THE EFFECT OF MULTIPLE–SCATTERING AND DENSITY DISTRIBUTION ON THE ELASTIC SCATTERING OF HIGH–ENERGY ELECTRONS

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Electron-nucleus elastic scattering is calculated by means of the multiple-scattering theory of Glauber and the eikonal approximation applied to different forms of nuclear charge distribution. The differences between the calculated cross-sections are examined and compared with the experimental data for ^{12}C, ^{16}O, ^{40}Ca and ^{208}Pb nuclei at different energies. A very good agreement with the experimental data, at higher angles, is obtained for densities other than Gaussian. The multiple-scattering shows even better agreement than the eikonal approach for lighter nuclei, especially at the diffraction dip.

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1. Introduction

The Glauber approximation multiple-scattering series has been widely used, in nuclear physics, for some 25 years [1]. Although Glauber theory has been successful in describing hadron-nucleus elastic scattering at different energies, it has been rarely used to investigate multiple scattering effects in electron-nucleus collisions [2].

Generally, in the study of electron-nucleus scattering the theoretical expression of the cross-section was obtained by solving the Dirac equation numerically. In such a case, the electron-nucleus many-body problem is approximated by a two-body problem which in turn is reduced to calculating the scattering by an effective potential obtained from a simple function for the nuclear charge distribution: $Z\rho(r)$, using the Dirac equation. It is of interest to study how much such calculations for the differential cross-section, according to the eikonal approach, differ from the multiple-scattering differential cross-section obtained by considering the individual Z protons of the nucleus with single particle distribution $\rho(r)$? Both cross-sections can be calculated using the Glauber approximation.

Instead of solving the Dirac equation numerically, Baker [3] showed that the Glauber approximation is in remarkably close agreement with the exact partial wave analysis, even at large angles. The expression for the scattering amplitude is an integral over a variable related to the classical impact parameter which replaces the sum over discrete partial waves, with the addition of a spinor term; so that the use of the Glauber approximation in electron (or positron) elastic scattering is appropriate.

Then, Franco [4] described the calculations for the differential cross-section, by considering the elastic scattering of an electron by a nucleus containing Z protons, taking into account the collisions with the individual protons.

The main objective of this paper is to compare the calculations based on the full multiple-scattering of the Glauber approximation and that obtained by means of the usual effective two-body eikonal solution over target nuclei (Z ranging from 12 to 82) and over an energy range (E ranging from 175 to 450 MeV).

Second aim of this paper is to study the effect of varying the charge density distribution in the above mentioned two theoretical approaches.

The paper is organized as follows: in Section 2 we present the general formalism, Section 3 contains the results of the calculations and the discussion. The conclusion is given in Section 4.

2. Formalism

To get the theoretical expressions for the differential cross-sections $\frac{d\sigma_0}{d\Omega}$ and $\frac{d\sigma_p}{d\Omega}$, according to the eikonal approach and the full multiple-scattering approach, respectively; we shall follow the method and notations as outlined by Franco [4].

The phase shift function $\chi_0(b)$, can be written as:

$$\chi_0(b) = 2Zn \ln \tfrac{b}{2a} + \delta_0(b)\,, \tag{1}$$

where $n = -\frac{e^2}{\hbar v}$ for incident electrons, b is the impact parameter and a is an arbitrary constant. The first term in equation (1) is simply the contribution due to a point charge. The function $\delta_0(b)$ represents the effect of the deviation of the nucleus from a point charge and will depend on the charge distribution $\rho(r)$.

The differential cross-sections may be written as

$$\frac{d\sigma_{0,p}}{d\Omega} = \cos^2 \frac{\theta}{2} \mid f_c(q) + h_{0,p}(q) \mid^2 , \tag{2}$$

where f_c is the quantum mechanical point Coulomb scattering amplitude and is given by:

$$f_c = -ik \int_0^\infty bJ_0(qb)\left(\frac{b}{2a}\right)^{2inZ} b\,db\,. \tag{3}$$

Then

$$f_c = -2nZkq^{-2}\exp\left\{-i[2nZ\ln(qa) - 2\arg\Gamma(1+inZ)]\right\}; \tag{4}$$

the factor $\cos^2\frac{\theta}{2}$ is produced by the spinors and $q = |\boldsymbol{k} - \boldsymbol{k}'| = 2k\sin\frac{\theta}{2}$ is the momentum transfer. Franco gave also an expression for h_0 which represents a correction due to the interaction of the electrons with the charge distribution of the nucleus as a whole, *viz.*:

$$h_0(q) = ik\int_0^\infty bJ_0(qb)\left(\frac{b}{2a}\right)^{2inZ}\{1 - \exp[i\delta_0(b)]\}db\,. \tag{5}$$

To get the function $h_p(q)$, the Glauber approximation to the elastic scattering differential cross-section $\frac{d\sigma_p}{d\Omega}$ in which collisions with the individual protons are explicitly incorporated in the description, we take [4]:

$$\frac{d\sigma_p}{d\Omega} = \cos^2\frac{\theta}{2}\left|\frac{k}{2\pi}\int e^{i\boldsymbol{q}\cdot\boldsymbol{b}}\left\{1 - \prod_{j=1}^{Z}[1 - \Gamma_j(\boldsymbol{b} - \boldsymbol{s}_j)]\right\}\right.$$
$$\left.\times \mid \psi(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_Z)\mid^2 d\boldsymbol{r}_1 d\boldsymbol{r}_2 \ldots d\boldsymbol{r}_Z d^2b\right|^2, \tag{6}$$

where $\Gamma_j(b) = 1 - (\left(\frac{b}{2a}\right)^{2in}$ and $\boldsymbol{s}_j$ is the projection of the coordinate $\boldsymbol{r}_j$ onto the plane of impact parameter vector $\boldsymbol{b}$. Assuming an independent particle model for the protons, we have

$$\mid \psi(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_Z)\mid^2 = \prod_{j=1}^{Z}\rho(\boldsymbol{r}_j)\,.$$

Then $\frac{d\sigma}{d\Omega}$ reduces to:

$$\frac{d\sigma_p}{d\Omega} = k^2\cos^2\frac{\theta}{2}\left|\int_0^\infty J_0(qb)b\left\{1 - \left[\int\left(\frac{|\boldsymbol{b} - \boldsymbol{s}|}{2a}\right)^{2in}\rho(\boldsymbol{r})d\boldsymbol{r}\right]^Z\right\}db\right|^2. \tag{7}$$

Comparing equations (2) and (7) and making use of (3), one gets

$$h_p(q) = ik \int_0^\infty bJ_0(qb)\Big\{\Big(\frac{b}{2a}\Big)^{2inZ} - \Big[\int \Big(\frac{|b-s|}{2a}\Big)^{2in} \rho(r)dr\Big]^Z\Big\}db\,. \tag{8}$$

Now, according to the shape considered for the density, we shall get different expressions for $h_{0,p}$ and consequently for $\frac{d\sigma_{0,p}}{d\Omega}$; f_c being the same as given by equation (4).

2.1. Gaussian density (G density)

A Gaussian density is given by [4]

$$\rho(r) = \Big(\frac{\beta^2}{\pi}\Big)^{3/2} \exp(-\beta^2 r^2)\,, \tag{9}$$

where β is a parameter related to the r.m.s. radius by the relation:

$$\beta = \sqrt{\frac{3}{2}}\Big(\frac{1}{\langle r^2\rangle^{1/2}}\Big)$$

and

$$a = \frac{1}{2\beta}\,. \tag{10}$$

Let $\gamma = \frac{q}{\beta}$, adopting the same procedure as Franco [4] and using the tables of Gradshteyn and Ryzhik [5], we get the following expressions for $h_{0,p}$

$$h_0(q) = ik\beta^{-2} \int_0^\infty xJ_0(\gamma x)x^{2inZ}\{1 - \exp[inZE_1(x^2)]\}dx\,, \tag{11}$$

where $x = \beta b$ and $E_1(x^2)$ is the exponential integral, and

$$h_p(q) = ik\beta^{-2} \int_0^\infty xJ_0(\gamma x)\{x^{2inZ} - [\Gamma(1+in)\,{}_1F_1(-in;1;-x^2)]^Z\}dx\,. \tag{12}$$

We note that f_c is given by Eq. (4) in terms of γ, and use is made of Eq. (10) for the arbitrary constant a.

2.2. Harmonic oscillator density (H.O. density)

A harmonic oscillator density can be written in the form [6]

$$\rho(r) = (A + Br^2)\exp[-\lambda^2 r^2], \tag{13}$$

where A and B are the harmonic oscillator parameters. Following the same procedure as in (2.1) and choosing the arbitrary constant a to be equal to $\frac{1}{2}\lambda$, with $\gamma = \frac{q}{\lambda}$ and $x = \lambda b$, we get the following expressions for $h_{0,p}$ and consequently for $\frac{d\sigma_{0,p}}{d\Omega}$

$$\begin{aligned} h_0(q) =& ik\lambda^{-2}\int_0^\infty xJ_0(\gamma x)x^{2inZ} \\ & \times \{1 - \exp[inZ(E_1(x^2) + 2F\exp(-x^2))]\}dx\,, \end{aligned} \tag{14}$$

where

$$F = \frac{\pi}{\lambda^2}\frac{D}{2\lambda^2}\,, \qquad D = \frac{B\sqrt{\pi}}{\lambda}$$

and

$$\begin{aligned} h_p(q) =& ik\lambda^{-2}\int_0^\infty xJ_0(\gamma x)\{x^{2inZ} - [(A_1 + B_1x^2)_1F_1(-in;1;-x^2) \\ & + C_{11}F_1(-in-1;1;-x^2) - 2C_1x^2{}_1F_1(-in;2;-x^2)]^Z\}dx\,, \end{aligned} \tag{15}$$

where

$$C = \frac{\sqrt{\pi}}{\lambda}\left(A + \frac{B}{2\lambda^2}\right), \qquad A_1 = \frac{C\pi}{\lambda^2}\Gamma(1+in)\,,$$

$$B_1 = \frac{D\pi}{\lambda^4}\Gamma(1+in)\,, \qquad C_1 = \frac{D\pi}{\lambda^4}\Gamma(2+in)\,.$$

2.3. Woods–Saxon density approximated by a sum of Gaussian densities (S.G. density)

The Woods–Saxon formula for the nuclear density is given by:

$$\rho(r) = \frac{\rho_0}{1 + \exp\frac{(r-R)}{\tau}}\,. \tag{16}$$

Since numerical calculations are slightly cumbersome with such density, Dalkarov *et al.* [7] approximated formula (16) by a sum of Gaussian densities, namely

$$\rho(r) = \sum_{m=1}^{12} C_m \exp - \frac{mr^2}{r_a^2}\,, \tag{17}$$

where r_a is a parameter related to the r.m.s. radius by:

$$r_a = \left\{ \frac{2}{3}\frac{\langle r^2 \rangle}{\pi^{3/2}} \frac{1}{\sum\limits_{m=1}^{12} \frac{C_m}{m^{5/2}}} \right\}^{1/5} .$$

The constants C_m are taken from Dalkarov *et al.* [7]. Taking $\eta = \frac{1}{r_a}$ and choosing the constant a to be equal to $\frac{1}{2}\eta$ with $\gamma = \frac{q}{\eta}$ and $x = \eta b$; we get the following formulae for $h_{0,p}$:

$$h_0(q) = ik\eta^{-2}\int_0^\infty x J_0(\gamma x) x^{2inZ}\left\{1 - \exp\left[inZ\left(\frac{\pi}{\eta^2}\right)^{3/2}\sum_{m=1}^{12}\frac{C_m}{m^{3/2}}E_1(mx^2)\right]\right\}dx \tag{18}$$

and

$$h_p(q) = ik\eta^{-2}\int_0^\infty xJ_0(\gamma x)$$
$$\times \left\{ x^{2inZ} - \left[\sum_{m=1}^{12}\frac{C_m}{m^{in+3/2}}\left(\frac{\pi}{\eta^2}\right)^{3/2}\Gamma(1+in)\,{}_1F_1(-in;1;-mx^2)\right]^Z \right\} dx\,. \tag{19}$$

3. Results and discussion

In the present work we give a comparison between the cross section obtained by means of the eikonal approximation and that deduced from Glauber's multiple scattering theory using different densities.

We applied the theoretical expressions for $\frac{d\sigma_{0,p}}{d\Omega}$ to study the scattering of high energy electrons from the even-even nuclei: ^{12}C, ^{16}O, ^{40}Ca and ^{208}Pb at different energies which are available for us in the literature. The only parameter needed in the performed calculations is the r.m.s. radius of the nucleus under consideration. Table I lists the values taken for this parameter.

TABLE I

The root mean square radius of the nuclei ^{12}C, ^{16}O, ^{40}Ca and ^{208}Pb

Nucleus	^{12}C	^{16}O	^{40}Ca	^{208}Pb
$\langle r^2 \rangle^{1/2}$(fm)	2.464	2.625	3.486	5.235
	Reuter *et al.* (1982)	Hofstadter (1964)	Dalkarov *et al.* (1985)	

All figures labelled (a) will refer to the eikonal approximation while those labelled (b) will refer to Glauber's multiple-scattering approximation.

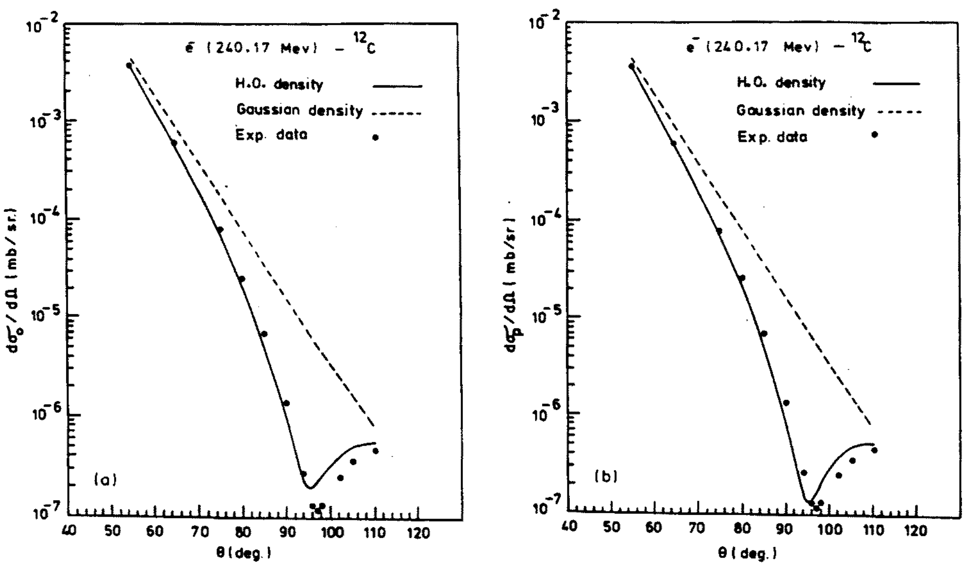

Fig. 1. The differential cross-section for ^{12}C at $E = 240.17$ MeV assuming Gaussian density and H.O. density; a — on the basis of the eikonal approximation, b — on the basis of Glauber's multiple-scattering theory.

^{12}C nucleus: Figures 1, 2 and 3 show the experimental data [8, 6, 9] together with the theoretical results at energies 240.17, 420 and 450 MeV, respectively. From these figures, we see that the G. density fails to reproduce the experimental data at any angle, while the agreement is good assuming a H.O. density. At 240 MeV, the minimum appearing at $\theta \approx 97°$ is well reproduced, especially when using Glauber's multiple-scattering, as can be seen from comparison between figures 1a and 1b. At higher energy the agreement is excellent, but there is no preference to either approach. In

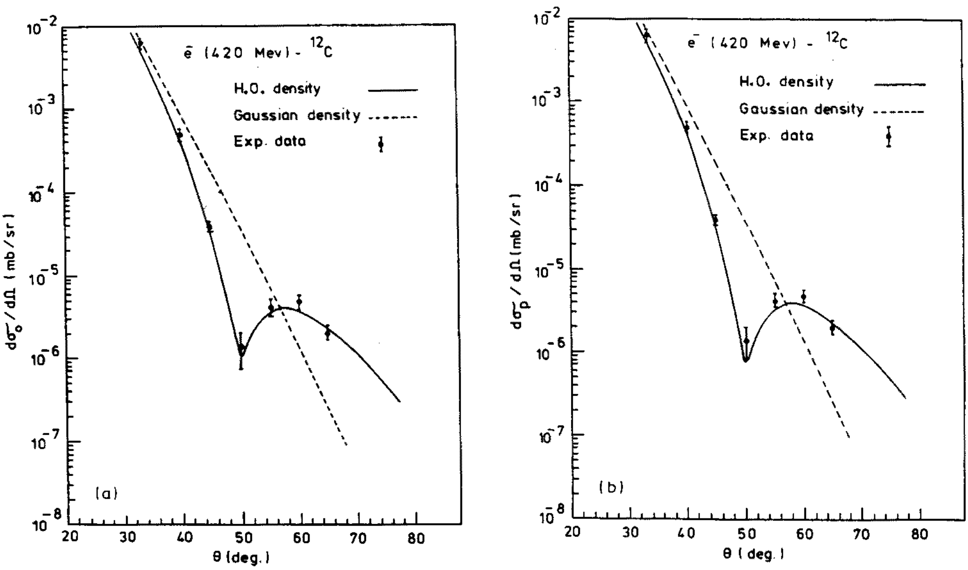

Fig. 2. The same as figure 1; a — on the basis of the eikonal approximation at energy $E = 420$ MeV, b — on the basis of Glauber's multiple-scattering at energy $E = 420$ MeV.

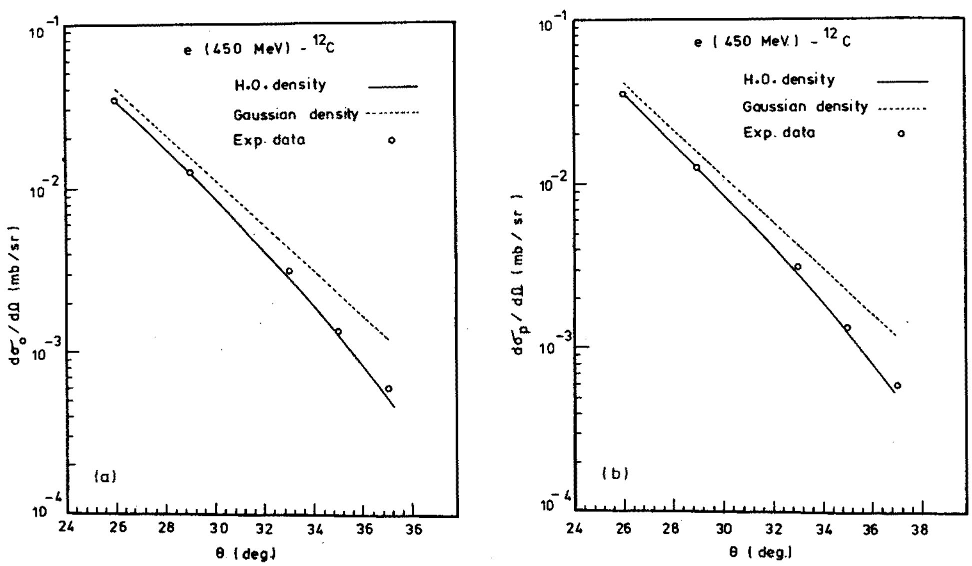

Fig. 3. The same as figure 1; a — on the basis of the eikonal approximation at energy $E = 450$ MeV, b — on the basis of Glauber's multiple-scattering at energy $E = 450$ MeV.

general, for this nucleus the only difference between the two approaches appears mainly around the dip which favours the full multiple-scattering at E=240 MeV.

^{16}O nucleus: A comparison between the theory and the experiment is shown in figures 4 and 5 at energies 240 and 420 MeV, respectively. At energy 240 MeV, a H.O. density reproduces well the experimental data [10] up to $\theta \approx 80°$, then predicts a minimum after that. At higher energy (420 MeV) the H.O. density succeeds to give an excellent agreement with the experimental data [6], as can be seen from figures 5. As predicted, the difference between the eikonal approximation and the full multiple-scattering approximation is clear at the minimum. The agreement attained favours the eikonal approximation.

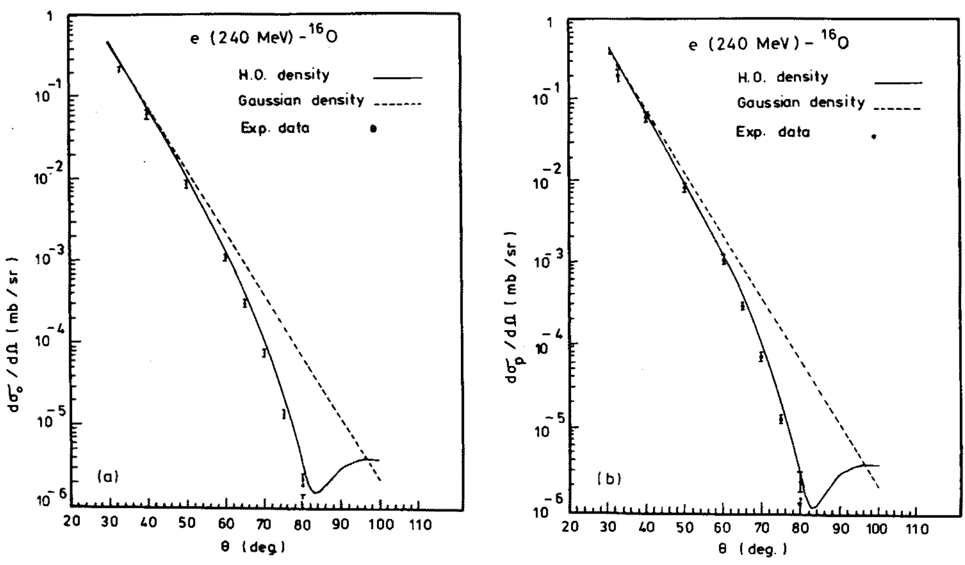

Fig. 4. The same as figure 1 for ^{16}O; a — on the basis of the eikonal approximation at energy E = 240 MeV, b — on the basis of Glauber's multiple-scattering at energy E = 240 MeV.

^{40}Ca nucleus: For this nucleus, the theoretical calculations are carried out assuming a S.G. density and a G. density at energies 183 and 250 MeV, respectively. Figures 6 and 7 show that the charge distribution density for ^{40}Ca nucleus is close to the one predicted by the Woods–Saxon model approximated by a sum of Gaussians. At energy 183 MeV and assuming a S.G. density, the agreement with the experimental data [11] is good up to $\theta \approx 72°$. We see from Fig. 7 that the agreement achieved with a S.G. density is excellent at all angles, except at the second minimum (corresponding to

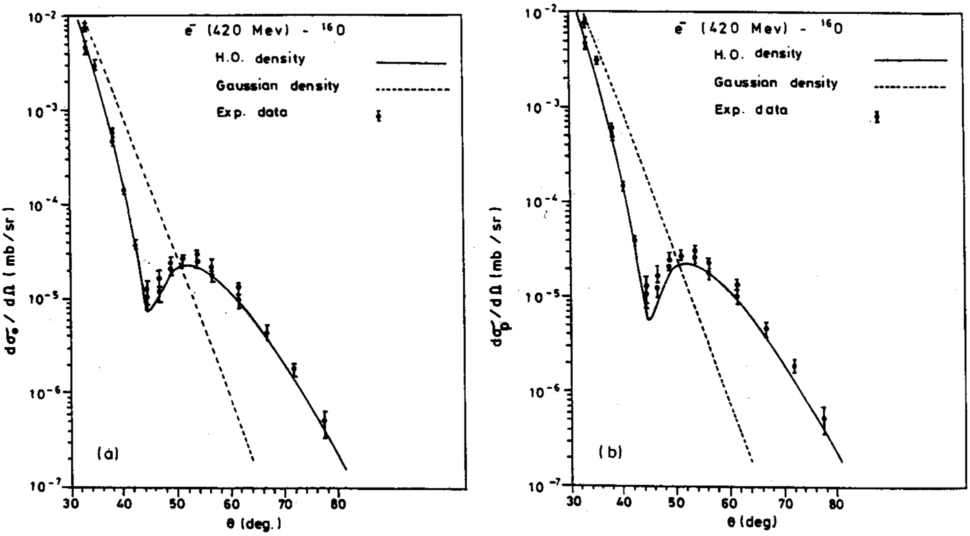

Fig. 5. The same as figure 1; a — on the basis of the eikonal approximation at energy $E = 420$ MeV, b — on the basis of Glauber's multiple-scattering at energy $E = 420$ MeV.

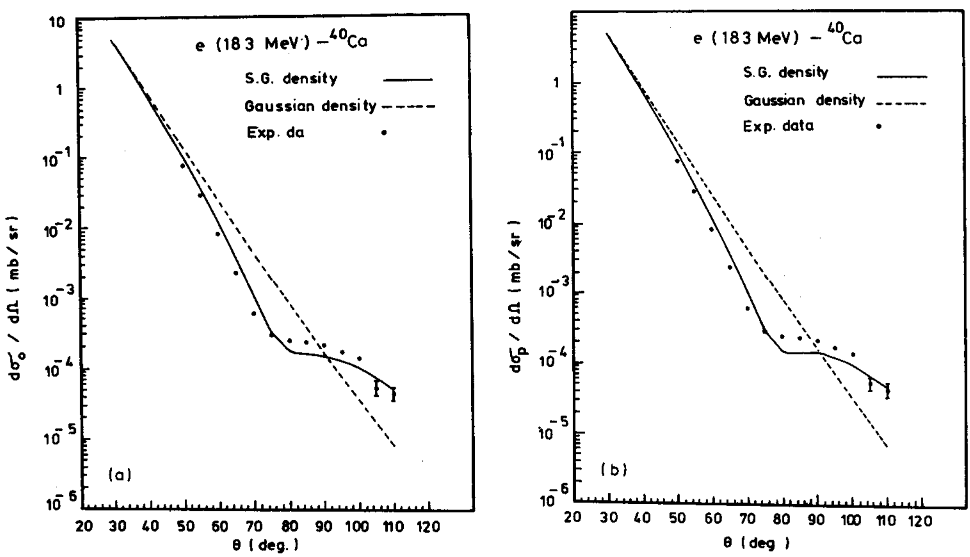

Fig. 6. The same as figure 1 for ^{40}Ca assuming a Gaussian density and S.G. density at energy $E = 183$ MeV; a — on the basis of the eikonal approximation at energy $E = 183$ MeV, b — on the basis of Glauber's multiple-scattering at energy $E = 183$ MeV.

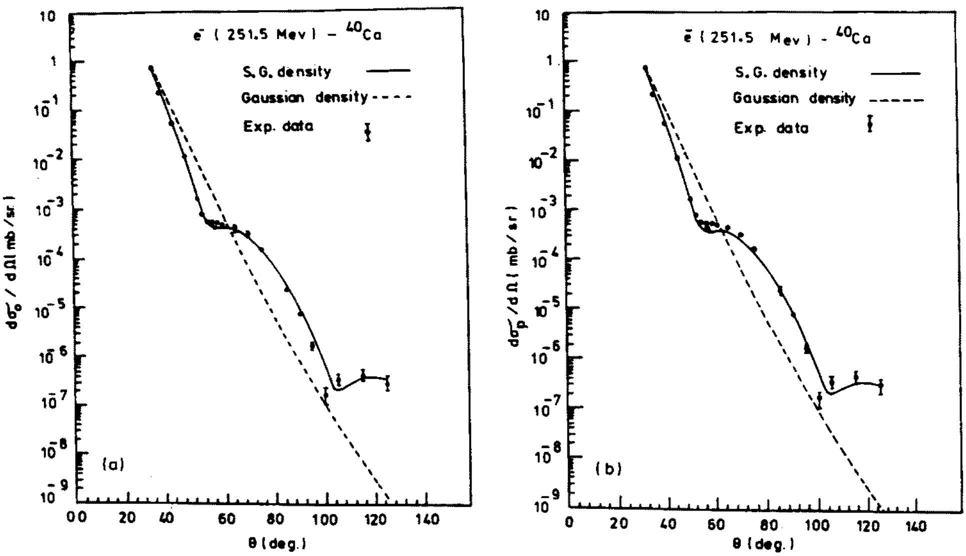

Fig. 7. The same as figure 6; a — on the basis of the eikonal approximation at energy $E = 251.5$ MeV, b — on the basis of Glauber's multiple-scattering at energy $E = 251.5$ MeV.

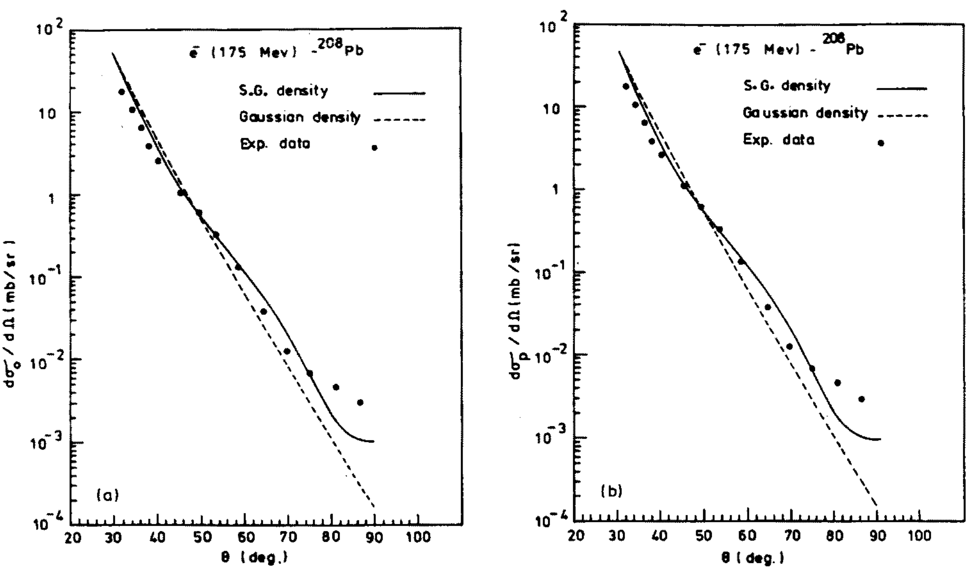

Fig. 8. The same as figure 6 for ^{208}Pb; a — on the basis of the eikonal approximation at energy $E = 175$ MeV, b — on the basis of Glauber's multiple-scattering at energy $E = 175$ MeV.

$\theta \approx 100^\circ$), where there is a slight difference with the experiment [12]. For this nucleus there is no difference between the two approaches, even at the minimum.

^{208}Pb nucleus: The elastic scattering cross-sections are calculated assuming a G. density and a S.G. density. Comparison with the experimental data [13], at $E = 175$ MeV (Figs 8). shows that the agreement attained with a S.G. density is good up to $\theta \approx 75^\circ$ only. This may be attributed to the presence of other phenomena, owing to the strong Coulomb field of this nucleus.

4. Conclusions

Elastic scattering of high energy electrons by different nuclei is investigated. Specifically, this is achieved on the basis of Glauber's full multiple-scattering theory and on the basis of the eikonal approximation which are developed previously in the literature. Comparison between the two approaches is formally studied, assuming different models of the nucleon density distribution. This study reveals that the difference between the two approaches is small, especially for small momentum transfer and for small impact parameter. This analysis also shows that the effect of varying the density is more appreciable than the effect of the approach applied. Clearly, the eikonal approximation converges more rapidly than the Glauber multiple-scattering approach.

Excellent agreement is achieved when we apply the theoretical expressions of the cross-section to some even-even nuclei. No preference between the two approaches has been realized, except at the diffraction minimum where the full multiple-scattering approach generally gives a better agreement between the theory and the experiment for lighter nuclei. The fit shows that the main discrepancies between the theory and the data appear for the heavy nucleus: ^{208}Pb. The calculations employ a point Coulomb amplitude appropriate to the Schrödinger equation, whereas the electron should be treated as a Dirac particle. The eikonal approximation for the Dirac case, and corrections to it are discussed by Wallace and Friar [14]. The corrections of order V^2 are different from the Schrödinger case, and such corrections may be relevant to heavy nuclei such as ^{208}Pb. This will be the subject of our future work.

In the meantime, work is in progress towards a study of correlation effects in the calculation of the elastic scattering of high energy electrons.

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