# A NOTE ON WEAK $\mathcal{H}$-SPACES HEAVENLY EQUATIONS AND THEIR EVOLUTION WEAK HEAVENLY EQUATIONS

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Using the weak heavens condition we find a non-linear partial differential equation which is shown to generalize the heavenly equation of Self-dual gravity. This differential equation we call "weak heavenly equation" ($\mathcal{WH}$-equation). For the two-dimensional case the $\mathcal{WH}$-equation is brought into the evolution (Cauchy–Kovalevski) form using a Legendre transformation. Finally, we find that this transformed equation ("evolution weak heavenly equation") does admit very simple solutions.

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## 1. Introduction

Complex Self-dual (SD) gravity has greatly stimulated our understanding of the new advances in other branches of mathematical physics; namely, Quantum Gravity, Conformal Field Theory, Integrable Models, Topological Field Theories, String Theory, *etc.* For this reason, any advance in this line is of great importance in the development of these branches.

Very recently a series of papers on SD gravity has been published [1–5], based on the seminal Grant's paper [6]. The first and the second heavenly equations and all hyper-heavenly equations with (and without) cosmological constant were brought into an evolution form called Cauchy–Kovalevski form. Some solutions for all these evolution equations also have been found.

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In all cases the relation between the heavenly equation and that of evolution has been achieved by a Legendre transformation applied to suitable coordinates.

On the other hand, the richness of structure in SD gravity allows us to explore what other possibilities might exist there. In this paper we explore one of these possibilities. We find, working with the spinorial formulation of SD gravity (using the weak heavens condition), a non-linear partial differential equation of the second order. This equation we call "Weak Heavenly equation" ($\mathcal{WH}$-equation); it involves a holomorphic function of its arguments. This equation contains the first heavenly equation and in fact generalizes the second one. Taking the dimensional reduction of the $\mathcal{WH}$-equation from 4 dimensions to the 2 dimensional case, we are able to find simple evolution equations and also some simple solutions to these equations.

The present paper is organized as follows. In Section 2 we find the differential equation of $\mathcal{WH}$-spaces using the basic spinorial formalism. This derivation is based on an unpublished work by one of us [7]. We show how the first heavenly equation results in this context and how the $\mathcal{WH}$-equation appears to be a generalization of the second Heavenly equation. Section 3 is devoted to a dimensional reduction of the weak heavenly equations considering only the two-dimensional case. We write the $\mathcal{WH}$-equation in the Cauchy-Kovalevski form and find some solutions by direct integration. Finally, in Section 4 some concluding remarks are given.

## 2. Differential equations of weak $\mathcal{H}$-spaces

### *2.1. Generalities*

In this section we start from the usual spinorial formulation for the two-form formalism for SD gravity on a four-dimensional complex Riemannian manifold $\mathcal{M}$ [8, 9]. We would like to integrate the equation

$$dS^{\dot{A}\dot{B}} + 2\alpha \wedge S^{\dot{A}\dot{B}} = 0 \tag{2.1}$$

for $S^{\dot{A}\dot{B}}$ being the spinor-valued anti-self-dual 2-form on $\mathcal{M}$ given by

$$S^{\dot{A}\dot{B}} = \tfrac{1}{2}\epsilon_{RS} g^{R\dot{A}} \wedge g^{S\dot{B}} , \tag{2.2}$$

with $\dot{A}, \dot{B} \in \{\dot{1}, \dot{2}\}$; $\alpha$ is a 1-form on $\mathcal{M}$, $\epsilon_{RS}$ is the usual Levi-Civita's matrix

$$(\epsilon_{RS}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (\epsilon^{RS})$$

and $g^{A\dot{B}}$ is a spinor-valued 1-form on $\mathcal{M}$ (the spinorial tetrad) defining the Riemannian metric by

$$g = -\tfrac{1}{2} g_{A\dot{B}} \otimes g^{A\dot{B}}, \tag{2.3}$$

where, as usually, $\otimes$ denotes the tensor product. We recall now that the "weak heavens" ("strong heavens") condition means to take the gauge condition to be $\Gamma_{\dot{A}\dot{B}} \neq 0$ ($\Gamma_{\dot{A}\dot{B}} = 0$) respectively.

At this level we can invoke one aspect of the Frobenius theorem[1] which implies the existence of the scalar spinorial functions on $\mathcal{M}$

$$\lambda^A{}_B,\ \tilde{\lambda}^A{}_B,\ q^A,\ \tilde{q}^A, \tag{2.4a}$$

where $\lambda^A{}_B$ and $\tilde{\lambda}^A{}_B$ are constrained by

$$\det(\lambda^A{}_B), \quad \det(\tilde{\lambda}^A{}_B) \neq 0. \tag{2.4b}$$

We can define now the spinorial tetrad to be

$$g^{A\dot{1}} = \phi^{-1} \mathrm{e}^{\sigma} dq^A; \quad g^{A\dot{2}} = \phi^{-1} \mathrm{e}^{-\sigma} d\tilde{q}^A, \tag{2.5}$$

where we have used a convenient parametrization and the fact that this tetrad is obviously meaningful only modulo the SL(2,C) transformations[2].

It is easy to see that substituting Eqs (2.5) into (2.1) we obtain for $\dot{A}\dot{B} = \dot{1}\dot{1}$ and $\dot{2}\dot{2}$

$$(\alpha - d\ln\phi + d\sigma) \wedge dq^1 \wedge dq^2 = 0 \tag{2.6a}$$

and

$$(\alpha - d\ln\phi - d\sigma) \wedge d\tilde{q}^1 \wedge d\tilde{q}^2 = 0, \tag{2.6b}$$

respectively.

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[1] We are using the Frobenius theorem as follows: Let $\omega_i, (i = 1, 2, \ldots, r \leq n)$ be functionally independent 1-forms on a $n$-dimensional manifold $\mathcal{N}$. Let $\Omega$ be the $r$-form given by $\Omega := \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_r \neq 0$. If there exists a 1-form $\theta$ such that it satisfies the relation $d\Omega = \theta \wedge \omega$, then there exist the functions $f^i_j$ and $g^i$ $(i, j = 1, 2, \ldots, r)$ which fulfill the relation $\omega^i = \sum_{j=1}^{r} f^i_j \cdot dg^i$, with $\det(f^i_j) \neq 0$.

[2] Notice that the parametrization can be written as $g^{A\dot{1}} = \phi^{-1} \mathrm{e}^{\sigma} l^A{}_B dq^B$ and $g^{A\dot{2}} = \phi^{-1} \mathrm{e}^{-\sigma} \tilde{l}^A{}_B d\tilde{q}^B$. We can take $g^{A\dot{B}} \to g'^{A\dot{B}} = L^A{}_R g^{R\dot{B}}$ with $(L^A{}_B) \in$ SL(2, C) and $\det(L^A{}_B) = 1$. Moreover, taking $L^A{}_B = -l_B{}^A$ we have $g'^{A\dot{1}} = \phi^{-1} \mathrm{e}^{\sigma} dq^A$, similarly for $g^{A\dot{2}}$ we take $L^A{}_B = -\tilde{l}_B{}^A$.

The pair $\{q^A, \tilde{q}^A\}$ constitutes a local chart on $\mathcal{M}$. Eqs (2.6a), (2.6b) and the equation for the volume form $\omega = -\phi^{-4} dq^1 \wedge dq^2 \wedge d\tilde{q}^1 \wedge d\tilde{q}^2$ imply that $\alpha$ reads

$$\alpha = d \ln \phi + \frac{\partial \sigma}{\partial q^A} dq^A - \frac{\partial \sigma}{\partial \tilde{q}^A} d\tilde{q}^A . \tag{2.7}$$

Taking the gauge of the spinorial tetrad (2.5) to be

$$g^{A\dot{1}} = \phi^{-1} \mathrm{e}^{\sigma} l^A{}_B dq^B , \quad g^{A\dot{2}} = \phi^{-1} \mathrm{e}^{-\sigma} \tilde{l}^A{}_B d\tilde{q}^B \tag{2.8}$$

with $\det(l^A{}_B) = 1 = \det(\tilde{l}^A{}_B)$, it is possible to express the metric (2.3) in the form

$$g = -\phi^{-2} \Omega_{AB} dq^A \otimes_s d\tilde{q}^B , \tag{2.9}$$

where $\otimes_s$ stands for the symmetric tensor product and $\Omega_{AB} := \epsilon_{RS} l^R{}_A \tilde{l}^S{}_B$. This definition satisfies

$$\det(\Omega_{AB}) = \tfrac{1}{2} \Omega_{AB} \Omega^{AB} = 1 . \tag{2.10}$$

The integrability condition of Eq. (2.1) with $\dot{A}\dot{B} = \dot{1}\dot{2}$ is given by

$$d\alpha \wedge S^{\dot{1}\dot{2}} = 0 \tag{2.11}$$

and can be reduced to the formula

$$\Omega^{AB} \frac{\partial^2 \sigma}{\partial q^A \partial \tilde{q}^B} = 0 . \tag{2.12}$$

On the other hand, Eq. (2.2), for $\dot{A}\dot{B} = \dot{1}\dot{2}$ is found to be equivalent to a pair of scalar conditions

$$\frac{\partial}{\partial q^A} (\mathrm{e}^{2\sigma} \Omega^A{}_B) = 0 , \quad \frac{\partial}{\partial \tilde{q}^B} (\mathrm{e}^{-2\sigma} \Omega_A{}^B) = 0 . \tag{2.13}$$

Manipulating these equations we obtain the following set of differential equations

$$\frac{\partial^2 \Omega^{AB}}{\partial q^A \partial \tilde{q}^B} + 4 \Omega^{AB} \frac{\partial \sigma}{\partial q^A} \frac{\partial \sigma}{\partial \tilde{q}^B} = 0 , \tag{2.14a}$$

$$\Omega^{AB} \frac{\partial^2 \sigma}{\partial q^A \partial \tilde{q}^B} = 0 . \tag{2.14b}$$

Therefore Eq. (2.12) is implicitly contained in Eqs (2.13) and it does not represent an independent condition.

All this is very close to the description of $\mathcal{H}$-spaces theory via the $\Omega$ formalism [8, 9]. To see this we make $\sigma$ = *constant* which implies that Eqs (2.13) are equivalent to the existence of a scalar holomorphic function $\Omega = \Omega(q_A, \tilde{q}_A)$ such that

$$\Omega_{AB} = \frac{\partial^2 \Omega}{\partial q^A \partial \tilde{q}^B} \,. \tag{2.15}$$

The condition det $(\Omega_{AB}) = 1$ reduces (2.15) directly to the first heavenly equation and the metric becomes conformally equivalent to the $\mathcal{H}$-space.

### *2.2. The Weak Heavenly Equation*

We shall now seek for an equivalent formulation of the structure described by Eqs (2.13) and (2.14). This formulation corresponds to the $\Theta$-formalism for the $\mathcal{H}$-spaces [8, 9]. First, we observe that Eqs (2.13) are equivalent to the existence of functions $P_A, \tilde{P}_A$ that satisfy

$$\mathrm{e}^{2\sigma}\Omega_{AB} = \frac{\partial}{\partial q^A}\tilde{P}_B \,, \quad \mathrm{e}^{-2\sigma}\Omega_{AB} = \frac{\partial}{\partial \tilde{q}^B} P_A \,. \tag{2.16}$$

From Eq. (2.16) and $\det(\Omega_{AB}) = 1$, we infer that

$$\det\left(\frac{\partial}{\partial q^A}\tilde{P}_B\right) = \mathrm{e}^{4\sigma} \quad \text{and} \quad \det\left(\frac{\partial}{\partial \tilde{q}^B} P_A\right) = \mathrm{e}^{-4\sigma} \,. \tag{2.17}$$

Which means that the Jacobians

$$\frac{\partial(\tilde{P}_1, \tilde{P}_2)}{\partial(q^1, q^2)} \quad \text{and} \quad \frac{\partial(P_1, P_2)}{\partial(\tilde{q}^1, \tilde{q}^2)}$$

are different from zero. Thus, in place of the local coordinate chart $\{q_A, \tilde{q}_A\}$ we can, alternatively, employ the charts $\{q_A, P_A\}$ and $\{\tilde{q}_A, \tilde{P}_A\}$.

For the first one $\{q_A, P_A\}$ the spinorial tetrad reads

$$g^{A\dot{1}} = \chi^{-1} dq^A \,, \tag{2.18a}$$

$$g^{A\dot{2}} = \chi^{-1}(dP^A - Q^A{}_B dq^B) \,, \tag{2.18b}$$

with $P_A = P_A(q_A, \tilde{q}_A)$, $\chi = \phi \mathrm{e}^{-\sigma}$ and $Q^A{}_B = \partial P^A / \partial q^B$. For the second, $\{\tilde{q}_A, \tilde{P}_A\}$, we have

$$g^{A\dot{1}} = -\tilde{\chi}^{-1}(d\tilde{P}^A - \tilde{Q}^A{}_B d\tilde{q}^B) \,, \tag{2.19a}$$

$$g^{A\dot{2}} = \tilde{\chi}^{-1} d\tilde{q}^A \,, \tag{2.19b}$$

where $\tilde{P}_A = \tilde{P}_A(q_A, \tilde{q}_A)$, $\tilde{\chi} = \phi e^{\sigma}$ and $\tilde{Q}^A{}_B = \partial\tilde{P}^A/\partial\tilde{q}^B$.

In both cases the metric takes the form

$$g = -\chi^{-2} dq^A \otimes_s (dP_A - Q_{AB} dq^B), \tag{2.20a}$$
$$g = -\tilde{\chi}^{-2} d\tilde{q}^A \otimes_s (d\tilde{P}_A - \tilde{Q}_{AB} d\tilde{q}^B). \tag{2.20b}$$

Thus, apart from the conformal factors the metric is determined entirely by the symmetric functions $Q_{(AB)}$ or $\tilde{Q}_{(AB)}$, respectively.

On the other hand, we would like to write Eqs (2.1) and (2.2) for $\dot{A}\dot{B} = \dot{1}\dot{1}$ in the context of the $\Theta$-formalism. These equations lead to

$$(\alpha - d\ln\chi) \wedge dq^1 \wedge dq^2 = 0 \tag{2.21}$$

which implies that $\alpha$ takes the form

$$\alpha = d\ln\chi + \alpha_A dq^A. \tag{2.22}$$

Working with the $S^{\dot{1}\dot{2}}$ from (2.2) we obtain

$$\left(2\alpha_A + \frac{\partial}{\partial P^A} Q^S{}_S\right) dP^A \wedge dq^1 \wedge dq^2 = 0. \tag{2.23}$$

From this equation it can be observed that

$$\alpha_A = -\tfrac{1}{2}\frac{\partial}{\partial P^A} Q^S{}_S, \tag{2.24}$$

and consequently, Eq. (2.22), amounts to

$$\alpha = d\ln\chi - \tfrac{1}{2}\frac{\partial}{\partial P^A} Q^S{}_S dq^A. \tag{2.25}$$

Finally, consider Eq. (2.1) for $\dot{A}\dot{B} = \dot{2}\dot{2}$. Cancelling the terms involving $d\ln\chi$ and employing the fact that $\det(Q^A{}_B) \neq 0$, one gets

$$-dQ_{AB} \wedge dP^A \wedge dq^B + d(\det(Q^A{}_B)) \wedge dq^1 \wedge dq^2$$
$$-\frac{\partial}{\partial P^C} Q^S{}_S dq^C \wedge \left[dP^1 \wedge dP^2 - Q_{AB} dP^A \wedge dq^B\right] = 0. \tag{2.26}$$

Or, equivalently

$$\left\{\frac{\partial Q^B{}_A}{\partial P^B} - \frac{\partial}{\partial P^A} Q^B{}_B\right\} dq^A \wedge dP^1 \wedge dP^2$$
$$+\left\{\frac{\partial}{\partial P^A}\det(Q^R{}_S) + Q_A{}^R \frac{\partial}{\partial P^R} Q^S{}_S - \frac{\partial Q_A{}^B}{\partial q^B}\right\} dP^A \wedge dq^1 \wedge dq^2 = 0. \tag{2.27}$$

After some manipulations we can conclude that Eqs (2.1) are fulfilled by $S^{\dot{A}\dot{B}}$ in the chart $\{q_A, P_A\}$ if and only if the structural functions ${Q_A}^B$ fulfill the differential conditions

$$\frac{\partial}{\partial P^B} {Q_A}^B = 0\,, \tag{2.28a}$$

$$ {Q_S}^B \frac{\partial}{\partial P^B} {Q_A}^S + \frac{\partial}{\partial q^B} {Q_A}^B = 0\,, \tag{2.28b}$$

with the corresponding 1-form $\alpha$ given by Eq. (2.25). A similar procedure can be exactly realized with the other equivalent chart $\{\tilde{q}_A, \tilde{P}_A\}$ leading to a similar set of equations.

A direct consequence of Eqs (2.28a) is the existence of the scalar holomorphic functions $\theta_A = \theta_A(q_A, P_A)$ which satisfy

$$Q_{AB} = \frac{\partial}{\partial P^B} \theta_A\,. \tag{2.29}$$

At this level is convenient to introduce the notational conventions

$$\begin{aligned} \partial_A &:= \frac{\partial}{\partial P^A}\,, \quad \partial^A := \frac{\partial}{\partial P_A}\,, \\ \check{\partial}_A &:= \frac{\partial}{\partial q^A}\,, \quad \check{\partial}^A := \frac{\partial}{\partial q_A}\,. \end{aligned} \tag{2.30}$$

So, it is possible to give a more concise form of Eqs (2.29) and (2.28b). Namely we get

$${Q_A}^B = -\partial^B \theta_A \tag{2.31}$$

and

$$\partial^B \theta^R \cdot \partial_B \partial_R \theta_A - \partial^B \check{\partial}_B \theta_A = 0\,, \tag{2.32}$$

respectively. We will consider Eqs (2.31) and (2.32) to be the *fundamental equations* of *weak heavens* ($\mathcal{WH}$) in the $\Theta$-formalism. Eq. (2.32) can be written as follows

$$\partial^B \{\partial_B \theta^R \cdot \partial_R \theta_A + \check{\partial}_B \theta_A\} = 0\,, \tag{2.33}$$

implying the existence of a scalar function $\phi_A$ which satisfies the conservation law $\partial^B \partial_B \phi_A = 0$ with

$$\partial_B \theta^R \cdot \partial_R \theta_A + \check{\partial}_B \theta_A = \partial_B \phi_A\,. \tag{2.34}$$

The structure of the $\mathcal{WH}$ (2.32) is intrinsically related to the properties of the 1-form (2.25) which satisfies the proposition

$$d\alpha = 0 \iff Q^S{}_S = -\partial^S \theta_S = \frac{\partial}{\partial q^A} \mathcal{R}(q) \cdot P^A + \mathcal{S}(q) , \tag{2.35}$$

where $\mathcal{R}$ and $\mathcal{S}$ are arbitrary functions of $q^A$.

*2.3. Reduction to the first Heavenly Equation*

This case corresponds to the metrics conformally equivalent to a right-flat space. To see this we consider the $\Omega$-formalism [8, 9]. According to Eq. (2.7) $d\alpha$ reads

$$d\alpha = -2 \frac{\partial^2 \sigma}{\partial q^A \partial \tilde{q}^B} dq^A \wedge d\tilde{q}^B . \tag{2.36}$$

So, we have

$$d\alpha = 0 \iff \frac{\partial^2 \sigma}{\partial q^A \partial \tilde{q}^B} = 0 , \tag{2.37}$$

which implies that $\sigma = \rho(q) + \tilde{\rho}(\tilde{q})$. With $\sigma$ of this form, we can equivalently spell out (2.13) in the form

$$\frac{\partial \Omega'^A{}_B}{\partial q^A} = 0 , \quad \frac{\partial \Omega'_A{}^B}{\partial \tilde{q}^B} = 0 , \tag{2.38}$$

where $\Omega'_{AB} = \exp(2(\rho - \tilde{\rho}))\Omega_{AB}$. This equation implies the existence of the scalar function $\Omega$ such that

$$\Omega'_{AB} = \frac{\partial^2 \Omega}{\partial q^A \partial \tilde{q}^B} \Rightarrow \Omega_{AB} = \exp(2(\tilde{\rho} - \rho)) \frac{\partial^2 \Omega}{\partial q^A \partial \tilde{q}^B} . \tag{2.39}$$

The metric is now

$$g = -\phi^{-2} \exp\left(2(\rho - \tilde{\rho})\right) \frac{\partial^2 \Omega}{\partial q^A \partial \tilde{q}^B} dq^A \otimes_s d\tilde{q}^B \tag{2.40}$$

while

$$\det\left(\frac{\partial^2 \Omega}{\partial q^A \partial \tilde{q}^B}\right) = \exp\left(4(\rho - \tilde{\rho})\right) . \tag{2.40a}$$

In the last step, we execute the coordinate transformation

$$q^A = q^A(q') , \quad \tilde{q}^A = \tilde{q}^A(\tilde{q}') \tag{2.41}$$

with Jacobians so adjusted that

$$\frac{\partial(q')}{\partial(q)} = \exp(4\rho) , \quad \frac{\partial(\tilde{q}')}{\partial(\tilde{q})} = \exp(-4\tilde{\rho}) . \tag{2.42}$$

This reduces (2.40a) to the first heavenly equation

$$\det(\frac{\partial^2 \Omega}{\partial q'^A \partial \tilde{q}'^B}) = 1 \, . \tag{2.43}$$

The metric in the new chart $\{q'^A, \tilde{q}'^A\}$ is now

$$g = -\phi'^{-2} \frac{\partial^2 \Omega}{\partial q'^A \partial \tilde{q}'^B} dq'^A \otimes_s d\tilde{q}'^B \, , \quad \phi' = \phi \exp(\rho - \tilde{\rho}) \, . \tag{2.44}$$

*2.4. Generalization of the second Heavenly Equation*

The SL(2, **C**)-gauge transformation induced by the coordinate transformation $q^A = q^A(q')$ leaves the tetrad (2.18$a, b$) form-invariant[3]

$$g^{A\dot{1}} \to g'^{A\dot{1}} = \chi'^{-1} dq'^A \, , \tag{2.45a}$$
$$g^{A\dot{2}} \to g'^{A\dot{2}} = \chi'^{-1} (dP'^A - Q'^A{}_B dq'^B) \, , \tag{2.45b}$$

if and only if the structural functions $\chi$ and $Q^A{}_B$ transform as follows:

$$\chi' = \chi \exp(2\rho) \, , \tag{2.46}$$
$$P'^A = \frac{\partial q'^A}{\partial q^B} P^B + \tau^A ; \quad \frac{\partial \tau^A}{\partial P^B} = 0 \tag{2.47}$$

and

$$Q'^A{}_B = \frac{\partial q'^A}{\partial q^R} Q^R{}_S \frac{\partial q^S}{\partial q'^B} + P^R \frac{\partial}{\partial q'^B} (\frac{\partial q'^A}{\partial q^R}) + \frac{\partial \tau^A}{\partial q'^B} \, . \tag{2.48}$$

Employing Eq. (2.25) in the primed version and Eq. (2.48) we can show, assuming equations (2.1) are fulfilled that, the 1-form $\alpha$ is invariant under the discussed coordinate-tetrad transformation *i.e.*

$$\alpha' = \alpha \, . \tag{2.49}$$

Working with the $\{q_A, P_A\}$ chart, we take $d\alpha = 0$, then according to (2.35) we have

$$Q^S{}_S = -\frac{\partial \mathcal{R}(q)}{\partial q^S} \cdot P^S - \mathcal{S}(q) \, . \tag{2.50a}$$

[3] The SL(2, **C**)-gauge transformations induced by $q^A = q^A(q')$, $\frac{\partial(q')}{\partial(q)} := \exp(4\rho) \neq 0$, are defined by: $\frac{\partial q^A}{\partial q'^B} = \exp(-2\rho) l^A{}_B$, $\frac{\partial q'^A}{\partial q^B} = -\exp(2\rho) l_B{}^A$, $\det(l^A{}_B) = 1$ *i.e.*, the matrix $(l^A{}_B) \in$ SL(2, **C**).

Working then with the chart $\{q'_A, P'_A\}$ and using (2.48) we arrive at the formula

$$Q'^S{}_S = P^S \frac{\partial}{\partial q^S}\left(\ln \frac{\partial(q')}{\partial(q)} - \mathcal{R}(q)\right) + \frac{\partial \tau^R}{\partial q'^R} - \mathcal{S}(q)\,. \tag{2.50b}$$

Since $\mathcal{R}(q)$ and $\mathcal{S}(q)$ are arbitrary functions, without any loss of generality we can take

$$\mathcal{R}(q) = \ln \frac{\partial(q')}{\partial(q)}\,, \quad \mathcal{S}(q) = \frac{\partial \tau^R}{\partial q'^R}\,. \tag{2.51}$$

Therefore assuming that $d\alpha = 0$ one infers that $Q^S{}_S$ vanishes. By (2.35), the inverse statement also holds. Thus we get

$$d\alpha = 0 \Leftrightarrow Q^S{}_S = 0\,. \tag{2.52}$$

From Eq. (2.31) one finds

$$Q^S{}_S = \partial^S \theta_S = 0\,. \tag{2.53}$$

Therefore there exists the scalar functions $\Theta$ satisfying

$$\theta_A = \partial_A \Theta = \frac{\partial}{\partial P^A}\Theta\,, \tag{2.54}$$

consequently

$$Q_{AB} = \partial_A \partial_B \Theta \tag{2.55}$$

is automatically symmetric. Substituting Eq. (2.55) into Eq. (2.28b) we get that the $W\mathcal{H}$ equation reduces to

$$-\partial^S \partial^R \Theta \cdot \partial_A(\partial_S \partial_R \Theta) - \partial^R \check{\partial}_R \partial_A \Theta = 0 \tag{2.56}$$

or

$$-\partial_A \left\{ \tfrac{1}{2} \partial^S \partial^R \Theta \cdot \partial_S \partial_R \Theta + \partial^R \check{\partial}_R \Theta \right\} = 0 \tag{2.57}$$

and the $W\mathcal{H}$-equation reduces to the scalar condition

$$\tfrac{1}{2} \partial^A \partial^B \Theta \cdot \partial_A \partial_B \Theta + \partial^A \check{\partial}_A \Theta = \Xi(q)\,. \tag{2.58}$$

But with $Q_{AB} = \partial_A \partial_B \Theta$, we can send now $\Theta \Rightarrow \Theta + \chi^A(q) P_A$, where $\chi^A(q)$ is chosen to be a solution to $\check{\partial}_A \chi^A = \Xi(q)$. This maintains the basic relation $Q_{AB} = \partial_A \partial_B \Theta$ reducing (2.58) to the second heavenly equation to its standard form [8, 9]

$$\tfrac{1}{2} \partial^A \partial^B \Theta \cdot \partial_A \partial_B \Theta + \partial^A \check{\partial}_B \Theta = 0\,. \tag{2.59}$$

The metric is now

$$g = -\chi^{-2} dq^A \otimes_s (dP_A - \partial_A \partial_B \Theta dq^B) \tag{2.60}$$

being thus conformally equivalent to the one of (strong) $\mathcal{H}$-spaces [8, 9].

The basic point of this argument is that the $W\mathcal{H}$-equation constitutes a legitimate generalization of the second heavenly equation. Detailed computations concerning connections, curvature, Bianchi identities, etc. can be found in Ref. [7].

## 3. The two-dimensional evolution Weak Heavenly Equations

In this Section we first consider the $W\mathcal{H}$-equation given by (2.32). Making a dimensional reduction of this equation by taking $\theta_A$ as functions depending only on the coordinates $P^A$, we get two equations for $A = 1, 2$

$$\frac{\partial\theta^1}{\partial P^2}\cdot\frac{\partial}{\partial P^1}\frac{\partial}{\partial P^1}\theta_1 + \frac{\partial\theta^2}{\partial P^2}\cdot\frac{\partial}{\partial P^1}\frac{\partial}{\partial P^2}\theta_1 - \frac{\partial\theta^1}{\partial P^1}\cdot\frac{\partial}{\partial P^2}\frac{\partial}{\partial P^1}\theta_1 - \frac{\partial\theta^2}{\partial P^1}\cdot\frac{\partial}{\partial P^2}\frac{\partial}{\partial P^2}\theta_1 = 0\,, \tag{3.1}$$

$$\frac{\partial\theta^1}{\partial P^2}\cdot\frac{\partial}{\partial P^1}\frac{\partial}{\partial P^1}\theta_2 + \frac{\partial\theta^2}{\partial P^2}\cdot\frac{\partial}{\partial P^1}\frac{\partial}{\partial P^2}\theta_2 - \frac{\partial\theta^1}{\partial P^1}\cdot\frac{\partial}{\partial P^2}\frac{\partial}{\partial P^1}\theta_2 - \frac{\partial\theta^2}{\partial P^1}\cdot\frac{\partial}{\partial P^2}\frac{\partial}{\partial P^2}\theta_2 = 0\,, \tag{3.2}$$

respectively.

Thus, we have two possibilities:

(a) We can take

$$\theta_1 = \theta^2 = \theta(P^1, P^2) = \theta(p, q)\,,$$
$$\theta^1 = -\theta_2 = 0\,.$$

(b) Alternatively, we can choose

$$\theta^1 = -\theta_2 = \theta(P^1, P^2) = \theta(p, q)\,,$$
$$\theta_1 = \theta^2 = 0\,.$$

The first case leads to the differential equation

$$\theta_{,q}\theta_{,pq} - \theta_{,p}\theta_{,qq} = 0 \tag{3.3}$$

while the second one leads to

$$\theta_{,p}\theta_{,pq} - \theta_{,q}\theta_{,pp} = 0\,. \tag{3.4}$$

It is easy to see that both differential equations have the underlying discrete symmetry $p \leftrightarrow q$.

In terms of differential forms the above equations can be written as

$$d\theta_{,q} \wedge d\theta = 0\,, \tag{3.5}$$
$$d\theta_{,p} \wedge d\theta = 0\,, \tag{3.6}$$

respectively.

In what follows we will study only the case $a$). The second case is completely similar due the existence of the above mentioned discrete symmetry.

*Case 1a)*

$$d\theta - \theta_{,p}dp - \theta_{,q}dq = 0\,, \tag{3.7a}$$
$$d\theta_{,q} \wedge d\theta = 0\,. \tag{3.7b}$$

Eq. (3.7a) can be rewritten as

$$d(\theta - q\theta_{,q}) + qd\theta_{,q} - \theta_{,p}dp = 0\,. \tag{3.8}$$

Performing now the following Legendre transformation in a spirit similar to [1]

$$t = -\theta_{,q} \Rightarrow q = q(t,p)\,,$$
$$\mathcal{H} = \mathcal{H}(t,p) = \theta(q(t,p),p) + t \cdot q(t,p)\,, \tag{3.9}$$

we easily find from (3.7 b) and (3.9)

$$d\mathcal{H} - \mathcal{H}_{,t}dt - \mathcal{H}_{,p}dp = 0\,, \tag{3.10a}$$

$$-\mathcal{H}_{,p}dt \wedge dp + t\mathcal{H}_{,tp}dt \wedge dp = 0\,. \tag{3.10b}$$

Thus, Eq. (3.10b) leads directly to

$$-\mathcal{H}_{,p} + t\mathcal{H}_{,tp} = 0\,. \tag{3.11}$$

It is a second order partial differential equation for a holomorphic function $\mathcal{H}(p,t)$. After integrating out with respect to $p$, the above equation can be rewritten in the form

$$-\mathcal{H} + t \cdot \mathcal{H}_{,t} = f(t)\,, \tag{3.12}$$

where $f(t)$ is a function only depending on $t$. This is a simple equation which has a simple solution

$$\mathcal{H}(t,p) = \tilde{\mathcal{H}}(t) = t \cdot \left( \int \frac{f(t)}{t^2}dt + K \right)\,, \tag{3.13}$$

where $K$ is a constant.

*Case 2a)* Alternatively, we can choose the coordinate $p$ in (3.5) to perform the Legendre transformation as

$$\begin{aligned} t &= -\theta_{,p} \Rightarrow p = p(t,q)\,, \\ \mathcal{H} &= \mathcal{H}(t,q) = \theta(q,p(t,q)) + t\cdot p(t,q)\,. \end{aligned} \tag{3.14}$$

Using (3.7b) and (3.14) we find

$$d\mathcal{H} - \mathcal{H}_{,t}dt - \mathcal{H}_{,q}dq = 0\,, \tag{3.15a}$$

$$d\mathcal{H}_{,q} \wedge d\Big(\mathcal{H} - t\mathcal{H}_{,t}\Big) = 0\,. \tag{3.15b}$$

Equivalently we can write

$$\mathcal{H}_{,qq}\mathcal{H}_{,tt} - \mathcal{H}^2_{,qt} + \frac{1}{t}\mathcal{H}_{,q}\mathcal{H}_{,qt} = 0\,. \tag{3.16}$$

This equation appears to be the second heavenly equation for the two variables $\{q,t\}$ with an additional term.

It is possible to find formal solutions to (3.16) in the sense of [1, 6], but instead of this, we would like to perform a second Legendre transformation on it. This in order to simplify this equation and to find simple solutions to it.

In terms of differential forms (3.16) gives

$$d\mathcal{H} - \mathcal{H}_{,q}dq - \mathcal{H}_{,t}dt = 0\,, \tag{3.17a}$$

$$d\mathcal{H}_{,t} \wedge d\mathcal{H}_{,q} + \frac{1}{2t}\cdot d(\mathcal{H}^2_{,q}) \wedge dq = 0\,. \tag{3.17b}$$

Performing the Legendre transformation

$$\begin{aligned} x &= -\mathcal{H}_{,q} \Rightarrow q = q(x,t)\,, \\ \mathcal{F} &= \mathcal{F}(x,t) = \mathcal{H}(q(x,t),t) + x\cdot q(x,t)\,, \end{aligned} \tag{3.18}$$

using (3.17b) and

$$d(\mathcal{H} - q\mathcal{H}_{,q}) + q d\mathcal{H}_{,q} - \mathcal{H}_{,t}dt = 0 \tag{3.19}$$

one gets

$$d\mathcal{F} - \mathcal{F}_{,x}dx - \mathcal{F}_{,t}dt = 0\,, \tag{3.20a}$$

$$-d\mathcal{F}_{,t} \wedge dx + \frac{1}{2t}dx^2 \wedge d\mathcal{F}_{,x} = 0\,. \tag{3.20b}$$

Eq. (3.20b) can be rewritten as

$$\mathcal{F}_{,tt} + \frac{x}{t}\mathcal{F}_{,xt} = 0\,. \tag{3.21}$$

This equation has a very simple solution given by

$$\mathcal{F}(x,t) = ax^K\left(\frac{b}{1-K}t^{1-K} + b'\right), \tag{3.22}$$

where $a, b, b'$ and $K$ are constants.

It is a remarkable fact that for the $\mathcal{WH}$-equation, its Cauchy–Kovalevski form coincides with a modification of the second heavenly equation (3.16).

A similar procedure can be exactly realized for the case $b$), leading to a similar set of weak evolution equations and their solutions.

## 4. Concluding remarks

In this paper we found a non-linear partial differential equation of the second order that we have called *Weak Heavenly equation* $\mathcal{WH}$. We showed that this equation is connected with the first and second heavenly equations. Actually, the $\mathcal{WH}$ equation appears as a natural generalization for the second heavenly equation. We claim that the $\mathcal{WH}$-equation is the fundamental equation of SD gravity. Also, a dimensional reduction for the $\mathcal{WH}$-equation from the four dimensional manifold $\mathcal{M}$ to the simply connected two-manifold $\Sigma$ is given. After the dimensional reduction we perform a Legendre transformation on the local coordinates $p, q$ of $\Sigma$ and find a very simple evolution $\mathcal{WH}$-equation which admits very simple solutions. Choosing appropriately a coordinate to perform the Legendre transformation we found a differential equation which looks like the second heavenly equation. However, this differential equation has an additional term. Making a new Legendre transformation on it we finally found a very simple differential equation. This equation also admits simple solutions.

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