. \tag{4}$$

Here $g^{\mu\nu}(x)$ is the metric tensor of the curved spacetime and $R(x)$ is the Ricci curvature scalar. The term $\zeta R\phi^2$ represents the coupling between the scalar field and the gravitational field. The coupling constant ζ may be put equal to zero and the case is called the minimal coupling. Then the gravity affects the Lagrangian density only via the metric tensor in the first term. In the case of the so-called conformal coupling ζ is given as a numerical factor dependent on the dimension of the spacetime. The corresponding action is

$$S = \int \mathcal{L}(x)\sqrt{-g(x)}\mathrm{d}^n x \tag{5}$$

with $g(x)$ being the determinant of $g_{\mu\nu}(x)$ and the Klein–Gordon equation takes the form

$$\left[\Box_n + m^2 + \zeta R(x)\right]\phi(x) = 0. \tag{6}$$

In the case of the Minkowski spacetime there exists a complete set of orthogonal mode solutions $u_k(x)$ such that the field $\phi(x)$ may be decomposed in the following way

$$\begin{aligned}\phi(x) &= \sum_k \left[a_k u_k(x) + a_k^\dagger \bar{u}_k(x)\right] \\ &= \sum_k \left[a_k \mathrm{e}^{-ikx} + a_k^\dagger \mathrm{e}^{ikx}\right].\end{aligned} \tag{7}$$

Here we choose $u_k(x) = e^{i\vec{k}\vec{x}-i\omega_k t}$, $\omega_k = \left(|\vec{k}|^2 + m^2\right)^{1/2}$. The modes are said to be of positive frequency with respect to the global time t if they are eigenfunctions of the operator $\frac{\partial}{\partial t}$, the global timelike Killing vector field in the Minkowski space

$$\frac{\partial}{\partial t} u_k(x) = -i\omega_k u_k(x) \tag{8}$$

with $\omega_k > 0$.

The quantization of the theory is then implemented by adopting commutation relations between the annihilation and creation operators to which the coefficients a_k, $a_k^\dagger$ in the expansion are promoted. They are used to define the vacuum state in the Hilbert–Fock space. In the Minkowski spacetime the vacuum state $|0\rangle$ is defined as the one annihilated by the operators a_k

$$a_k|0\rangle = 0 \qquad \forall k\,.$$

It is therefore essential for the construction of the Hilbert–Fock space to know which solutions are of positive frequency.

In a curved spacetime the construction of a vacuum state can proceed in the same way but there appears an ambiguity in defining the positive frequency solutions. In the generic case there are no global timelike Killing vector fields and thus we are given no natural mode decomposition.

A way to avoid the difficulty in the particular case of the de Sitter spacetime is to treat it as a hypersurface in the Minkowski space of one higher dimension. Then the positive frequency solutions may be obtained by projecting the well-defined positive frequency solutions from the higher dimensional flat spacetime onto the curved spacetime embedded in it. The possibility arises from unique properties of the de Sitter spacetime.

The above idea was given by Staruszkiewicz [3] and first applied to the projection from the $(1+3)$-dimensional Minkowski spacetime onto the $(1+2)$-dimensional de Sitter hyperboloid. A similar task was done by Wyrozumski [4] who introduced one more dimension thus covering physically the most interesting case of the $(1+3)$-dimensional de Sitter spacetime.

In this paper we generalize the construction in order to obtain the positive frequency solutions of the Klein–Gordon equation in the de Sitter spacetime of arbitrary dimension. As the construction is performed by projection of the Poincare invariant solutions of the Minkowski space onto the de Sitter hyperboloid the results are de Sitter invariant (although we do not provide an exact proof of the fact here) and may apply in multidimensional cosmological models.

In Section 2 we consider a massless scalar field and solve the d'Alembert equation in the $(1+n)$-dimensional Minkowski spacetime. Then in Section 3 we determine which of the solutions are of positive frequency. In Section 4 the Klein–Gordon equation in the n–dimensional de Sitter spacetime is considered and we show its equivalence to the angular part of the d'Alembert equation in hyperspherical coordinates in the Minkowski spacetime. Then in Section 5 we give the explicit formula for the restriction of the positive frequency solutions to the de Sitter hyperboloid. In Section 6 we discuss the mass spectrum obtained for the projected field and finally in Section 7 compare the results with the special 4-dimensional case.

Units are chosen so that $G = \hbar = c = 1$ throughout the paper and the notation for special functions follows that in [6].

2. Solution of d'Alembert equation in $(1 + n)$ dimensions

We consider $(1 + n)$-dimensional flat Minkowski spacetime. Let us parametrize the spacetime by the pseudospherical coordinates $(r,\ \xi,\ \theta_1,\ \theta_2,\ \ldots,\ \theta_{n-1})$ in the following way

$$
\begin{aligned}
x^0 &= r \sinh \xi \,, \\
x^1 &= r \cosh \xi \cos \theta_1 \,, \\
x^2 &= r \cosh \xi \sin \theta_1 \cos \theta_2 \,, \\
x^3 &= r \cosh \xi \sin \theta_1 \sin \theta_2 \cos \theta_3 \,, \\
&\vdots \\
x^i &= r \cosh \xi \left(\prod_{k=1}^{i-1} \sin \theta_k \right) \cos \theta_i \,, \quad 2 \le i \le n-1 \,, \\
&\vdots \\
x^n &= r \cosh \xi \left(\prod_{k=1}^{n-1} \sin \theta_k \right) ,
\end{aligned}
\tag{9}
$$

where the variables vary in the ranges

$$
\begin{aligned}
0 &< r < +\infty \,, \\
-\infty &< \xi < +\infty \,, \\
0 &\le \theta_j < \pi \,, \qquad 1 \le j \le n-2 \,, \\
0 &\le \theta_{n-1} < 2\pi \,.
\end{aligned}
\tag{10}
$$

The metric is then given by

$$
ds^2 = r^2 (d\xi)^2 - (dr)^2 - r^2 \cosh^2 \xi \left[(d\theta_1)^2 + \sum_{i=2}^{n-1} \prod_{k=1}^{i-1} \sin^2 \theta_k (d\theta_i)^2 \right] . \tag{11}
$$

The hypersurface $r =$ const is the n–dimensional de Sitter hyperboloid

$$S_{1,n-1} = \left\{ (x^0, x^1, x^2, \ldots, x^n) \in \mathcal{R}^{n+1} : -(x^0)^2 + \sum_{i=1}^{n} (x^i)^2 = r^2 \right\}. \quad (12)$$

The massless scalar field ϕ in the $(1+n)$-dimensional Minkowski space-time obeys the d'Alembert equation

$$\Box_{n+1} \phi = 0 \,. \quad (13)$$

After computing the d'Alembert operator in the pseudospherical coordinates (9) we get

$$\begin{aligned} \Box_{n+1} = {} & \frac{1}{r^2} \frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial r^2} - \frac{1}{r^2 \cosh^2 \xi} \frac{\partial^2}{\partial \theta_1^2} \\ & - \sum_{i=2}^{n-1} \frac{1}{r^2 \cosh^2 \xi \prod_{k=1}^{i-1} \sin^2 \theta_k} \frac{\partial^2}{\partial \theta_i^2} \\ & + (n-1) \frac{\tanh \xi}{r^2} \frac{\partial}{\partial \xi} - n \frac{1}{r} \frac{\partial}{\partial r} - (n-2) \frac{\cot \theta_1}{r^2 \cosh^2 \xi} \frac{\partial}{\partial \theta_1} \\ & - \sum_{j=2}^{n-2} (n-j-1) \frac{\cot \theta_j}{r^2 \cosh^2 \xi \prod_{l=1}^{j-1} \sin^2 \theta_l} \frac{\partial}{\partial \theta_j} \,. \end{aligned} \quad (14)$$

Let us separate the radial part of ϕ

$$\phi(r, \xi, \theta_1, \theta_2, \ldots, \theta_{n-1}) = R(r) \, \tilde{\phi}(\xi, \theta_1, \theta_2, \ldots, \theta_{n-1}) \,. \quad (15)$$

It obeys the equation

$$-\frac{r^2}{R} \frac{\mathrm{d}^2 R}{\mathrm{d}r^2} - \frac{nr}{R} \frac{\mathrm{d}R}{\mathrm{d}r} = \text{const} = A \,, \quad (16)$$

where A is a separation constant, or

$$\frac{1}{r^{n-2}} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^n \frac{\mathrm{d}}{\mathrm{d}r} R \right) + A\, R = 0 \,. \quad (17)$$

The solution of this equation is

$$R_\nu(r) = r^\nu \quad (18)$$

with ν being an arbitrary complex number and we have

$$A_\nu = -\nu[\nu + (n-1)] \,. \quad (19)$$

By fixing ν we get solutions of the $(n+1)$-dimensional d'Alembert equation that are homogeneous functions of r of degree ν.

The d'Alembert equation (13) takes now the form similar to that of the Klein–Gordon equation (3) in one lower dimensional space

$$\tilde{\Box}_{n+1}\tilde{\phi} + A_\nu \tilde{\phi} = 0\,, \tag{20}$$

where $\tilde{\Box}_{n+1}$ is the angular part of $\Box_{n+1}$, independent of r, and A_ν plays the role of the square of the mass for $\tilde{\phi}$.

Now we separate the ξ-dependent part from $\tilde{\phi}$

$$\tilde{\phi}(\xi, \theta_1, \theta_2, \ldots, \theta_{n-1}) = X(\xi)\tilde{\tilde{\phi}}(\theta_1, \theta_2, \ldots, \theta_{n-1}) \tag{21}$$

and obtain from the d'Alembert equation

$$\cosh^2\xi \frac{\mathrm{d}^2 X}{\mathrm{d}\xi^2} + (n-1)\sinh\xi\cosh\xi\frac{\mathrm{d}X}{\mathrm{d}\xi} + (A_\nu \cosh^2\xi + B)X = 0\,, \tag{22}$$

where B is another separation constant. Introducing the so-called conformal time η

$$\eta = 2\arctan(\mathrm{e}^{\xi}) \tag{23}$$

we have

$$\frac{\mathrm{d}^2 X}{\mathrm{d}\eta^2} - (n-2)\cot\eta\frac{\mathrm{d}X}{\mathrm{d}\eta} + \left(\frac{A_\nu}{\sin^2\eta} + B\right) X = 0\,. \tag{24}$$

We search for the solution of this equation in the form

$$X(\eta) = \sin^{(n-1)/2}\eta P(\eta) \tag{25}$$

suggested by the $n = 4$ case [4]. This way the equation (24) may be reduced to the form of the associated Legendre equation

$$(1-\mu^2)\frac{\mathrm{d}^2 P}{\mathrm{d}\mu^2} - 2\mu\frac{\mathrm{d}P}{\mathrm{d}\mu} + \left[s(s+1) - \frac{\varrho^2}{1-\mu^2}\right] P = 0 \tag{26}$$

with $\mu = -\cos\eta$,

$$\varrho^2 = \frac{(n-1)^2}{4} - A_\nu = \left(\nu + \frac{n-1}{2}\right)^2 \tag{27}$$

and the free coefficient

$$\begin{aligned} B + \frac{(n-1)^2}{4} - \frac{n-1}{2} &= \left(l + \frac{n-3}{2}\right)\left(l + \frac{n-1}{2}\right) \\ &= \left(k - \frac{1}{2}\right)\left(k + \frac{1}{2}\right)\,, \end{aligned} \tag{28}$$

where we put $k = l + \frac{n-2}{2}$.

The general solution of the Legendre equation may be expressed as a linear combination of the associated Legendre functions of the first and second kind [6]

$$P(-\cos\eta) = DP^{\varrho}_{k-1/2} + EQ^{\varrho}_{k-1/2}, \tag{29}$$

where D, E are constant coefficients. The η-dependent part of the solutions of the d'Alembert equation (25) thus takes the form

$$X_k(\eta) = \sin^{(n-1)/2}\eta \left[DP^{\varrho}_{k-1/2}(-\cos\eta) + EQ^{\varrho}_{k-1/2}(-\cos\eta)\right]. \tag{30}$$

The constant B was introduced as a separation constant for the angular (*i.e.* independent of r and η) part of the d'Alembert equation. Thus we have the following eigenvalue equation

$$L^2_n Y_B(\Omega_n) = BY_B(\Omega_n), \tag{31}$$

where $Y_B(\Omega_n)$ (denoted as $\bar{\bar{\phi}}$ before) are hyperspherical functions (or hyperspherical harmonics) and L^2_n is the Laplacian on the unit $(n-1)$-sphere S^{n-1}. The eigenvalues B reflect the dimensionality of the space [5]

$$B = l(l+n-2). \tag{32}$$

The hyperspherical harmonics are given by [7]

$$Y(m_k, \theta_k) = \mathrm{e}^{\pm i m_{n-2}\theta_{n-1}} \times \prod_{k=0}^{n-3} (\sin\theta_{k+1})^{m_{k+1}} C^{m_{k+1}+(n-2)/2-k/2}_{m_k - m_{k+1}}(\cos\theta_{k+1}), \tag{33}$$

where m_k stands for the set of $(n-1)$ integer constants $(m_0, m_1, \ldots, m_{n-2})$ such that

$$m_0 \geq m_1 \geq \ldots \geq m_{n-2} \geq 0$$

and θ_k denotes the set of $(n-1)$ angles $(\theta_1, \theta_2, \ldots, \theta_{n-1})$ introduced in the parametrization (9). C^{α}_{β} are Gegenbauer polynomials.

The general solution of the equation (20) is given by the linear combination

$$\tilde{\phi}(x) = \sum_{m_0=1}^{\infty} \sum_{m_2=0}^{m_1-1} \sum_{m_3=0}^{m_2} \cdots \sum_{m_{n-2}=0}^{m_{n-3}} \alpha(m_k)\phi(m_k, \eta, \theta_k), \tag{34}$$

where $\alpha(m_k)$ are constants and

$$\phi(m_k,\eta,\theta_k) = \sin^{(n-1)/2}\eta\left[DP^{\varrho}_{m_0-1/2}(-\cos\eta) + EQ^{\varrho}_{m_0-1/2}(-\cos\eta)\right]$$
$$\times e^{\pm i m_{n-2}\theta_{n-1}} \prod_{k=0}^{n-3} (\sin\theta_{k+1})^{m_{k+1}} C^{m_{k+1}+(n-2)/2-k/2}_{m_k - m_{k+1}}(\cos\theta_{k+1})\,. \quad (35)$$

Here the η-dependent solutions (30) and the hyperspherical harmonics (33) were combined according to (21).

These basic functions $\phi(m_k,\eta,\theta_k)$ include both the parts of positive and negative frequency and in what follows we will specify the positive frequency solutions.

3. Positive frequency solutions in (1+n) dimensions

The field $\phi(x)$ may be represented as a $(1+n)$-dimensional Fourier integral

$$\phi(x) = \int d^{n+1}k\delta(kk)\theta(k^0)\phi(k)e^{-ikx} + \text{cc}\,. \quad (36)$$

In the $(1+n)$–dimensional flat spacetime the definition of a positive frequency solution reads

$$\phi(x) = \int d^{n+1}k\delta(kk)\theta(k^0)\phi(k)e^{-ikx} \quad (37)$$

which may be evaluated performing the integral over k^0 and we are left with

$$\phi(x) = \int_{k^0=\underline{k}} \frac{d^n k}{2k^0}\phi(k)e^{-ikx}\,, \quad (38)$$

where $\underline{k}$ is the length of the spacial part of the $(1+n)$-dimensional vector $k = (k^0, k^1, \ldots, k^n)$

$$\underline{k} = \left(\sum_{i=1}^{n}(k^i)^2\right)^{\frac{1}{2}}.$$

As it has already been stated $\phi(x)$ is homogeneous of degree ν

$$\phi(\lambda x) = \lambda^{\nu}\phi(x)\,. \quad (39)$$

Given this we can determine the homogeneity degree for $\phi(k)$

$$\phi(\lambda k) = \lambda^{1-n-\nu}\phi(k)\,. \quad (40)$$

We may parametrize the light cone in momentum space by $(k^0, k_{\theta_1}, k_{\theta_2}, \ldots, k_{\theta_{n-1}})$

$$\begin{aligned} k^1 &= k^0 \cos k_{\theta_1} , \\ k^2 &= k^0 \sin k_{\theta_1} \cos k_{\theta_2} , \\ &\vdots \\ k^n &= k^0 \sin k_{\theta_1} \sin k_{\theta_2} \ldots \sin k_{\theta_{n-1}} . \end{aligned} \tag{41}$$

There exists a symmetry between momentum space and coordinate space which can be easily seen if one compares the above parametrization (41) and the definition of coordinates (9). The spaces are related to each other by the Fourier transform. We make use of the fact here stating that the angular part of basic solutions of the equation $\Box\phi = 0$ in momentum space must have the same form as in coordinate space.

Since the dependence on k^0 is given by the homogeneity degree (40) we have

$$\begin{aligned} \phi(m_k, k) =& (k^0)^{1-n-\nu} \mathrm{e}^{\pm m_{n-2} k_{\theta_{n-1}}} \prod_{k=0}^{n-3} (\sin k_{\theta_{k+1}})^{m_{k+1}} \\ & \times C_{m_k - m_{k+1}}^{m_{k+1}+(n-2)/2-k/2} (\cos k_{\theta_{k+1}}) . \end{aligned} \tag{42}$$

The Jacobian for the variables change as given in the parametrization (41) reads [7]

$$\begin{aligned} J &= \frac{\partial(k^1, k^2, \ldots, k^n)}{\partial(k^0, k_{\theta_1}, \ldots, k_{\theta_{n-1}})} \\ &= (k^0)^{n-1} \sin^{n-2} k_{\theta_1} \sin^{n-3} k_{\theta_2} \ldots \sin^2 k_{\theta_{n-3}} \sin k_{\theta_{n-2}} \end{aligned} \tag{43}$$

and the exponential term in (38) may be expanded as follows

$$\mathrm{e}^{i\vec{k}\vec{x}} = \mathrm{e}^{i|\vec{k}||\vec{x}|\cos\gamma} = 2^l \Gamma(l) \sum_{p_0=0}^{\infty} (l + p_0) i^{p_0} \frac{J_{l+p_0}(|\vec{k}||\vec{x}|)}{(|\vec{k}||\vec{x}|)^l} C_{p_0}^l (\cos\gamma) , \tag{44}$$

where $\vec{k}$ and $\vec{x}$ are n-dimensional vectors, γ is the angle between them and J_{l+p_0} is the Bessel function of the first kind. For $l = \frac{n}{2} - 1$ the Gegenbauer polynomial may be expressed as a series of hyperspherical harmonics (33) [7]

$$C_{p_0}^{\frac{n}{2}-1} = \sum_{p_k} G(p_k, n) \bar{Y}(p_k, k_{\theta_k}) Y(p_k, \theta_k) , \tag{45}$$

where p_k are constants analogous to m_k,

$$\sum_{p_k} = \sum_{p_1=0}^{p_0} \cdots \sum_{p_{n-2}=0}^{p_{n-3}}$$

and $G(p_k, n)$ are normalization constants. The integration over k_{θ_i} in (38) is easily performed using orthogonality relations for spherical harmonics and Gegenbauer polynomials [6]

$$\int_{-1}^{1} d(\cos k_{\theta_i})(\sin k_{\theta_i})^{2\lambda-1} C_{\mu}^{\lambda}(\cos k_{\theta_i}) C_{\mu'}^{\lambda}(\cos k_{\theta_i}) = \delta_{\mu,\mu'} \,. \tag{46}$$

The final result of the integration over the angular variables is just $Y(m_k, \theta_k)$ and we are left with the integral with respect to k^0

$$I = \int_0^{\infty} \frac{dk^0}{(k^0 r \cosh\xi)^{n/2-1}(k^0)^{1+\nu}} \mathrm{e}^{-ik^0 r \sinh\xi} J_{n/2-1+m_0}(k^0 r \cosh\xi) \,, \tag{47}$$

where we put $k^0 x^0 = k^0 r \sinh\xi$. Changing ξ into η defined by (23) we may put the integral in the form of Legendre polynomials [6]

$$I = r^{\nu} \mathrm{e}^{-i\pi(\varrho+1/2)/2} \sin^{(n-1)/2}\eta \times \left[P^{\varrho}_{m_0-1/2}(-\cos\eta) - \frac{2i}{\pi} Q^{\varrho}_{m_0-1/2}(-\cos\eta) \right] \,, \tag{48}$$

where ϱ is given by (27) and we assume

$$-\frac{n+1}{2} < \mathrm{Re}\nu < 0 \,, \tag{49}$$

for the integral to be convergent.

Finally the basic positive frequency solutions of the $(1+n)$-dimensional d'Alembert equation (13) in the coordinates (9) are

$$\phi(m_k, x) = \mathcal{N} r^{\nu} \mathrm{e}^{-i\pi(\varrho+1/2)/2} \sin^{(n-1)/2}\eta \times \left[P^{\varrho}_{m_0-1/2}(-\cos\eta) - \frac{2i}{\pi} Q^{\varrho}_{m_0-1/2}(-\cos\eta) \right] Y(m_k, \theta_k) \,, \tag{50}$$

where $Y(m_k, \theta_k)$ are the hyperspherical harmonics given by (33) and $\mathcal{N}$ is a normalization constant. The equation (50) is the final result of performing the integral (38) and by comparing it with (15), (34) and (35) we can easily check that (50) is a solution of the d'Alembert equation in the coordinates (9). It confirms that the assumption (42) was correct.

We may now compare the result with the special case $n = 4$ [4] and find that while performing the construction the n-dependent separation constants A_ν (19) and B (32) are absorbed into the indices of the Legendre functions while the Legendre functions of the first and second kind enter the η-dependent part of solutions with the similar coefficients as for $n = 4$.

4. The Klein–Gordon equation in the n-dimensional de Sitter spacetime

Let us now consider a massive scalar field $\psi(x)$ in the n-dimensional de Sitter spacetime (12). As it has already been stated the field obeys the Klein–Gordon equation (6)

$$\left[\Box_n + m^2 + \zeta(n)R(n)\right]\psi = 0\,. \tag{51}$$

In the case of the conformal coupling, which we consider here, the coupling constant $\zeta(n)$ depends on the dimension of the space in the following way [1]

$$\zeta(n) = \frac{1}{4}\frac{n-2}{n-1}\,. \tag{52}$$

The curvature scalar of the de Sitter spacetime is constant (*i.e.* independent of x) and also reflects the dimensionality of the space [5]

$$R(n) = \frac{n(n-1)}{r^2} \tag{53}$$

with r being the de Sitter radius.

We may parametrize the de Sitter hyperboloid by the coordinates (t, θ_k)

$$\begin{aligned}
x^0 &= r\sinh\frac{t}{r}\,,\\
x^1 &= r\cosh\frac{t}{r}\cos\theta_1\,,\\
x^2 &= r\cosh\frac{t}{r}\sin\theta_1\cos\theta_2\,,\\
&\vdots\\
x^n &= r\cosh\frac{t}{r}\left(\prod_{k=1}^{n-1}\sin\theta_k\right)\,,
\end{aligned} \tag{54}$$

where t is the global timelike coordinate and therefore we have $\psi = \psi(t, \theta_k)$ in the equation (51). The de Sitter hyperboloid can be viewed as the n-dimensional hypersurface of constant r in the $(1+n)$-dimensional flat spacetime and if parametrized by (ξ, θ_k) the equation (51) may be compared with (20). The d'Alembert operators $\Box_n$ and $\tilde{\Box}_{n+1}$ both act in n-dimensional spaces actually as $\tilde{\Box}_{n+1}$ is only the angular part of $\Box_{n+1}$. Thus the exact equivalence between the equations (51) and (20) may be obtained by putting

$$\begin{aligned} &\psi(t, \theta_k) = \tilde{\phi}(\xi, \theta_k)\,, \\ &t = r\xi\,, \\ &m^2 + \zeta R = \frac{A_\nu}{r^2}\,. \end{aligned} \tag{55}$$

5. Positive frequency solutions in the n-dimensional de Sitter spacetime

It was shown that the massive scalar field in the n-dimensional de Sitter space may be introduced via a restriction of the massless scalar field in one higher dimensional flat spacetime in which the hyperboloid is embedded. The fact allows us to project the positive frequency solutions obtained for the flat spacetime onto the de Sitter space. The corresponding prescription

$$\psi(t, \theta_k) = r^{-\nu}\phi(r, \xi, \theta_k) \tag{56}$$

with

$$t = r\xi\,,$$

can be found by comparing (55) and (15). Therefore the basic positive frequency solutions of the Klein–Gordon equation (51) in the n-dimensional de Sitter space are given by

$$\begin{aligned} \psi(m_k, x) = \mathcal{N} \mathrm{e}^{-i\pi(\varrho+1/2)/2} \sin^{(n-1)/2}\eta \\ \times \left[P^{\varrho}_{m_0-1/2}(-\cos\eta) - \frac{2i}{\pi} Q^{\varrho}_{m_0-1/2}(-\cos\eta) \right] Y(m_k, \theta_k)\,, \end{aligned} \tag{57}$$

where we used (50) and (56). From (23) we have the definition of the global time on the hyperboloid

$$x^0 = t = r \ln\tan\frac{\eta}{2}\,. \tag{58}$$

6. The mass spectrum

The last of equations (55) imposes a restriction on the mass of the field. We have

$$m^2 = \frac{A_\nu}{r^2} - \zeta(n)R(n)$$

which we do not want to take any negative values in order to avoid tachyons. We calculate m^2 using (19), (52), (53) and have

$$m^2 = -\frac{1}{r^2}\left[\nu(\nu + (n-1)) + \frac{n(n-2)}{4}\right] \geq 0\,. \tag{59}$$

With $\nu = \alpha + i\beta$, $\alpha, \beta \in \mathcal{R}$ we obtain real values of m^2 for $\beta = 0$ or $\alpha = (1-n)/2$. For $\beta = 0$ we get from (59) the following range for α

$$-\frac{n}{2} \leq \alpha \leq 1 - \frac{n}{2} \tag{60}$$

which is a stronger restriction than the convergence condition (49). These values of α correspond to the following range of m^2

$$0 \leq m^2 \leq \frac{1}{4r^2} \tag{61}$$

which is independent of the dimension of the space. The largest value of the mass

$$m^2 = \frac{1}{4r^2}\,, \tag{62}$$

is obtained for $\alpha = (1-n)/2$. For the same value of α, if we assume $\beta \neq 0$ (but still real values of m^2), we get

$$m^2 = \frac{1}{4r^2} + \frac{\beta^2}{r^2} > \frac{1}{4r^2}\,. \tag{63}$$

The remarkable value $\nu = (1-n)/2$ for which we get the mass (62) also appears to be the geometrical dimension of the field (*i.e.* the dimension corresponding to the requirement for the action to be dimensionless).

Considering the form of the equation (20) and the dependence of the mode solutions on A_ν, we might want to re-express the above results in terms of

$$M^2 = \frac{A_\nu}{r^2} = m^2 + \zeta R\,. \tag{64}$$

The conditions for the imaginary part of M^2 to vanish are the same as for m^2. For $\beta = 0$, $M^2 \geq 0$ is equivalent to

$$1 - n \leq \alpha \leq 0 \tag{65}$$

(which for $n \geq 3$ is a weaker condition than (49)) and

$$0 \leq M^2 \leq \frac{(n-1)^2}{4r^2}\,. \qquad (66)$$

The maximum of M^2 is again obtained for $\alpha = (1-n)/2$. For the same value of α, when $\beta \neq 0$ we similarly get

$$M^2 = \frac{(n-1)^2}{4r^2} + \frac{\beta^2}{r^2}\,. \qquad (67)$$

As was shown in [9] in the analysis of the Lorentz-type SO$(n,1)$ group representations, $(n-1)^2/4r^2$ is the minimum of the eigenvalues spectrum of the Laplace–Beltrami operator on the hyperboloid of type (12). The spectrum corresponds to the continuous series of the most degenerate representations of an arbitrary noncompact SO$(n,1)$ group.

7. Discussion

The mass value (62) obviously does not depend on the dimension of the de Sitter space and is exactly the same as for $n = 4$ [4]. For $n = 4$ it has been revealed that this value is in a sense critical as it divides two types of solutions that have different asymptotical behaviour for $x^0 \to \pm\infty$ [10]. One can therefore suppose that the same result also holds in higher dimensions although the detailed analysis goes beyond the framework of this paper.

We considered the massless field in the Minkowski spacetime in order to find the positive frequency solutions of the d'Alembert equation and then to project them onto the de Sitter hyperboloid embedded in the Minkowski space. The fact that the field was massless was of essential importance. If we considered a massive field then its homogeneity in coordinates would not imply the homogeneity in momenta of its Fourier transform and the construction could not be performed within the framework proposed here.

The problem of vacuum construction in curved spaces does not have unique solution. However the one presented here seems simpler and more natural (due to its connection with the unique solution of the problem in the flat Minkowski spacetime) compared to the solution presented *e.g.* in [11]. We also obtain the complete mass spectrum contrary to the case in [5] which applies only to the mass region above the critical value (62).

The positive frequency solutions of the Klein–Gordon equation obtained for the de Sitter space of an arbitrary dimension have the similar structure as the solutions for $n = 4$ [4].

I am grateful to Professor A. Staruszkiewicz and Dr T. Wyrozumski at the Institute of Physics, Jagellonian University in Cracow for inspiration and discussions.

REFERENCES

[1] N.D. Birrell, P.C.W. Davies, *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge 1982.
[2] S.W. Hawking, G.F.R. Ellis, *The Large Scale Structure of Space-time*, Cambridge University Press, Cambridge 1973.
[3] A. Staruszkiewicz, *Ann. Phys.*, NY **190**, 354 (1988).
[4] T. Wyrozumski, *Classical Quantum Gravity* **5**, 1607 (1988).
[5] J.H. Redmount, S. Takagi, *Phys. Rev.* **D37**, 1443 (1988).
[6] J.S. Gradshteyn, J.M. Ryzhik, *Tables of Integrals, Sums, Series and Transforms*, Nauka, Moscow 1971 (in Russian).
[7] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Higher Transcendental Functions* vol. II, McGraw–Hill, New York 1953.
[8] G.N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge 1922.
[9] R. Raczka, N. Limic, J. Niederle, *J. Math. Phys.*, **7**, 1861 (1966); N. Limic, J. Niederle, R. Raczka, *J. Math. Phys.*, **7**, 2026 (1966); N. Limic, J. Niederle, R. Raczka, *J. Math. Phys.*, **8**, 1079 (1967).
[10] E.A. Tagirov, *Ann. Phys.*, NY **76**, 561 (1973).
[11] H. Rumpf, *Phys. Lett.*, **61B**, 272 (1976).